Strong Essential Submodules And Strong Uniform Modules

Nada Khalid Abdullah

Department of Mathematics , College of Education for Pure Science , University of Tikrit , Tikrit , Iraq

Abstract

A non-zero submodule K of an R-module M is called essential if K ∩ L ≠ (0) for each non-zero submodule L of M . And an R-module M is called uniform if each non-zero submodule of M is an essential . In this paper we give generalization of essential submodule and uniform module that are strong essential submodule and strong uniform module. A non-zero submodule N of M is called strong essential if N ∩ P ≠ (0) for each non-zero strongly prime submodule P of M . And an R-module M is called strong uniform if each non-zero submodule of M is a strong essential .

Keywords: Prime submodules , Strongly prime submodules, Essential submodules, Semi-Essential submodules, Uniform modules, Semi-Uniform module .

Introduction

Let R be a commutative ring with unity and let M be unitary R-module . In this work we assume that every submodule of M contained in strongly prime submodule of M . A non-zero submodule K of M is called essential if K ∩L ≠ (0) for e ach non-zero submodule L of M , [9] . And a proper submodule P of M is called prime if for each m ∈ M and r ∈ R whenever rm ∈ P , then either m ∈ P or r ∈ [P:M], [10] . Abdullah , N. K. (2005) gave in her thesis [1] generalization of essential submodule , she name it semi-essential submodule that is a non-zero submodule N of M is semi-essential if N ∩ P ≠ (0) for each non-zero prime submodule P of M . A proper submodule B of M is called semi-prime if for each m ∈ M and r ∈ R with $r^k x$ ∈ B , k ∈ Z^+, then rx ∈ B , [11] . Ahmed , M. A. (2009) generalize essential submodule to weak essential submodule that is a non-zero submodule W of M is weak essential if W∩B ≠ (0) for each non-zero semi-prime submodule B of M, [6] . A proper submodule P of M is called strongly prime if I_x^py ⊆ P for x, y ∈ M implies that either x ∈ P or y ∈ P, whenever I_x^p= { r ∈ R : rM ⊆ P + Rx }, [11] . A non-zero submodule N of M is called strong essential if N ∩ P ≠ (0) , for each non-zero strongly prime submodule P of M .

An R-module M is called uniform if each non-zero submodule of M is an essential, [5, P.85]. [1] generalize uniform module to semi-uniform module that is an R-module M is called semi-uniform if for each non-zero submodule of M is semi-essential . [6] generalize uniform module to weak uniform module that is an R-module M is called weak uniform if for each non-zero submodule of M is weak essential. In this work we present a strong uniform module concept as a generalization of semi-uniform module, that is a module M is called a strong uniform if each non-zero submodule of it is strong essential . Also we generalize some characterizations and properties of semi-essential submodules to strong essential submodules and semi-uniform modules to strong uniform modules .

S_1 : Strong Essential Submodules

In this section we introduce strong essential submodules and give a characterization for strong essential submodules. Also, we generalize some properties of semi-essential submodules to strong essential submodules . And we study the relation between prime submodule and strongly prime submodule. Recall that a proper submodule B of M is called prime if for each x ∈ M and r ∈ R whenever rm ∈ B, then either x ∈ B or r ∈[B:M], where [B:M] ={r ∈ R: rM ⊆B}, [10] . A proper submodule B of M is called semi-prime if for each x ∈ M and r ∈ R with $r^k x$ ∈ B, k ∈ Z^+ , then rx ∈ B, [11]. It ś clear that each strongly prime submodule is a prime submodule, but the converse is not always correct . For example; let p ∈ spec (R). Then (p,p) is a prime submodule of an R-module R×R, but It ś not strongly prime submodule, because $I_{(1,0)}^{(p,p)}$ (1,0) ⊆ P(1,0) ⊆ (1,0) , and (1,0) ∉ (p,p) .

(1-1) : Definition

A non-zero submodule N of M is called strong essential if N ∩ P ≠ (0) for each non-zero strongly prime submodule P of M .

(1-2) : Examples and Remarks

1/ Let M = Q ⊕ Q and let N = xQ , where x = (1,1) . The only strongly prime submodules of M are pQ ⊕ (0) and (0) ⊕ pQ , where p is a prime number . Then N ∩ (pQ ⊕ (0)) ≠ (0) and N ∩ ((0) ⊕ pQ) ≠ (0) . Hence N is a strong essential submodule of M .

2/ Every essential submodules are strong essential submodule . But the converse is not true in general , as we see in the following example .

3/ Let M = Q ⊕ Q . If N = Qx , where x = $(1,\frac{1}{2})$, the only strongly prime submodules of M are of the form pQ ⊕ (0) and (0) ⊕ pQ, where p is a prime number. Then N ∩ (pQ ⊕ (0)) ≠ (0) and N ∩ ((0) ⊕ pQ) ≠ (0) , hence N is a strong essential submodule of M . But it ś not an essential submodule of M , because N ∩ yZ = (0) , where y = (0,1) .

4/ On the other hand , every strong essential submodule is semi-essential submodule . But the converse is not true in general , as we see in the following example

5/ Let $M = R\times R$. If $N = Rx$, where $x = (1,0)$, the only prime submodule of M is of the form R(p,p) , where p is a prime number .Thus N is a semi-essential submodule of M , because $N \cap R(p,p) \neq (0)$. But N is not strong essential submodule of M , because $N \cap R(0,p) = (0)$, whenever R(0,p) is strongly prime submodule of M .

Under some conditions we can get strong essential submodules from semi-essential submodules .

Recall that an integral domain R with quotient field K , and a torsion free R-module M . A prime submodule P of M is called strongly prime if for each $y \in K$ and $x \in M_T$, $yx \in P$ gives $x \in P$ or $y \in (P: M)$, whenever $T = R/\{0\}$, [7] .

(1-3) : Proposition

Let R be an integral domain with quotient field K , and let M be a torsion free R-module . If N is a non-zero submodule of M , then N is a strong essential submodule of M if and only if N is a semi-essential submodule of M .

The following proposition is a characterization of strong essential submodules . Compare with [9] .

(1-4) : Proposition

Let M be an R-module , a non-zero submodule N of M is a strong essential if and only if for each non-zero strongly prime submodule P of M , there exists $x\in P$ and $r\in R$ such that $0 \neq rx \in N$.

Proof : Suppose that for each non-zero strongly prime submodule P of M , there exists $x\in P$ and $r\in R$ such that $0 \neq rx \in N$. Not that $rx\in P$, therefore $0 \neq rx \in N\cap P$. Thus $N\cap P \neq (0)$, implies that N is a strong essential submodule of M .

Conversely ; suppose N is a strong essential submodule of M . Then $N\cap P \neq (0)$ for each non-zero strongly prime submodule P of M , thus there exists $0 \neq x \in N\cap P$. This implies that $x \in N$ and hence $0 \neq 1.x \in N$.

(1-5) : Remark

Let M be an R-module and let N_1 and N_2 are submodules of M such that N_1 is a submodule of N_2 . If N_1 is a strong essential submodule of M , then N_2 is strong essential submodule of M .

Proof : Suppose that $N_2\cap P = (0)$ for some strongly prime submodule P of M . Since N_1 is a submodule of N_2 , thus $N_1\cap P = (0)$. But N_1 is a strong essential submodule of M , therefore $P = (0)$. Hence N_2 is a strong essential submodule of M .

(1-6) : Proposition

Let M be an R-module and let N_1 and N_2 are submodules of M . Then

1/ If $N_1\cap N_2$ is a strong essential submodule of M , then both N_1 and N_2 are strong essential submodules of M .

2/ If N_1 is an a essential submodule of M and N_2 is a a strong essential submodule of M , then $N_1\cap N_2$ is also strong essential submodule of M .

Proof : 1/ Follows immediately from remark (1-5) .

2/ Let P be a strongly prime submodule of M . Since N_2 is a strong essential submodule of M, thus $N_2 \cap P\neq (0)$. And since N_1 an essential submodule of M , then $N_1\cap(N_2\cap P) \neq (0)$, so $(N_1\cap N_2)\cap P \neq (0)$. Which implies that $N_1\cap N_2$ is a strong essential submodule of M .

(1-7) : Corollary

Let M be an R-module and let N_1 and N_2 are submodules of M , such that either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$ .Then $N_1\cap N_2$ is a strong essential submodule of M if and only if N_1 and N_2 are strong essential submodules of M .

S_2 : Strong Essential Homomorphisms

This section is devoted to study strong essential homomorphisms, we start this section by the following definition .

(2-1) : Definition

Let M_1 and M_2 are two R-modules . An R-homomorphism $f : M_1 \rightarrow M_2$ is called strong essential homomorphism if $f(M_1)$ is a a strong essential submodule of M_2 .

(2-2) : Remark

Let M be an R-module and let N be a submodule of M , Then N is a a strong essential submodule of M if and only if the inclusion homomorphism $i : N \rightarrow M$ is strong essential homomorphism .

Before we give the following proposition we need the following result .

(2-3) : Lemma

Let M_1 and M_2 are two R-modules and let $f : M_1 \rightarrow M_2$ be an epimorphism . Assume P is a submodule of M_1 such that $\ker f \subseteq P$, Then P is a strongly prime submodule of M_1 if and only if f(P) is a strongly prime submodule of M_2 .

Proof : Let P be a strongly prime submodule of M_1 , and let $I_{x_1}^{f(P)}x_2 \subseteq f(P)$, where x_1 , $x_2 \in M_2$. It ś mean that $(f(P)+Rx_1: M_2)\, x_2 \subseteq f(P)$. There exists m_1, $m_2 \in M_1$ such that $f(m_1) = x_1$ and $f(m_2) = x_2$. Implies that $(f(P) +R f(m_1):M_2)\, f(m_2) \subseteq f(P)$, and there exists $r \in R$ and $y \in P$ such that $(f(y) + rf(m_1))\, f(m_2) = f(y)$, implies $(f(y)+f(rm_1))\, f(m_2)=f(y)$, so $f(y+rm_1)f(m_2)=f(y)$ and $f((y+rm_1)m_2) - f(y) = 0$. which is implies $(y+rm_1)m_2 - y \in \ker f$. Since $\ker f \subseteq P$, hence $(y+rm_1)m_2 - y \in P$. And since $y \in P$, thus $(P + Rm_1)m_2 \subseteq P$, it ś mean that $I_{m_1}^{P}m_2 \subseteq P$. And since P is a strongly prime submodule of M_1, then either $m_1\in P$ or $m_2\in P$. If $m_1\in P$, implies $x_1=f(m_1)\in P$, and if $m_2 \in P$ implies $x_2=f(m_2)\in P$. Therefore f(P) is a strongly prime submodule of M_2 .

Conversely; suppose f(P) is a strongly prime submodule of M_2 and $I_{m_1}^{P}m_2 \subseteq P$, where m_1, $m_2 \in M_1$. It ś mean that $(P + Rm_1)m_2 \subseteq P$, hence $f((P + Rm_1)m_2) \subseteq f(P)$, and $f(P + Rm_1)f(m_2) \subseteq f(P)$. There exists $x_1, x_2 \in M_2$ such that $x_1 = f(m_1)$ and $x_2 = f(m_2)$. Hence $(f(P) + Rx_1)\, x_2 \subseteq f(P)$, so $I_{x_1}^{f(P)}x_2 \subseteq f(P)$. Since f(P) is a strongly prime submodule of M_2, hence either $x_1\in f(P)$ or $x_2 \in f(P)$. If $x_1 \in f(P)$, implies $f(m_1) \in f(P)$ and $f^{-1}(f(m_1))\in f^{-1}(f(P))$, thus $m_1\in P$. And if $x_2 \in f(P)$ implies $f(m_2) \in f(P)$ so $m_2 \in P$. Therefore P is a strongly prime submodule of M_1 .

(2-4) : Proposition

Let M_1 and M_2 are two R-modules, and let f : $M_1 \rightarrow M_2$ be an epimorphism . Then

1/ If N_1 is a strong essential submodule of M_1, then $f(N_1)$ is a strong essential submodule of M_2 .

2/ If N_2 is a strong essential submodule of M_2 , such that ker $f \subseteq P_1$ for each strongly prime submodule P_1 of M_1. Then $f^{-1}(N_2)$ is a strong essential submodule of M_1 .

Proof: 1/ Let P_2 be a strongly prime submodule of M_2, then $f^{-1}(P_2)$ is strongly prime submodule of M_1 (lemma (2-3)) . N_1 is a strong essential submodule of M_1, thus $N_1 \cap f^{-1}(P_2) \neq (0)$, and hence $f(N_1) \cap P_2 \neq (0)$. Therefore $f(N_1)$ is a strong essential submodule of M_2.

2/ Let $f^{-1}(N_2) \cap P_1 = (0)$, where P_1 is a strongly prime submodule of M_1 . Since ker $f \subseteq P_1$, then $f(P_1)$ is a strongly prime submodule of M_2 (lemma (2-3)) . This implies that $N_2 \cap f(P_1)=(0)$. Since N_2 is a strong essential submodule of M_2 , then $f(P_1)=(0)$. Thus $P_1 \subseteq$ ker $f \subseteq f^{-1}(N_2)$, and hence $f^{-1}(N_2) \cap P_1 = (0)$, this implies that $P_1 = (0)$. Therefore $f^{-1}(N_2)$ is a strong essential submodule of M_1 .

Before we give the following theorem , we need the following lemma .

(2-5) : Lemma

Let M_1 and M_2 are two R-modules , and let P_2 be a strongly prime submodule of M_2 such that $Hom_R(M_1 , P_2) \subseteq Hom_R(M_1, M_2)$. Then $Hom_R(M_1, P_2)$ is a strongly prime submodule of $Hom_R(M_1 , M_2)$.

Proof: Let $f_1, f_2 \in Hom_R(M_1, M_2)$ such that $I_{f_1}^{HomR(M1,P2)} f_2 \subseteq Hom_R(M_1, P_2)$. Then for each $x_1 \in M_1$, we get $I_{f1(x)}^{P2} f_2(x) \subseteq P_2$. Since P_2 is a strongly prime submodule of M_2 , then either $f_1(x) \in P_2$ or $f_2(x) \in P_2$. If $f_1(x) \in P_2$ implies that $f_1 \in Hom_R(M_1, P_2)$, and if $f_2(x) \in P_2$ implies that $f_2 \in Hom_R(M_1, P_2)$. Therefore $Hom_R(M_1, P_2)$ is a strongly prime submodule of $Hom_R(M_1 , M_2)$.

(2-6) : Theorem

Let M_1 and M_2 are two R-modules and let $Hom_R (M_1,N_2)$ be a proper submodule of $Hom_R(M_1,M_2)$ for each submodule N_2 of M_2. If $Hom_R(M_1 ,N_2)$ is strong essential submodule of M_2, then N_2 is strong essential submodule of M_2 .

Proof: Let P_2 be a non-zero strongly prime submodule of M_2 , then by lemma (2-5) , we get $Hom_R (M_1 , P_2)$ is strongly prime submodule of $Hom_R (M_1 , M_2)$. Since $Hom_R (M_1 , N_2)$ is strong essential submodule of $Hom_R (M_1 , M_2)$, then by proposition (1-4) , there exists $0 \neq f \in Hom_R (M_1 , P_2)$, and $0 \neq r \in R$ such that $0 \neq rf \in Hom_R (M_1,N_2)$, that is $rf(m) \in N_2$ for each $m \in M_1$. So for each non-zero strongly prime submodule P_2 of M_2 , we find $f(m) \in P_2$ for each $m \in M_1$. And $r \in R$ with $0 \neq rf(m) \in N_2$. Hence N_2 is strong essential submodule of M_2 .

(2-7) : Corollary

Let M be an R-module and let N be a submodule of M. If $Hom_R (M, N)$ is a strong essential submodule of $Hom_R (M, M)$, then N is a strong essential submodule of M .

S_3: Strong Essential Submodules In Multiplication Modules

Recall that an R-module M is called a multiplication R-module if every submodule N of M is of the form EM for some ideal E of a ring R , [4] . In this section we gives a condition under which a submodule N of a faithful multiplication R-module M becomes strong essential submodule .

Recall that an R-module M is called a faithful if annM = 0 , [4] . Recall that if R is an integral domain and K is a quotient field , then the prime ideal B of R is called strongly prime ideal if each x , y ∈ K, and x, y ∈ B , then either x ∈ B or y ∈ B .

Now, we start this section by the following definition.

(3-1) : Definition

A non-zero ideal I of a ring R is called strong essential ideal if $I \cap B \neq (0)$ for each non-zero strongly prime ideal B of R , [8] .

Before we give the following proposition, we need the following well-know lemma, which appear in [7, Prop. (2-10)] .

(3-2) : Lemma

Let R be an integral domain and let M be a faithful multiplication R-module. Then B is a strongly prime ideal of R if and only if BM is a strongly prime submodule of M .

(3-3) : Proposition

Let R be an integral domain and let M be a faithful multiplication R-module. N is a submodule of M such that N = IM for some ideal I of R . Then N is a strong essential submodule of M if and only if I is a strong essential ideal of R .

Proof : Let N be a strong essential submodule of M and let $I \cap B = (0)$ where B is a non-zero strongly prime ideal of R. Since M is a multiplication R-module, then $(I \cap B)M = IM \cap BM = (0)$. By lemma (3-2) BM is strongly prime submodule of M, since IM=N is a strong essential submodule of M, thus BM=(0). And since M is a faithful R-module , then B=(0). Therefore I is strong essential ideal of R .

Conversely; let $N \cap P = (0)$ where P is a non-zero strongly prime submodule of M. Since M is a multiplication R-module , then there exists a strongly prime ideal B of R such that P = BM (lemma (3-2)) . Hence $N \cap P = IM \cap BM = (I \cap B)M = (0)$. But M is a faithful R-module , thus $I \cap B = (0)$. And since I is a strong essential ideal of R, hence B=(0) , whence BM=P=(0). Therefore N is strong essential submodule of M .

Also , we can put another condition to get a strong essential submodule IM of
a multiplication R-module M from strong essential ideal I of R , by the following proposition .

Before we give the following propositions, we need the following well-know lemma which appear in [13, Prop. (1-14)(1)] .

(3-4) : Lemma

Let M be a multiplication R-module with $ann_R M \subseteq B$ and $BM \neq M$ for each strongly prime ideal B of R . If

B is a strongly prime ideal of R , then BM is strongly prime submodule of M .

(3-5) : Proposition

Let M be a faithful multiplication R-module with $ann_R M \subseteq B$ and BM≠M for each strongly prime ideal B of R . If I is a strong essential ideal of R , then IM is a strong essential submodule of M .

Proof: Let IM ∩ P= (0) where P is a non-zero strongly prime submodule of M. Since M is a multiplication R-module , then there exists a strongly prime ideal B of R such that P = BM (lemma (3-4)) . Hence IM∩P= IM∩BM=(I∩B)M=(0). Since $ann_R M \subseteq B$, then I ∩ B= (0) . And since I is a strong essential ideal of R, hence B=(0), whence BM=P=(0). Therefore IM is strong essential submodule of M .

(3-6) : Corollary

Let M be a faithful multiplication R-module with $ann_R M \subseteq B$ and BM≠M for each strongly prime ideal B of R, If I is an ideal of R . Then I is a strong essential ideal of R if and only if IM is a strong essential submodule of M.

By [1 , prop.(2-1-1] and prop.(1-3) , we can get the following result .

(3-7) : Proposition

Let R be an integral domain with quotient field K , and let M be a torsion free faithful multiplication R-module. If N is a submodule of M such that N = IM for some ideal I of R . Then N is a strong essential submodule of M if and only if I is a strong essential ideal of R .

(3-8) : Lemma [13 , Th. (1-12)]

Let M be a multiplication R-module , and let P be a submodule of M . Then P is a strongly prime submodule of M if and only if (P:M) is strongly prime ideal of R .

(3-9) : Proposition

Let R be an integral domain, and M is a faithful multiplication R-module. N is a submodule of M . Then N is a strong essential submodule of M if and only if (N:(x)) is a strong essential ideal of R, for each x ∈ M .

Proof : Let N be a strong essential submodule of M . By prop.(3-3) (N:M) is a strong essential ideal of R . But for each x ∈ M , (N:M) ⊆ (N:(x)) . Since M is a multiplication R-module , thus (N:M)M ⊆ (N:(x))M , [4] . This implies that (N:(x))M is strong essential submodule of M (remark (1-5)) . Hence (N:(x)) is strong essential ideal of R , (prop,(3-3)) .

Conversely ; let (N:(x)) is strong essential ideal of R, for each x∈M , and let B be a non-zero strongly prime ideal of R . Hence (N:(x)) ∩ B ≠ (0), by lemma (3-2) we get BM is strongly prime submodule of M , Since M is a multiplication R-module, then (N:(x))M ∩ BM ≠ (0). Thus N ∩ BM ≠ (0) that is N is a strong essential submodule of M .

By the same way we can prove the following proposition .

(3-10) : Proposition

Let M be a faithful multiplication R-module with $ann_R M \subseteq B$ and BM≠M for each strongly prime ideal B of R . If N is a submodule of M . Then N is a strong essential submodule of M if and only if (N:(x)) is strong essential ideal of R , for each x ∈ M .

From propositions (3-3) and (3-9) , we have the following theorem .

(3-11) : Theorem

Let R be an integral domain and let M be a faithful multiplication R-module. If N is a submodule of M such that N = IM for some ideal I of R . Then the following statements are equivalent .

1/ N is a strong essential submodule of M .

2/ I is a strong essential ideal of R .

3/ (N:(x)) is a strong essential ideal of R , for each x∈M .

Proof : (1) ⇒ (2) By prop. (3-3).

(2) ⇒ (3) Let I be a strong essential ideal of R, since M is a faithful multiplication R-module, then I = (IM:M) , [7]. But (IM:M)⊆(IM:(x)) for each x ∈ M, and (IM:M) is strong essential ideal of R , also we consider (IM:(x)) is strong essential ideal of R (remark (1-5)). Thus (N:(x)) is strong essential ideal of R .

(3) ⇒ (1) By prop. (3-9) .

From propositions (3-5) and (3-10) , we have the following theorem .

(3-12) : Theorem

Let M be a faithful multiplication R-module with $ann_R M \subseteq B$ and BM≠M for each strongly prime ideal B of R . If N is a submodule of M such that N = IM for some ideal I of R . Then the following statements are equivalent .

1/ N is a strong essential submodule of M .

2/ I is a strong essential ideal of R .

3/ (N:(x)) is a strong essential ideal of R, for each x∈M .

The proof is similar to the proof of theorem (3-11) , hence we omitted .

S_4 : Strong Uniform R-modules

Recall that a non-zero module M is called uniform if every non-zero submodule of M is an essential, [3, p. 85]. Abdullah, N. K. generalize uniform module, she name it semi-uniform module that is an R-module M in which every non-zero submodule is semi-essential, [1]. In this section, we introduce another generalization of uniform module to strong uniform modules .

(4-1) : Definition

A non-zero R-module M is called strong uniform if each non-zero submodule of M is strong essential . And a ring R is called strong uniform if each non-zero ideal of R is strong essential .

The following theorem gives a relation between strong uniform module and strong uniform ring .

(4-2) : Theorem

Let R be an integral domain and let M be a faithful multiplication R-module. Then M is strong uniform R-module if and only if R is strong uniform ring.

Proof : Let M be a strong uniform R-module, and let I be an ideal of R such that I ∩ B = (0) for each strongly prime ideal B of R. Since M is a multiplication R-module, then (I ∩ B)M = (0), [2, prop.(1-3), P.10], so IM ∩ BM = (0). By lemma (3-2) we get BM is a strongly prime submodule of M. But M is a strong uniform R-module, hence IM is a strong essential submodule of M. Therefore BM = (0), and since M is a faithful R-module, then B = (0). Thus I is a strong essential ideal of R, Whence R is strong uniform ring .

Conversely ; let R be a strong uniform ring , and let N be a submodule of M such that N ∩ P = (0) for each strongly prime submodule P of M . Thus (N∩P:M) = (0), so (N:M) ∩ (P:M) = (0) , but (P:M) strongly prime ideal of R (lemma (3-8)) . Since R is strong uniform ring, then (N:M) is strong essential ideal of R . Hence (P:M) = (0), which implies that P = (0), that is N is strong essential submodule of M. Therefore M is strong uniform R-module .

(4-3) : Proposition

Let M_1 and M_2 are two R-modules , and let f : $M_1 \rightarrow M_2$ be an epimorphism . Then

1/ If M_1 is strong uniform R-module , then M_2 is also strong uniform R-module .

2/ If M_2 is strong uniform R-module such that ker f ⊆ P for each strongly prime submodule P of M_1 , then M_1 is strong uniform R-module .

Proof :

1/ Let N_2 be a submodule of M_2 , then $f^{-1}(N_2)$ is a submodule of M_1 . Since M_1 is strong uniform R-module, thus $f^{-1}(N_2)$ is a strong essential submodule of M_1. By prop. (2-4) (1), we get $f(f^{-1}(N_2)) = N_2$ is a strong essential submodule of M_2 . Therefore M_2 is strong uniform R-module .

2/ / Let N_1 be a submodule of M_1, then $f(N_1)$ is a submodule of M_2. Since M_2 is strong uniform R-module, thus $f(N_1)$ is a strong essential submodule of M_2. By prop. (2-4) (2), we get $f^{-1}(f(N_1)) = N_1$ is a strong essential submodule of M_1. Therefore M_1 is strong uniform R-module .

(4-4) : Corollary

Let M_1 and M_2 are two R-modules, and let f: $M_1 \rightarrow M_2$ be an epimorphism , such that ker f ⊆ P for each strongly prime submodule P of M_1 . Then M_1 is strong uniform R-module if and only if M_2 is strong uniform R-module .

Recall that an R-module M is called pseudo valuation module (provided PVM) , if every prime submodule of M is strongly prime submodule , [7] .

(4-5) : Remark

Let M be a pseudo valuation module . If M is a semi-uniform R-module , then M is strong uniform R-module .

Recall that a trace of R-module M is defined by T(M) $=\sum_{f\in M^*} f(M)$ [14], where $M^* = Hom(M,R)$.

Recall that an R-module M is called torsion less module if ∩ ker f =(0) , where f ∈ Hom(M,R) [9] .

We end this section by the following proposition, compare with [1,prop.(2-3-11)] .

(4-6) : Proposition

Let M be a faithful multiplication R-module . And for each f ∈ M^*, f is an onto and ker f ⊆ P for each strongly prime submodule P of M . If T(M) is a strong uniform module , then M is strong uniform module .

Proof : Let N be a non-zero submodule of M , and let P be a strongly prime submodule of M, such that N∩P =(0) . Since M is multiplication R-module. Thus (N:M)M ∩ (P:M)M = (0), and then (N:M) ∩ (P:M) = (0). Then M is torsion less module [9], so $\bigcap_{f\in M^*} \ker f = (0)$, and there are f, g ∈ M^* such that f(N)≠(0) and g(P) ≠(0). In fact if f(N)≠(0) for each f ∈ M^*, then $N \subseteq \bigcap_{f\in M^*} \ker f = (0)$, which is a contradiction , similarly g(P) ≠(0). Note that g(P) is strongly prime ideal of R ⟨lemma(2-3)⟩ , and is strongly prime ideal of T(M) .

(N:M) ⊇ (N:M)f(M) = f⟨(N:M)M⟩ = f(N) ⊆ T(M) ,

and (P:M) ⊇ (P:M)g(M) = g⟨(P:M)M⟩ = g(P) ⊆ T(M).

Then f(N)∩g(P) ⊆ (N:M) ∩ (P:M) = ⟨(N∩ P):M⟩ = (0:M) = annM .

Hence f(N)∩g(P)=(0). This is a contradiction, whence $N \cap P \neq (0)$. Therefore N is a strong essential submodule of M , and hence M is a strong uniform module .

References

1/ Abdullah, N. K. (2005), Semi-essential submodules and semi-uniform modules , M. Sc. Thesis, Univ. of Tikrit .

2/ Ahmed, A. A. (1992), On submodules of multiplication modules, M. Sc. Thesis , Univ. of Baghdad .

3/ Ahmed, M. A. (2009), Weak essential submodules , Dep. of Math. Um-Salama Sci. Journal , Vol. 6 (1) , p. 214-221 .

4/ Athab, E. A. (1996), Prime and semi prime submodules , M. Sc. Thesis , Univ. of Baghdad .

5/ Launs, J. and McDonald, B. R. (1980), Prime modules and one sided ideals in ring theory and Algebra III, Proceedings of the third Oklahoma conference : 301-344 .

6/ Elbast , A. and Smith , P. F. (1988) , Multiplication modules , Comm. Algebra Vol. 16 : 755-774 .

7/ Goodearl, K. R. (1976), Ring theory, Marcel Dekker , New York .

8/ Hedstron, J. R. and Houston, E. G. (1978) , Pseudo valuation domains , Pac J. of Math. , 75(1) : 137-147 .

9/ Kasch, F. (1982), Modules and rings , Academic press, London .

10/ Lu , C. P. (1984) , Prime submodules of modules , Comment Math. Univ. St. Paul. 33 : 61-69 .

11/ Moghaderi , J. and Nekooei , R. (2010) , Strongly prime submodules and pseudo-valuation modules , Rec. 31 May , com. Math. Subject class.: 31c13c99 .

12/ Naghipour, A. R. (2000), Strongly prime submodules, Dep. Of Math. Univ. of Shahrekord, Iran, P. 1-9 .
13/ Naoum, A. G. and Hadi, I. M. A. (2009), On strongly prime submodules, Iraqi Journal of science , P. 35-40 .
14/ Zimmermann Huisgen, B. (1975), Endomorphism rings of self generators, pacific, J. Math. 61 : 587-602 .

المقاسات الجزئية الجوهرية بقوة والمقاسات المنتظمة بقوة

ندى خالد عبد الله الدبان

قسم الرياضيات ، كلية التربية للعلوم الصرفة ، جامعة تكريت ، تكريت ، العراق

الملخص

المقاس الجزئي الغير صفري K من المقاس M يسمى بالمقاس الجزئي الجوهري اذا كان لكل مقاس جزئي L من M . ويسمى المقاس M مقاس منتظم اذا كان كل $K \cap L \neq (0)$ مقاس جزئي من M جوهري . لقد عمم الباحث هذين المفهومين الى مقاس جزئي جوهري بقوة ومقاس منتظم بقوة , حيث عرف المقاس الجزئي الغير صفري N بالمقاس الجوهري بقوة لكل مقاس جزئي اولي بقوة P من M . كما عرف الباحث المقاس N $\cap P \neq (0)$ M بانه منتظم بقوة اذا كان كل مقاس جزئي من M جوهري بقوة. ثم عمم بعض صفات المقاس الجزئي الجوهري الى المقاس الجزئي الجوهري بقوة ، فضلا عن ذلك قدمنا تعميم لبعض خصائص المقاس المنتظم الى المقاس المنتظم بقوة، وذلك بوضع بعض الشروط الواجب توفرها للحصول على التعميم.