Study Some Properties of a Circulant Matrix

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Abstract

The aim of this paper is to study the properties of idempotent, nilpotent and stability to a circulant matrix $A \in R^{3\times 3}$ which generated by the first row, finally we showed the relation between this properties with eigenvalues of this matrix.

Keywords: A circulant matrix, idempotent, nilpotent, stability.

1- Introduction:

The existence of steady states depends all on eigenvalues of the system matrix having negative real parts. The eigenvalues can be found directly if the matrix is 2×2 , but for most larger matrices other methods are needed. Many numerical solvers have programs for approximating eigenvalues and there are several simple tests that you can carry out with pencil and paper .These tests will give you information about the location of the eigenvalues, but not their precise values [2].

A special matrices are a square matrices which have the same number of rows and columns such that diagonal matrix if $a_{ij}=0$ for all $i\neq j$ and upper triangular matrix if $a_{ij}=0$ for all $i>j$ and lower triangular matrix if $a_{ij}=0$ for all $i<j$ and symmetric matrix if $a_{ij}=a_{ji}$ for all i and j, also there more special matrices such that the transposed A^T where a_{ij} *and* a_{ji} are swapped. A skew-symmetric is a square matrix A such that $a_{ij}=-a_{ji}$ [4].

Take a circulant matrix a sin [6] which has the form:

$$M=\begin{bmatrix} 1 & a & \dots & a^{n-1} & a^n \\ a & 1 & \dots & a^{n-2} & a^{n-1} \\ \vdots & & & & \vdots \\ a^{i-1} & \dots & \dots & \dots & a^{n+1-i} \\ \vdots & & & & \vdots \\ a^{n-1} & a^{n-2} & \dots & 1 & a \\ a^n & a^{n-1} & \cdots & a & 1 \end{bmatrix}$$

This is generated by the element $a^{|i-j|}$ and found the determinant and inverse of this matrix .

In [7] a special matrices have been studied in which a square matrix M over any field such that $M.M=[0]_{n\times n}$ and called its 2-nilpotent matrix, also we take a square matrix N over any field such that $N.N=I_{n\times n}$ and call it involutory matrix.

In this paper, we take a circulant matrix which has the form:

$$A=\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \dots(1)$$

where a,b,c are real numbers, and it's generated by the first row, and we study of some properties to this matrix .

2- Definitions:

Definition 1[4]: A matrix A such that $A^m=0$ for some positive integer m is called nilpotent , a matrix A such that $A^2=A$ is called idempotent .

Definition 2[3]: Let A be an $n\times n$ matrix. An eigenvalue of A is a root of the characteristic equation of A given by $f(\lambda)=\det(A-\lambda I)$. An eigenvector of A associated to the eigenvalue λ is a non-zero vector v for which $Av=\lambda v$. The multiplicity of λ as an eigenvalue is the multiplicity of λ as a root of the characteristic equation.

3- Helping Results:

The first two lemmas give us information about the roots of an equation and can be applied to the characteristic equation of a matrix.

Lemma 1 (Coefficient Test) [2]:

Suppose that

$$f(\lambda)=\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+a_3\lambda^{n-3}+\dots\dots+a_{n-1}\lambda+a_n \quad \dots\dots(2)$$

where the coefficients are real. If any coefficient of $f(\lambda)$ is either zero or negative, then at least one root has a nonnegative real part.

Lemma 2 (Routh– Hurwitz Test) [2]:

All the roots of the indicated equation have negative real parts precisely when the given conditions are met.

- $\lambda^2+a_1\lambda+a_2$: $a_1>0$, $a_2>0$ (3)
- $\lambda^3+a_1\lambda^2+a_2\lambda+a_3$: $a_1>0,\ a_3>0,\ a_1a_2-a_3>0$. …(4)
- $\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4$:

$a_1>0,\ \ a_1a_2-a_3>0,\ \ (a_1a_2-a_3)a_3-a_1^2a_4>0$, $a_4>0$. .....(5)

Lemma 3 (Trace Test) [2]:

Suppose that A is an $n\times n$ matrix of real constants and that trace $A=a_{11}+a_{22}+\dots\dots+a_{nn}$ is negative (positive). Then at least one eigenvalue of A has a negative (positive) real part. If the trace is zero, then either all eigenvalues have zero real parts, or there is a pair of eigenvalues whose real parts have positive signs.

Proposition 1[2]:

1) The coefficient test does not apply to the characteristic equation with a positive coefficient.

2) The Routh – Hurwitz test applies to characteristic equation of degree no more than four.

3) The trace test applies directly to matrix, so there is no need to find the characteristic equation.

Theorem 1 [3]:

Let A be an $n \times n$ matrix of constants. A critical point for the system $\dot{x} = Ax$ is:

(i) asymptotically stable if all eigenvalues of A has negative real parts

(ii) unstable if A has at least one eigenvalue with positive real parts.

Remark 1[3]: If A is a diagonal or upper triangular matrix, then the eigenvalues of A are displayed along the diagonal

Theorem 2[1]:

The eigenvalues of idempotent matrices are only zero and one.

4- Main result:

Corollary1: The matrix A has the following properties

(i) $A - A^T = -(A - A^T)^T$ and this is skew – symmetric;

(ii) The inverse of the matrix A always exists except when $a = b = c$;

(iii) $(A - A^T)(A - A^T) = M = M^T$;

(iiii) $$A^{-1} = \begin{bmatrix} e & f & g \\ g & e & f \\ f & g & e \end{bmatrix}$$

where
$$e = \frac{a^2 - bc}{a^3 + b^3 + c^3 - 3abc},$$
$$f = \frac{c^2 - ab}{a^3 + b^3 + c^3 - 3abc},$$
$$g = \frac{b^2 - ac}{a^3 + b^3 + c^3 - 3abc}.$$

Theorem 3: The matrix A is not nilpotent.

Proof: since A has three distinct eigenvalues, therefore we can write A as $A = PDP^{-1}$ where D is diagonal matrix, the elements of diagonal it's eigenvalues of A, P is the eigenvector matrix with respect to eigenvalues,

Now, suppose $\exists\ m \in Z_+$ such that

$$A^m = 0 \quad \Rightarrow \quad A^m = PD^mP^{-1} = 0$$
$$P \neq 0 \quad \Rightarrow \quad D^m = 0$$
$$\Rightarrow \lambda_i = 0 \quad \forall i = 1,2,3 \qquad C\,!$$

hence A is not nilpotent, the proof is complete.

Proposition 2: A matrix A is not idempotent matrix.

Proof: The eigenvalues of idempotent matrix are only one or zero (theorem2), but A has eigenvalues not only one or zero, so that the matrix A is not idempotent matrix.

Theorem 4: Let

$$A_1 = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \quad \& \quad A_2 = \begin{bmatrix} a_1 & b_1 & c_1 \\ c_1 & a_1 & b_1 \\ b_1 & c_1 & a_1 \end{bmatrix}$$

then

$A_1A_2 = A_2A_1$ *always* $\forall\ A_1 \,\&\, A_2 \in C$.

Proof: Since $A_1 = PD_1P^{-1}$ & $A_2 = PD_2P^{-1}$

$$A_1A_2 = PD_1P^{-1}PD_2P^{-1}$$

Then
$$= PD_1D_2P^{-1}$$
$$= PD_2D_1P^{-1}$$
$$= PD_2P^{-1}PD_1P^{-1} = A_2A_1$$

Hence $A_1A_2 = A_2A_1$ always.

Proposition 3: The matrix A has one real eigenvalue λ_1 and two complex conjugate eigenvalues λ_2, λ_3

Such that $\lambda_1 = a + b + c$ and

$$\lambda_{2,3} = \frac{1}{2}\left[2a - b - c \pm i\sqrt{3}(b - c)\right].$$

Proof: The characteristic equation of matrix A yield the following form:

$\lambda^3 - 3a\lambda^2 + (3a^2 - 3bc)\lambda - (a^2 + b^2 + c^2 - 3abc) = 0$

….(6)

By relation between the roots and coefficients for a cubic equation [5] we get:

$$\lambda_1 + \lambda_2 + \lambda_3 = 3a$$
$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 3a^2 - 3bc$$
$$\lambda_1\lambda_2\lambda_3 = a^3 + b^3 + c^3 - 3abc$$

Therefore satisfied the coefficients of equation (6) , consequently $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of matrix A, the proof is complete.

Theorem 5: The solutions of matrix A depended on the values a , b ,c as follows:

(i) if $a < -(b + c), a < 0$ then matrix A is asymptotically stable**;**

(ii) if $a > -(b + c)$ then matrix is unstable .

Proof: By proposition (2) we have $\lambda_1 = a + b + c$, $\operatorname{Re}\lambda_{2,3} = a - \frac{1}{2}(b + c)$ and By theorem (1) lead to the conclusion that the matrix A is asymptotically stable if the real parts of those eigenvalues are negative.

So,

$$a + b + c < 0 \quad \text{and} \quad a - \frac{1}{2}(b + c) < 0$$
$$\Rightarrow \quad a < -(b + c) \quad \text{and} \quad a < \frac{b + c}{2}$$

Obviously, if $a < \frac{b + c}{2}$ then λ_1 is not negative and $\operatorname{Re}\lambda_{2,3} < 0$, namely the stability is impossible for $a < \frac{b + c}{2}$, consequently, stability may appear only under the condition of $a < -(b+c)$ the proof is complete.

Corollary2:

(i) if $a = b = c$ the matrix A has only real eigenvalues,

$\lambda_1 = 3a$, $\lambda_{2,3} = 0$;

(ii) if $b = c$ then, $\lambda_1 = a+b+c$, $\lambda_{2,3} = a - \frac{1}{2}(b+c)$

(vanished the imaginary parts).

5 - Illustrative Example:

Example1: Investigate for idempotent, nilpotent and stability for the following matrix

$$A = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}$$

Solution: The matrix A is asymptotically stable (theorem (5) first condition) ,where a = -3 and - (b + c) = - 2 , hence $a < -(b+c)$;

Second method: The eigenvalues of matrix A are $\lambda_1 = -1$, $\lambda_{2,3} = -4$, therefore satisfied theorem (1) first condition, hence the matrix A is asymptotically stable, and by theorem (2) The eigenvalues of matrix A are not zero and one therefore, the matrix A is neither idempotent nor nilpotent**.**

Example2: Investigate for idempotent, nilpotent and stability of the following matrix

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Solution: The matrix A is unstable (theorem (5) second condition) ,where a = -1 and - (b + c) = - 2 ,hence $a > -(b+c)$;

Second method: The eigenvalues of matrix A are $\lambda_1 = 1$, $\lambda_{2,3} = -2$,

Therefore theorem (1) satisfies the second condition . Hence matrix A is unstable, and by theorem (2) The eigenvalues of matrix A are not zero .Therefore ,the matrix A is neither idempotent nor nilpotent.

6 - Conclusion:

In this paper, we have investigated the idempotent, nilpotent and stability of a circulant matrix by using eigenvalues of this matrix. By this method we justified the same results which have been found by previous methods. An illustrative example shows the effectiveness and feasibility of this method.

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دراسة بعض الخصائص للمصفوفة الدائرية

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الملخص

الهدف من هذا البحث هو دراسة خواص, متساوية القوى, معدومة القوى والاستقرارية لمصفوفة دائرية سعة 3×3 وعناصرها تنتمي إلى حقل الأعدادالحقيقية, وهذه المصفوفة مولدة من الصف الأول, وأخيرا بينا العلاقة بين خواص هذه المصفوفة والقيم الذاتية لها .