On Interval-valued Anti Fuzzy Prime bi-ideal of Semigroup

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Abstract:
In this paper, we introduce the notions of interval-valued anti fuzzy prime (resp. strongly prime and semiprime) bi-ideals of semigroup. By using these ideas, we characterize those semigroup for which each interval anti-fuzzy bi-ideal is semiprime and strongly prime.

1. Introduction: The theory of fuzzy sets proposed by Zadeh [6], in his classic paper of 1965, deal with the applications of fuzzy technology in information processing. Mordeson, Malik and Kuroki gave a systematic exposition fuzzy semigroups in [2], where one can find theoretical results on fuzzy semigroups. Shabir and Kanwal [5], introduced the concept of prime bi-ideal, strongly prime bi-ideals and semiprime bi-ideal of a semigroup and studied those semigroups.

In this paper we introduced the concept of interval valued anti fuzzy prime, strongly prime and semiprime bi-ideal of semigroups and give characterization of semigroups in terms of these notions we characterize those semigroups for which each interval-valued anti fuzzy bi-ideal is semiprime and strongly irreducible.

2. Basic Concepts of Semigroups:

A semigroup is a non-empty set K together with an associative binary operation "." . An element 0 of a semigroup K is called a zero element of K if $x0=0x=0$ for all $x\in K$. A semigroup which contains a zero element is called a semigroup with zero.

Anon empty subset A of a semigroup K is called a subsemigroup of K if $ab\in A$ for all $a,b\in A$. A subsemigroup B of a semigroup K is called a bi-ideal of S if $BKB\subseteq B$. A non empty subset A of a semigroup K is called left (right) ideal of K if $KA\subseteq A\,(AK\subseteq A)$. A is called two sided ideal of K if it is both a left and right ideal of K .A bi-ideal B of a semigroup K is called prime (strongly prime) if $B_1B_2\subseteq B\;(B_1B_2\cap B_2B_1\subseteq B)$ implies $B_1\subseteq B$ or $B_2\subseteq B$ for any bi-ideal B_1 B_2 of K [3]. A bi-ideal B of a semigroup K is called semiprime if implies $B_1\subseteq B$ for any bi-ideal B_1 of K [4].

An element of a semigroup K is called a regular element if there exists an element $x\in K$ s.t $axa=a$. A semigroup K is called regular if every element of K is regular. An element x of semigroup K is called S-weakly regular element if there exist $a,b\in K$ s.t $x=xax^2b$.

2-1 Definition [3]: Let μ^L and μ^u subsets of a non empty set X s.t $0\le\mu^L(x)\le\mu^u(x)\le 1$ for all $x\in X$, we define $\hat{\mu}:X\to D([0,1])$ by $\hat{\mu}(x)=\left[\mu^L(x),\mu^u(x)\right]$ then $\hat{\mu}$ is called interval-valued fuzzy subset of X where $D([0,1])$ is the power set of the set $[0,1]$.

Now we define a some set theoretical operation on interval-valued fuzzy set for interval-valued fuzzy sets $\hat{\mu},\hat{\lambda}$ of X . We define the interval-valued fuzzy subset $\hat{\mu}\wedge\hat{\lambda},\hat{\mu}\vee\hat{\lambda}$ and $\hat{\mu}^c$ as follows:

$$\left(\hat{\mu}\wedge\hat{\lambda}\right)(x)=\left[\mu^l(x)\wedge\lambda^l(x),\hat{\mu}^u(x)\wedge\hat{\lambda}^u(x)\right]$$

$$\left(\hat{\mu}\vee\hat{\lambda}\right)(x)=\left[\mu^l(x)\vee\lambda^l(x),\hat{\mu}^u(x)\vee\hat{\lambda}^u(x)\right]$$

$$\mu^c(x)=\left[1-\hat{\mu}^u(x),1-\mu^1(x)\right]\quad\forall x\in X$$

For any $x,y\in X;\hat{\mu}(x)\le\lambda(y)$ implies that $\mu^1(x)\le\lambda^1(y)$ for $\mu^u(x)\le\lambda^u(y)$ also we have $\left(\hat{\mu}\cap\hat{\lambda}\right)(x)=\hat{\mu}(x)\wedge\hat{\lambda}(x)$ and $\left(\hat{\mu}\cup\hat{\lambda}\right)(x)=\hat{\mu}(x)\vee\hat{\lambda}(x)$ where $\left(\hat{\mu}\cap\hat{\lambda}\right)$ is called the intersection of the interval-valued fuzzy subset $\hat{\mu}$ and $\hat{\lambda}(\hat{\mu}\cup\hat{\lambda}\;\; union)$.

Let $\hat{\mu}$ and $\hat{\lambda}$ be two interval-valued fuzzy subsets of a semigroup

Then the anti-product $\hat{\mu}*\hat{\lambda}$ is the interval-valued anti fuzzy set is defined by

$$\left(\mu*\lambda(x)\right)=\begin{cases}\underset{x=yz}{\wedge}\{\mu(y)\vee\lambda(z)\}\\ [1,1]\quad if\ x\ is\ not\ expressible\ as\ x=yz\end{cases}$$

Lemma 2-2: Let $\hat{\lambda},\hat{\mu}$ and $\hat{\delta}$ be the interval-valued anti fuzzy subset of K, if $\lambda\supseteq\mu$ then $\lambda*\delta\supseteq\mu*\delta$ and $\delta*\lambda\supseteq\delta*\mu$.

Proof: Let $x\in K$, then $(\lambda*\delta)(x)=(\mu*\delta)(x)=[1,1]$ if x is not expr-essible as $x=yz$ o.w

$$(\lambda*\delta)(x)=\underset{x=yz}{\wedge}\{\lambda(y)\vee\delta(z)\}$$

$$(\lambda*\delta)(x)\ge\underset{x=yz}{\wedge}\{\mu(y)\vee\delta(z)\}=(\mu*\delta)(x)$$

Similarly we can show that $\delta*\lambda\supseteq\delta*\mu$ □

Lemma 2-3: Let λ,μ and δ be the interval-valued anti fuzzy subset of K then the following properties hold

1. $\lambda*(\mu\cap\delta)=(\lambda*\mu)\cap(\lambda*\delta)$
   $(\mu\cap\delta)*\lambda=(\mu*\lambda)\cap(\delta*\lambda)$
2. $\lambda*(\mu\cup\delta)\supseteq(\lambda*\mu)\cup(\lambda*\delta),\{(\mu\cup\delta)*\lambda\supseteq(\mu*\lambda)\cup(\delta*\lambda)\}$

Proof: Let $x\in K$, if x is not expressible as $x=yz$ then

$$(\lambda*(\mu\cap\delta))(x)=[1,1]=((\lambda*\mu)\cap(\lambda*\delta))(x)$$

Otherwise

$$(\lambda*(\mu\cap\delta))(x)=\underset{x=yz}{\wedge}\{\lambda(y)\vee(\mu\cap\delta)(z)\}$$
$$=\underset{x=yz}{\wedge}\{\lambda(y)\vee((\mu)(z)\wedge\delta(z))\}$$
$$=\underset{x=yz}{\wedge}\{(\lambda(y)\vee\mu(z))\wedge((\lambda(y)\vee\delta(z))\}$$
$$=\underset{x=yz}{\wedge}\{\lambda(y)\vee\mu(z)\}\wedge\underset{x=yz}{\wedge}\{\lambda(y)\vee\delta(z)\}$$
$$=(\lambda*\mu)(x)\cap(\lambda*\delta)(x)$$

Similarly we can show that

$$(\mu\cap\delta)*\lambda=(\mu*\lambda)\cap(\delta*\lambda)$$

$x\in K$, then 2. Let $(\lambda*(\mu\cup\delta))(x)=[1,1]=(\lambda*\mu)\cup(\lambda*\delta)$ If x is not expressible as $x=yz$, otherwise

$$(\lambda*(\mu\cup\delta))(x)=\underset{x=yz}{\wedge}\{\lambda(y)\vee(\mu\cup\delta)(z)\}$$
$$=\underset{x=yz}{\wedge}\{\lambda(y)\vee\mu(z)\vee\delta(z)\}$$
$$=\underset{x=yz}{\wedge}\{\{\lambda(y)\vee(\mu)(z)\}\vee\{\lambda(y)\vee\delta(z)\}\}$$
$$\geq\underset{x=yz}{\wedge}\{\lambda(y)\vee(\mu)(z)\}\vee\underset{x=yz}{\wedge}\{\lambda(y)\vee\delta(z)\}$$
$$(\lambda*(\mu\cup\delta))(x)\geq(\lambda*\mu)(x)\cup(\lambda*\delta)(x)$$

Similarly we can prove that

$$(\mu\cup\delta)*\lambda\supseteq(\mu*\lambda)\cup(\delta*\lambda)\ \square$$

If A is nonempty subset of K, then the interval-valued characteristic function of A is denoted by $\hat{C}_A$ and is defined by [1]

$$\hat{C}_A(x)=\begin{cases}[1,1] & if\ x\in A\\ [0,0] & o.w\end{cases}$$

Definition 2-4: Let K be a semigroup. An interval-valued anti fuzzy subset $\hat{\mu}$ of K is called an interval-valued anti fuzzy subsemigroup of K if

$$\hat{\mu}(x)\vee\hat{\mu}(y)\geq\hat{\mu}(xy)\quad\forall x,y\in K$$

Definition 2-5: An interval-valued anti fuzzy subsemigroup $\hat{\mu}$ of K a semigroup K is called an interval-valued anti fuzzy bi-ideal of K if

$$\hat{\mu}(xyz)\leq\hat{\mu}(x)\vee\hat{\mu}(z),\quad\forall x,y,z\in K$$

Definition 2-6: An interval-valued anti-fuzzy subset $\hat{\mu}$ of a semigroup K is called an interval-valued anti fuzzy left (right) ideal of K if

$$\hat{\mu}(y)\geq\hat{\mu}(xy),(\hat{\mu}(x)\geq\hat{\mu}(xy)),\forall x,y\in K$$

Lemma 2-7: Let $\hat{\mu}$ be an interval-valued anti fuzzy subset of a semigroup K then $\hat{\mu}$ is an interval-valued, anti-fuzzy bi-ideal of K iff $\hat{\mu}\subseteq\hat{\mu}*\hat{\mu}$ and $\hat{\mu}*\hat{k}*\hat{\mu}\supseteq\hat{\mu}$ where K is an interval-valued anti fuzzy subset of K mapping .

Proof: Assume that $\hat{\mu}$ is an interval-valued anti fuzzy bi-ideal of a semigroup K .

Let a be any element of K, if $\hat{\mu}*\hat{\mu}(a)=[1,1]$ then $\hat{\mu}*\hat{\mu}\geq\hat{\mu}(a)$, if a is expressible as $a=xy$ then

$$\hat{\mu}*\hat{\mu}(a)=\underset{a=xy}{\wedge}\{\mu(x)\vee\mu(y)\}=\hat{\mu}(a)$$

Thus $\hat{\mu}*\hat{\mu}\supseteq\hat{\mu}$

Now if $\hat{\mu}*\hat{k}*\hat{\mu}(a)=[1,1]$ then $\hat{\mu}*\hat{k}*\hat{\mu}\geq\hat{\mu}(a)$

Otherwise $\exists$ elements x,y,p and q s.t $a=xy$ and $x=pq$.

Therefore

$$(\hat{\mu}*\hat{k}*\hat{\mu})(a)=\underset{a=xy}{\wedge}\{(\hat{\mu}*\hat{k})(x)\wedge\hat{\mu}(y)\}$$
$$=\underset{a=xy}{\wedge}\{\{\underset{x=pq}{\wedge}(\hat{\mu}(p)\vee\hat{k}(q))(x)\}\vee\hat{\mu}(y)\}$$
$$=\underset{a=xy}{\wedge}\{\{\underset{x=pq}{\wedge}(\hat{\mu}(p)\vee[0,0])\}\vee\hat{\mu}(y)\}$$
$$=\underset{a=xy}{\wedge}\ \underset{x=pq}{\wedge}(\hat{\mu}(p)\vee\hat{\mu}(y))$$
$$\geq\hat{\mu}(pqy)=\hat{\mu}(a)$$

Thus

$$(\hat{\mu}*\hat{k}*\hat{\mu})(a)\geq\hat{\mu}(a)$$

Hence $\hat{\mu}*\hat{k}*\hat{\mu}\supseteq\hat{\mu}$

Conversely assume that $\hat{\mu}*\hat{\mu}\supseteq\hat{\mu}$ and $\hat{\mu}*\hat{k}*\hat{\mu}\supseteq\hat{\mu}$, let x,y and z be an elements of K , then $(\hat{\mu}*\hat{\mu})(xy)\geq\hat{\mu}(xy)$, also

$$\underset{xy=ab}{\wedge}\{\hat{\mu}(a)\vee\hat{\mu}(b)\}=\{\hat{\mu}*\hat{\mu}\}(xy)\geq\hat{\mu}(xy)$$

Thus $\{\hat{\mu}(x)\vee\hat{\mu}(y)\}\geq(\hat{\mu}*\hat{\mu})(xy)\geq\hat{\mu}(xy)$

Hence $\hat{\mu}$ is an interval-valued anti fuzz sub semi group of K , also

$$\hat{\mu}(xyz)\leq(\hat{\mu}*\hat{k}*\hat{\mu})(xyz)=\underset{xyz=bc}{\wedge}\{(\hat{\mu}*\hat{k})(b)\vee\hat{\mu}(c)\}$$
$$\leq\{(\hat{\mu}*\hat{k})(xy)\vee\hat{\mu}(z)\}\leq\underset{xy=pq}{\wedge}\{\hat{\mu}(p)\vee\hat{k}(q)\vee\hat{\mu}(z)\}$$
$$\leq\{\hat{\mu}(x)\vee\hat{k}(y)\}\vee\hat{\mu}(z)=\hat{\mu}(x)\vee\hat{\mu}(z)$$

Thus $\hat{\mu}(xyz)\leq\hat{\mu}(x)\vee\hat{\mu}(z)$. Hence $\hat{\mu}$ is an interval-valued anti fuzzy bi-ideal of K .

Lemma 2-8: Every interval-valued anti fuzzy left (right) ideal of a semigroup K is interval-valued anti fuzzy bi-ideal.

Proof: Let $\hat{\mu}$ be an interval-valued anti fuzzy left (right) of $\hat{k}$ and $x,y,w\in K$

Then $\hat{\mu}(xwy)=\hat{\mu}((xw)y)\leq\hat{\mu}(y)\leq\max\{\hat{\mu}(x),\hat{\mu}(y)\}$

Thus $\hat{\mu}$ is an interval-valued anti fuzzy bi-ideal of K . The right case is proved in an analogous way $\square$

Lemma 2-9: A semigroup K is regular, then $\hat{\lambda}\cup\hat{\mu}=\hat{\lambda}*\hat{\mu}$ for each interval-valued anti fuzzy right ideal $\hat{\lambda}$ and each interval-valued anti fuzzy left ideal $\hat{\mu}$ of K .

Proof:

Assume that K is regular. Let $\hat{\lambda}$ be any interval-valued anti fuzzy right ideal and $\hat{\mu}$ be any interval-valued anti fuzzy left ideal of K . Then we have $\hat{\lambda}\subseteq\hat{\lambda}*\hat{k}\subseteq\hat{\lambda}*\hat{\mu}$ and $\hat{\mu}\subseteq\hat{k}*\hat{\mu}\subseteq\hat{\lambda}*\hat{\mu}$. Thus

$\hat{\lambda}\cup\hat{\mu}\subseteq\hat{\lambda}*\hat{\mu}$, let a be any element of K. Then since K is regular, $\exists$ an element $x\in K$ s.t $a=axa$.

Hence we have

$$\left(\hat{\lambda}*\hat{\mu}\right)(a)=\underset{a=yz}{\wedge}\left\{\hat{\lambda}(y)\vee\hat{\mu}(z)\right\}$$

$$\left(\hat{\lambda}*\hat{\mu}\right)\le\hat{\lambda}(ax)\vee\hat{\mu}(a)\le\hat{\lambda}(a)\vee\hat{\mu}(a)=\left(\hat{\lambda}\cup\hat{\mu}\right)(a)$$

And so $\hat{\lambda}*\hat{\mu}\subseteq\hat{\lambda}\cup\hat{\mu}$, therefore $\hat{\lambda}\cup\hat{\mu}=\hat{\lambda}*\hat{\mu}$ □

Lemma 2-10: If K is S-weakly regular semigroup then $\lambda*\mu\subseteq\lambda\bigcup\mu$ for each interval-valued anti fuzzy bi-ideal λ and each interval-valued anti fuzzy right ideal μ of K .

Proof: Suppose that K is S-weakly regular. Let λ be an interval-valued anti fuzzy bi-ideal of K and μ interval-valued anti fuzzy right ideal of K Let $a\in K$, then $\exists x,y\in K$, s.t $a=a\times a^2y=a\times aay$, thus

$$(\lambda*\mu)(a)=\underset{a=st}{\wedge}\{\lambda(s)\vee\hat{\mu}(t)\}\le\{\lambda(a\times a)\vee\mu(ay)\}$$

$$\le\lambda(a)\vee\mu(a)=(\lambda\vee\mu)(a)$$

Hence $\lambda*\mu\subseteq\lambda\bigcup\mu$.

Definition 2-11: Let K be a semigroup and $\hat{\mu}$ an interval-valued anti fuzzy bi-ideal of K . Then $\hat{\mu}$ is called an interval-valued anti fuzzy prime (strongly prime) bi-ideal of K if:

For any interval-valued anti fuzzy bi-ideals $\hat{\lambda}$ and $\hat{\delta}$ of of K . $\hat{\lambda}*\hat{\delta}\subseteq\hat{\mu}\left(\hat{\lambda}*\hat{\delta}\bigcup\hat{\delta}*\hat{\lambda}\subseteq\hat{\mu}\right)$ implies $\hat{\lambda}\subseteq\hat{\mu}$ or $\hat{\delta}\subseteq\hat{\mu}$.

Definition 2-12: An interval-valued anti fuzzy bi-ideal $\hat{\mu}$ of K is called an interval-valued anti fuzzy semi-prime bi-ideal of K if: for any interval-valued anti fuzzy bi-ideal $\hat{\lambda}$ of K , $\hat{\lambda}*\hat{\lambda}\subseteq\hat{\mu}\Rightarrow\hat{\lambda}\subseteq\hat{\mu}$.

Definition 2-13: Let K be a semigroup and $\hat{\mu}$ an interval-valued anti fuzzy bi-ideal of K . Then $\hat{\mu}$ is called an interval-valued anti fuzzy irreducible (strongly irreducible) bi-ideal of K if:

For any interval-valued anti fuzzy bi-ideals $\hat{\lambda}$ and $\hat{\delta}$ of K , $\hat{\lambda}\bigcup\hat{\delta}=\hat{\mu}(\hat{\lambda}\bigcup\hat{\delta}\subseteq\hat{\mu})\Rightarrow\hat{\lambda}=\hat{\mu}$ or $\hat{\delta}=\hat{\mu}$ $\left(\hat{\lambda}\subseteq\hat{\mu}\ \ or\ \ \hat{\delta}\subseteq\hat{\mu}\right)$.

Also

Proposition 2-14: An interval-valued anti fuzzy strongly irreducible, semi-prime bi-ideal of a semigroup K is an interval-valued anti fuzzy strongly prime bi-ideal of K .

Proof: Let $\hat{\mu}$ be an interval-valued anti fuzzy strongly irreducible semi-prime bi-ideal of K , let $\hat{\lambda}$ and $\hat{\delta}$ be interval-valued anti fuzzy bi-ideals of K s.t $\hat{\lambda}*\hat{\delta}\bigcup\hat{\delta}*\hat{\lambda}\subseteq\hat{\mu}$ Since $\left(\hat{\lambda}\bigcup\hat{\delta}\right)^2\subseteq\hat{\lambda}*\hat{\delta}$ and also $\left(\hat{\lambda}\bigcup\hat{\delta}\right)^2\subseteq\hat{\delta}*\hat{\lambda}$, so

$\left(\hat{\lambda}\bigcup\hat{\delta}\right)^2\subseteq\hat{\lambda}*\hat{\delta}\bigcup\hat{\delta}*\hat{\lambda}\subseteq\hat{\mu}$, since $\hat{\mu}$ is an interval-valued anti fuzzy semi-prime bi-ideal, so $\hat{\lambda}\bigcup\hat{\delta}\subseteq\hat{\mu}$. As $\hat{\mu}$ is irreducible so $\hat{\lambda}\subseteq\hat{\mu}$ or $\hat{\delta}\subseteq\hat{\mu}$. That is $\hat{\mu}$ an interval-valued anti fuzzy strongly prime bi-ideal of K .

Theorem 2-15: If K is a regular semigroup and $\hat{\lambda}$ is interval valued anti fuzzy bi-ideal of K then $\left(\hat{\lambda}*\hat{k}*\hat{\lambda}\right)\subseteq\hat{\lambda}$.

Proof: Let $\hat{\lambda}$ be any interval-valued anti fuzzy bi-ideal of K and a be any element of K , since K is regular, so $\exists$ an element $x\in K$ s.t $a=axa$.

Hence we have

$$\left(\hat{\lambda}*\hat{k}*\hat{\lambda}\right)(a)=\underset{a=yz}{\wedge}\left\{\left(\hat{\lambda}*\hat{k}\right)(y)\vee\hat{\lambda}(z)\right\}$$

$$\le\left(\hat{\lambda}*k\right)(\mathrm{ax})\vee\hat{\lambda}(\mathrm{a})$$

$$\le\left(\underset{ax=pq}{\wedge}\left\{\hat{\lambda}(p)\vee\hat{k}(q)\right\}\right)\vee\hat{\lambda}(a)$$

$$\le\hat{\lambda}(a)\vee[0,0]\vee\hat{\lambda}(a)$$

$$\le\hat{\lambda}(a)$$

And so $\hat{\lambda}*k*\hat{\lambda}\subseteq\hat{\lambda}$ □

Lemma 2-16: Let K be a semigroup and $\hat{\mu},\hat{\lambda}$ be interval valued anti fuzzy bi-ideal of K , then $\mu*\lambda$ is an interval valued anti fuzzy bi-ideal of K .

Proof: Let $\hat{\mu}$ and $\hat{\lambda}$ be interval valued anti fuzzy bi-ideal of K , then

$$(\mu*\lambda)*(\mu*\lambda)=(\mu*\lambda*\mu)*\lambda\ge(\mu*K*\mu)*\lambda\ge\mu*\lambda$$

$$(\mu*\lambda)*K*(\mu*\lambda)\ge(\mu*K)*K*(\mu*\lambda)$$
$$=\mu*(K*K)*(\mu*\lambda)\ge\mu*K*(\mu*\lambda)$$
$$=(\mu*K*\mu)*\lambda\ge\mu*\lambda$$

Thus $\mu*\lambda$ is an interval valued anti fuzzy bi-ideal of K □

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حول قيم عكس الضبابية الاوليه للمثاليات الثنائية لشبه الزمرة في فترة

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الملخص

في هذا البحث قدمنا فكره قيم عكس الضبابية الاوليه في الفترة (الاوليه بقوة, شبه الاوليه) للمثاليات الثنائية لشبه الزمره . باستخدام هذه الافكار وصفنا شبه الزمرة والتي فيها قيم عكس الضبابية للمثاليات الثنائية في فترة تكون شبه اوليه واولية بقوة.