Notes on Extension of Fuzzy Complex Sets

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Abstract
The aim of this paper is to modify and improve the corresponding weakness results of multi-fuzzy complex numbers as an extension of fuzzy complex numbers, next we introduce and study the generalized multi-fuzzy complex numbers $\tilde{\mathbb{C}}$ and get some results. Lastly, we discuss the derivative of functions mapping complex numbers $\mathbb{C}$ into $\tilde{\mathbb{C}}$ as an extension of fuzzy complex derivatives.

Keywords: Fuzzy complex numbers; generalized multi-fuzzy complex sets; multi-fuzzy complex derivatives

1. Introduction
Rejun, *et al.* [1], Buckley [2] and Quan [3] have done some works on fuzzy complex numbers and given some characterization of fuzzy complex numbers, next Buckley and Qu [4] developed the concept of fuzzy complex analysis and considered the definition of the derivative of a fuzzy function which maps the open interval (a, b) into the fuzzy subset of the real $\mathcal{F}(\mathbb{R})$ in [5], to generalize a fuzzy function maps (a, b) into the set of fuzzy subsets of the complex case $\mathcal{F}(\mathbb{C})$. In view of Buckley's work, some consummate author's extensively studied fuzzy complex numbers, and continuities and differentiation of complex fuzzy functions like [6-17]

Zadeh [18] developed the concept of fuzzy sets from crisp sets and defined fuzzy subset $\tilde{A}$ on the universal set X, which is a mapping $\mu_{\tilde{A}}(x): X \to [0, 1]$. One of the basic notions of fuzzy subsets is the Zadeh's extension principle. This extension first implied in [18] in an elementary presentation and was finally in [19] and [20] are presented. This principle provides a method for extending crisp mathematical notions to fuzzy quantities as the arguments of the function. Let $g: A_1 \times A_2 \times \ldots \times A_n \to B$ given by $y = g(a_1, a_2, \ldots, a_n)$ and $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ are n fuzzy sets on X_i for $i = 1,2, \ldots n$. Here the extension set $\tilde{A} = g(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ is defined by

$$\mu_{\tilde{A}}(y) = \bigvee_{a_1,a_2,\ldots,a_n \mid y=g(a_1,a_2,\ldots,a_n)} (\mu_{\tilde{A}_1}(a_1) \wedge \mu_{\tilde{A}_2}(a_2) \wedge \ldots \wedge \mu_{\tilde{A}_n}(a_n))$$

Let A be a non empty set, $\mathbb{N}^0$ the set of all natural numbers excluding zero, $\{CL_n : n \in \mathbb{N}^0\}$ a family of complete lattices and CL_n^A consisting of all the mappings from A to CL_n. Also, let $I^\star$, $I_\star$, $I_0^\star$ and $I_1^\star$ denotes for the unit intervals [0,1], (0,1), (0,1] and [0,1), respectively. Yager [21] defined fuzzy multisets as a fuzzy bag $\tilde{\chi}$ drawn from A characterized by a function $\mu_{\tilde{\chi}} : A \to \chi$, where χ is the set of all crisp bags drawn from $I^\star$. Next, Sebastian and Ramakrishnan [22] introduced the concept of multi-fuzzy sets in terms of ordered sequences of characteristic functions as a set $\tilde{\Gamma} = \{(a, \mu_1(a), \mu_2(a), \ldots, \mu_n(a), \ldots) : a \in A\}$, where $\mu_n \in CL_n^A$ for $n \in \mathbb{N}^0$. Also, in [23] Atanassov defined intuitionistic fuzzy set $\tilde{A}$ in the universal set X as an object of the form $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ define the degree of membership and the degree of non membership of elements $x \in X$ to the fuzzy subset $\tilde{A}$ in X, respectively, and for every $x \in X$, $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$.

Buckley [2] defined a fuzzy complex number $\tilde{z}$ by its membership function $\mu_{\tilde{z}}(z): \mathbb{C} \to I^\star$ satisfies:

1. $\mu_{\tilde{z}}(z)$ is continuous;
2. $\{z : \mu_{\tilde{z}}(z) > \alpha\}$, $0 \leq \alpha < 1$, is open, bounded, connected and simply connected;
3. $\{z : \mu_{\tilde{z}}(z) = 1\}$ is non-empty, compact, arcwise connected and simply connected.

2. Multi-fuzzy Complex Numbers
Dey and Pal [24, 25] defined multi-fuzzy complex set as the set of ordered sequences $\tilde{\mathbb{C}} = \{(z, \mu_{\tilde{z}}^1(z), \mu_{\tilde{z}}^2(z), \mu_{\tilde{z}}^3(z), \ldots, \mu_{\tilde{z}}^n(z), \ldots): z \in \mathbb{C}\}$, where $\mu^n \in CL_n^{\mathbb{C}}$ for $n \in \mathbb{N}^0$. A weak α-cut of $\tilde{Z} = (z, \mu_{\tilde{z}}^1(z), \mu_{\tilde{z}}^2(z), \mu_{\tilde{z}}^3(z), \ldots, \mu_{\tilde{z}}^n(z), \ldots)$ in $\tilde{\mathbb{C}}$ is $\tilde{Z}^\alpha = \{z \in \mathbb{C} : \mu_{\tilde{z}}^n(z) > \alpha \text{ for all } n \in \mathbb{N}^0\}$, $\alpha \in I_1^\star$, and $\tilde{Z}^1 = \{z \in \mathbb{C} : \mu_{\tilde{z}}^n(z) = 1 \text{ for all } n \in \mathbb{N}^0\}$. A strong α-cut of $\tilde{Z} = (z, \mu_{\tilde{z}}^1(z), \mu_{\tilde{z}}^2(z), \mu_{\tilde{z}}^3(z), \ldots, \mu_{\tilde{z}}^n(z), \ldots)$ is $\tilde{Z}^{\alpha^+} = \{z \in \mathbb{C} : \mu_{\tilde{z}}^n(z) \geq \alpha \text{ for all } n \in \mathbb{N}^0\}$, $\alpha \in I_0^\star$, and $\tilde{Z}^{0^+} = \overline{\bigcup_{0<\alpha\leq 1} \tilde{Z}^{\alpha^+}}$ for all $n \in \mathbb{N}^0$. If the sequence of multi-membership complex function have only k-terms, k is called the dimension of $\tilde{\mathbb{C}}$. In case of $k = 2$, a multi-fuzzy complex set $\tilde{\mathbb{C}}$ called an Atanassov intuitionistic fuzzy complex set if $\mu_1(z, \tilde{Z}) + \mu_2(z, \tilde{Z}) \in I^\star$. In view of Sebastian and Ramakrishnan work [22], Dey and Pal [24] defined multi-fuzzy complex numbers as a member $\tilde{Z} = (z, \mu_{\tilde{z}}^1(z), \mu_{\tilde{z}}^2(z), \mu_{\tilde{z}}^3(z), \ldots, \mu_{\tilde{z}}^n(z), \ldots)$ of $\tilde{\mathbb{C}}$ if and only if:

1. $\mu_{\tilde{z}}^n(z)$ is continuous for all $n \in \mathbb{N}^0$;
2. $\tilde{Z}^\alpha$, $\alpha \in I_1^\star$, is open, bounded, connected and simply connected; and
3. $\tilde{Z}^1$ is non-empty, compact, arcwise connected and simply connected.

Let $\tilde{Z}', \tilde{Z}'' \in \tilde{\mathbb{C}}$. If we denote the extended addition and multiplication by $\oplus$ and $\odot$, respectively, then by the Zadeh's principle, one obtains

$$\mu_{\tilde{Z}'\oplus\tilde{Z}''}^n(y) = \bigvee_{z',z'' \mid y=z'+z''} (\mu_{\tilde{Z}'}^n(z') \wedge \mu_{\tilde{Z}''}^n(z'')), \text{for } n \in \mathbb{N}^0$$

and

$$\mu^n_{\tilde{Z}'\odot\tilde{Z}''}(y)=\bigvee_{\substack{z',z''\mid y=z'.z''\\ \in\mathbb{N}^0}}\left(\mu^n_{\tilde{Z}'}(z')\wedge\mu^n_{\tilde{Z}''}(z'')\right), \text{for } n$$

The negation of $\tilde{Z}$, $-\tilde{Z}$, the reciprocal of $\tilde{Z}$, $\tilde{Z}^{-1}$, the conjugate of $\tilde{Z}$, $\bar{\tilde{Z}}$, and the modulus of $\tilde{Z}$, $|\tilde{Z}|$, is defined respectively, as follows:

$-\tilde{Z}=(z,\mu^1_{-\tilde{z}}(z),\mu^2_{-\tilde{z}}(z),\dots,\mu^n_{-\tilde{z}}(z),\dots)$
$=(-z,\mu^1_{\tilde{z}}(-z),\mu^2_{\tilde{z}}(-z),\dots,\mu^n_{\tilde{z}}(-z),\dots)$,
$\tilde{Z}^{-1}=\left(z,\mu^1_{\tilde{z}^{-1}}(z),\mu^2_{\tilde{z}^{-1}}(z),\dots,\mu^n_{\tilde{z}^{-1}}(z),\dots\right)$
$=(z^{-1},\mu^1_{\tilde{z}}(z^{-1}),\mu^2_{\tilde{z}}(z^{-1}),\dots,\mu^n_{\tilde{z}}(z^{-1}),\dots)$,
$\bar{\tilde{Z}}=\left(z,\mu^1_{\bar{\tilde{z}}}(z),\mu^2_{\bar{\tilde{z}}}(z),\dots,\mu^n_{\bar{\tilde{z}}}(z),\dots\right)$
$=(\bar{z},\mu^1_{\tilde{z}}(\bar{z}),\mu^2_{\tilde{z}}(\bar{z}),\dots,\mu^n_{\tilde{z}}(\bar{z}),\dots)$, and
$|\tilde{Z}|=\vee\{\mu^n_{|\tilde{z}|}(r):n\in\mathbb{N}^0\}$, where $\mu^n_{|\tilde{z}|}(r)=\vee\{\mu^n_{\tilde{z}}(z): r \text{ is the modulus of } z\in\mathbb{C}\}$ for $n\in\mathbb{N}^0$.

Theorem 2 in [24] has some weakness of introduced and is not true as shown in example below:

Example 1. Let us consider $CL_n=I^\star$ for $n\in\mathbb{N}^0$. Then the set of fuzzy complex numbers can be represented as a multi-fuzzy complex set $\tilde{\mathbb{C}}=\{(z,\mu^1_{\tilde{z}}(z),\mu^2_{\tilde{z}}(z)):z\in\mathbb{C}\}$. Let $\mu^1_{\tilde{z}}(z),\mu^2_{\tilde{z}}(z)$ be linearly depended with $\mu^1_{\tilde{z}}(z)+\mu^2_{\tilde{z}}(z)=1$ for all $z\in\mathbb{C}$. Then the multi-fuzzy complex set represents the set of crisp fuzzy complex numbers with membership value $\mu^1_{\tilde{z}}(z)$ and non-membership value $\mu^2_{\tilde{z}}(z)$. Let $\tilde{Z}$, $\tilde{W}$ and $\tilde{X}$ are three multi-fuzzy complex numbers whose $\tilde{Z}^0=\{x+iy:64<x^2+y^2<100\}\backslash\{x+iy:\ x^2\le 0.01, y\le 0\}$, $\tilde{W}^0=\{x+iy:\ x^2<1,y^2<1\}$, and $\tilde{X}^0=\left\{x+iy:\left|\left(x-\frac{\sqrt{2}}{2}\right)+i\left(y+\frac{\sqrt{2}}{2}\right)\right|<0.1\right\}$. Then, $\tilde{Z}^0+\tilde{W}^0=\{x_1+y_1:x_1\in\tilde{Z}^0,y_1\in\tilde{W}^0\}$ and $\tilde{Z}^0\tilde{X}^0=\{x_2+y_2:x_2\in\tilde{Z}^0,y_2\in\tilde{X}^0\}$ are not simply connected. This implies that extended basic arithmetic operations on multi-fuzzy complex numbers are not satisfied and the simply connected condition is inappropriate for the definition of multi-fuzzy complex numbers. Hence, we redefine multi-fuzzy complex number $\tilde{Z}=(z,\mu^1_{\tilde{z}}(z),\mu^2_{\tilde{z}}(z),\dots,\mu^n_{\tilde{z}}(z),\dots)$ by its multi-membership complex functions $\mu^n_{\tilde{z}}(z)$ as follows:

Definition 2. A member $\tilde{Z}=(z,\mu^1_{\tilde{z}}(z),\mu^2_{\tilde{z}}(z),\dots,\mu^n_{\tilde{z}}(z),\dots)$ of $\tilde{\mathbb{C}}$ is a multi-fuzzy complex number if and only if:

1. $\mu^n_{\tilde{z}}(z)$ is continuous for all $n\in\mathbb{N}^0$;
2. $\tilde{Z}^\alpha$ is open, bounded, and connected for $\alpha\in I_1^\star$; and
3. $\{z\in\mathbb{C}:\ \mu^n_{\tilde{z}}(z)=1 \text{ for all } n\in\mathbb{N}^0\}$ is non-empty, compact, and arcwise connected.

So, according to the modified version of definition of multi-fuzzy complex numbers we correct Theorem 2 in [24] as follows:

Theorem 3. The set of multi-fuzzy complex numbers are closed under the extended basic arithmetic operations.

Proof: The conditions of Definition 2 can prove by the original proof in [24]. So, we omitted the proof here.

In the next, we define and study multi-fuzzy numbers as an extension of fuzzy number $\tilde{a}$ in [26].

Definition 4. A number $\tilde{A}=(a,\mu^1_{\tilde{a}}(a),\mu^2_{\tilde{a}}(a),\dots,\mu^n_{\tilde{a}}(a),\dots)$ called multi-fuzzy number characterized by a grade of multi-membership $\mu^n_{\tilde{a}}(a)\in I^\star$ if and only if:

1. $\mu^n_{\tilde{a}}(a)$ is continuous for all $n\in\mathbb{N}^0$;
2. There are $a\in\mathbb{R}$ and $b\in\mathbb{R}$ such that $c\le a\le b\le d$;
3. $\mu^n_{\tilde{a}}(a)$ is increasing on the interval $[c,a]$ for all $n\in\mathbb{N}^0$;
4. $\mu^n_{\tilde{a}}(a)$ is decreasing on the closed interval $[b,d]$ for all $n\in\mathbb{N}^0$;
5. $\mu^n_{\tilde{a}}(a)=0$ outside some interval $[c,d]$ for $n\in\mathbb{N}^0$;
6. $\mu^n_{\tilde{a}}(a)=1$ for the interval $[a,b]$ for all $n\in\mathbb{N}^0$.

Lemma 5. Let $\tilde{Z}=(z,\mu^1_{\tilde{z}}(z),\mu^2_{\tilde{z}}(z),\dots,\mu^n_{\tilde{z}}(z),\dots)$ be a multi-fuzzy complex number. Then $|\tilde{Z}|^\alpha$ is open for $\alpha\in I_1^\star$.

Proof: According to Lemma 1 in [24], $\tilde{Z}^\alpha$ is open for $\alpha\in I_1^\star$. So, we need only to show that $|\tilde{Z}|^\alpha=|\tilde{Z}^\alpha|$, for all $\alpha\in I_1^\star$. If $r\in|\tilde{Z}^\alpha|$ for $\alpha\in I_1^\star$, then there is a z such that $r=|z|$ and $\mu^n_{\tilde{z}}(z)>\alpha$ for all $n\in\mathbb{N}^0$. So that, $\vee\{\mu^n_{\tilde{z}}(z):\ |z|=r\}>\alpha$ for $n\in\mathbb{N}^0$ and this implies that $r\in|\tilde{Z}|^\alpha$. Conversely, let $r\in|\tilde{Z}|^\alpha$, then there exists z such that $|z|=r$ and $\mu^n_{\tilde{z}}(z)>\alpha$ for all $n\in\mathbb{N}^0$. Hence, $r\in|\tilde{Z}^\alpha|$. In addition, if $r\in|\tilde{Z}|^1$, then there is a $z_k\in\tilde{Z}^0$ so that $r=|z_k|$ and $\mu^n_{\tilde{z}}(z_k)>1-\frac{1}{k}$ for all $n\in\mathbb{N}^0$ and each $k\in\mathbb{N}^0\backslash\{1\}$. This implies z_k is in the closure of $\tilde{Z}^0$ so there is a subsequence $r_{k_i}\longrightarrow z$ with $|z|=r$ and $\mu^n_{\tilde{z}}(z)\ge 1$ for all $n\in\mathbb{N}^0$. So $r\in|\tilde{Z}^1|$. For the other side, suppose that $r\in|\tilde{Z}^1|$, then there exists z such that $r=|z|$ and $\mu^n_{\tilde{z}}(z)=1$ for all $n\in\mathbb{N}^0$. Then $\vee\{\mu^n_{\tilde{z}}(z):\ |z|=r\}=1$ for all $n\in\mathbb{N}^0$ and $r\in|\tilde{Z}|^1$.

Lemma 6. Let $r_i\in|\tilde{Z}|^0$ for $i\in\mathbb{N}^0$ with $r_i\longrightarrow r$ and $\mu^n_{|\tilde{z}|}(r)$ converges to $\lambda_n\in I^\star$ for $n\in\mathbb{N}^0$. Then $\mu^n_{|\tilde{z}|}(r)\ge\lambda_n$ for all $n\in\mathbb{N}^0$.

Proof: The proof is similar to that in [24, Lemma 2] showing $w_i\in\tilde{W}^0$ for $i\in\mathbb{N}^0$ converges to w and $\mu^n_{\tilde{w}}(w)$ converges to λ_n for $n\in\mathbb{N}^0$ then $\mu^n_{\tilde{w}}(w)\ge\lambda_n$ for all $n\in\mathbb{N}^0$. So we omitted the proof here.

Theorem 7. If $\tilde{Z}=(z,\mu^1_{\tilde{z}}(z),\mu^2_{\tilde{z}}(z),\dots,\mu^n_{\tilde{z}}(z),\dots)$ is a multi-fuzzy complex number then $|\tilde{Z}|$ is a multi-fuzzy number.

Proof: Let $c=\wedge\{|z|:z\in\tilde{Z}^0\}$, $a=\wedge\{|z|:z\in\tilde{Z}^1\}$, $b=\vee\{|z|:z\in\tilde{Z}^1\}$, and $d=\vee\{|z|:z\in\tilde{Z}^0\}$. It is obvious that $\mu^n_{|\tilde{z}|}(r)=1$ on $[a,b]$ for all $n\in\mathbb{N}^0$.

Now we discuss that $\mu^n_{|\tilde{z}|}(r)$ is continuous for $n\in\mathbb{N}^0$. The proof of this condition is analogous to that in [24, Theorem 2] target to show $\mu^n_{\tilde{w}}(w)$ is continuous for all $n\in\mathbb{N}^0$ so we sketch briefly. Let $|\tilde{Z}|^0\ni r_i\longrightarrow r$, there exist a subsequence $\mu^n_{|\tilde{z}|}(r_{i_k})\longrightarrow\lambda_n$ for all

$n \in \mathbb{N}^0$. We have, by Lemma 5, $|\tilde{Z}|^\alpha$ is open for $\alpha \in I^\star$, so $\{r : \mu^n_{|\tilde{z}|}(r) \le \beta\}$ is closed for all real β. Hence, $\mu^n_{|\tilde{z}|}(r)$ is lower semi-continuous for $n \in \mathbb{N}^0$ and $\lim \wedge \mu^n_{|\tilde{z}|}(r_i) \ge \mu^n_{|\tilde{z}|}(r)$ for $n \in \mathbb{N}^0$. Also, by Lemma 6, $\lim \wedge \mu^n_{|\tilde{z}|}(r_i) \ge \lambda_n$ for $n \in \mathbb{N}^0$. So that, $\lim \wedge \mu^n_{|\tilde{z}|}(r_i) = \lambda_n = \mu^n_{|\tilde{z}|}(r)$ for all $n \in \mathbb{N}^0$ and there is a subsequence $\mu^n_{|\tilde{z}|}(r_{i_k}) \longrightarrow \lim \vee \mu^n_{|\tilde{z}|}(r_i)$. Again, by Lemma 6, $\lim \vee \mu^n_{|\tilde{z}|}(r_i) \le \mu^n_{|\tilde{z}|}(r)$ for $n \in \mathbb{N}^0$. Therefore, $\lim \vee \mu^n_{|\tilde{z}|}(r_i) = \mu^n_{|\tilde{z}|}(r) = \lim \wedge \mu^n_{|\tilde{z}|}(r_i)$ for $n \in \mathbb{N}^0$, so that $\lim \mu^n_{|\tilde{z}|}(r_i) = \mu^n_{|\tilde{z}|}(r)$ for $n \in \mathbb{N}^0$ and means $\mu^n_{|\tilde{z}|}(r)$ is continuous for all $n \in \mathbb{N}^0$.

Next, we show that $\mu^n_{|\tilde{z}|}(r)$ is increasing on $[c, a]$ for $n \in \mathbb{N}^0$. For this first we discuss that $\mu^n_{|\tilde{z}|}(r) = \vee\{\mu^n_{\tilde{z}}(z) : |z| \le r\}$ for $r \in [c, a]$ and $n \in \mathbb{N}^0$. For fixed r assume there exists z_0 such that $|z_0| < r$ and $\mu^n_{|\tilde{z}|}(r)$ do not exceeds $\mu^n_{\tilde{z}}(z)$ for all $n \in \mathbb{N}^0$. We know that $\{z \in \mathbb{C} : |z| = r\} \cap \{z \in \mathbb{C} : \mu^n_{\tilde{z}}(z) > \alpha \text{ for all } n \in \mathbb{N}^0\}$ is empty for $\mu^n_{|\tilde{z}|}(r) < \alpha$ and $n \in \mathbb{N}^0$. Also, $z_0 \in \{z : \mu^n_{\tilde{z}}(z) > \alpha_0 \text{ for all } n \in \mathbb{N}^0\}$ for some $\mu^n_{|\tilde{z}|}(r) < \alpha_0$ and $n \in \mathbb{N}^0$. Since $\{z \in \mathbb{C} : \mu^n_{\tilde{z}}(z) > \alpha \text{ for all } n \in \mathbb{N}^0\}$ are connected $\{z \in \mathbb{C} : \mu^n_{\tilde{z}}(z) > \alpha \text{ for all } n \in \mathbb{N}^0\} \subseteq \{z : |z| < r\}$ for α exceeds α_0. Therefore, $a < c$, a contradiction. If $[c, e] < [f, a]$, then $\mu^n_{|\tilde{z}|}(e) \le \mu^n_{|\tilde{z}|}(f)$ for all $n \in \mathbb{N}^0$ since $\{z \in \mathbb{C} : |z| \le e\}$ is a subset of $\{z \in \mathbb{C} : |z| \le f\}$.

Finally, we show $\mu^n_{|\tilde{z}|}(r)$ is decreasing on $[b, d]$ for $n \in \mathbb{N}^0$. First we argue that $\mu^n_{|\tilde{z}|}(r) = \vee\{\mu^n_{\tilde{z}}(z) : |z| \ge r\}$ for $r \in [b, d]$ and $n \in \mathbb{N}^0$. For fixed value of r assume there is z_0 so that $|z_0| > r$ and $\mu^n_{\tilde{z}}(z)$ exceeds $\mu^n_{|\tilde{z}|}(r)$ for all $n \in \mathbb{N}^0$. We know that $\{z \in \mathbb{C} : |z| = r\} \cap \{z \in \mathbb{C} : \mu^n_{\tilde{z}}(z) > \alpha \text{ for all } n \in \mathbb{N}^0\}$ is empty for $\mu^n_{|\tilde{z}|}(r) < \alpha$ and $n \in \mathbb{N}^0$. Also, $z_0 \in \{z : \mu^n_{\tilde{z}}(z) > \alpha_0 \text{ for all } n \in \mathbb{N}^0\}$ for some $\mu^n_{|\tilde{z}|}(r) < \alpha_0$ and $n \in \mathbb{N}^0$. Since $\{z \in \mathbb{C} : \mu^n_{\tilde{z}}(z) > \alpha \text{ for all } n \in \mathbb{N}^0\}$ are connected $\{z \in \mathbb{C} : \mu^n_{\tilde{z}}(z) > \alpha \text{ for all } n \in \mathbb{N}^0\} \subseteq \{z : |z| < r\}$ for α exceeds α_0. Therefore, $a < c$, a contradiction. If $d \le e < f \le b$, then $\mu^n_{|\tilde{z}|}(e) \ge \mu^n_{|\tilde{z}|}(f)$ for all $n \in \mathbb{N}^0$ owing to $\{z \in \mathbb{C} : |z| \le e\}$ is a superset of $\{z \in \mathbb{C} : |z| \le f\}$.

Lemma 8 [24]. Let $\tilde{Z}_1$, $\tilde{Z}_2$ be multi-fuzzy complex numbers. Then

1. $|\tilde{Z}_1 \oplus \tilde{Z}_2|^\alpha \le |\tilde{Z}_1|^\alpha + |\tilde{Z}_2|^\alpha$ for $\alpha \in I^\star$,
2. $|\tilde{Z}_1 \odot \tilde{Z}_2|^\alpha = |\tilde{Z}_1|^\alpha \cdot |\tilde{Z}_2|^\alpha$ for $\alpha \in I^\star$.

In the next, we generalize the results of Lemma 8.

Theorem 9. Let $\tilde{Z}_1 = (z, \mu^1_{\tilde{z}_1}(z), \mu^2_{\tilde{z}_1}(z), \dots, \mu^n_{\tilde{z}_1}(z), \dots)$, $\tilde{Z}_2 = (z, \mu^1_{\tilde{z}_2}(z), \mu^2_{\tilde{z}_2}(z), \dots, \mu^n_{\tilde{z}_2}(z), \dots)$, $\tilde{Z}_3 = (z, \mu^1_{\tilde{z}_3}(z), \mu^2_{\tilde{z}_3}(z), \dots, \mu^n_{\tilde{z}_3}(z), \dots)$, …, $\tilde{Z}_k = (z, \mu^1_{\tilde{z}_k}(z), \mu^2_{\tilde{z}_k}(z), \dots, \mu^n_{\tilde{z}_k}(z), \dots)$ be any k number of multi-fuzzy complex numbers. Then for $\alpha \in I^\star$,

1. $|\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3 \oplus \dots \oplus \tilde{Z}_k|^\alpha \le |\tilde{Z}_1|^\alpha + |\tilde{Z}_2|^\alpha + |\tilde{Z}_3|^\alpha + \dots + |\tilde{Z}_k|^\alpha$,
2. $|\tilde{Z}_1 \odot \tilde{Z}_2 \odot \tilde{Z}_3 \odot \dots \odot \tilde{Z}_k|^\alpha = |\tilde{Z}_1|^\alpha \cdot |\tilde{Z}_2|^\alpha \cdot |\tilde{Z}_3|^\alpha \cdot \dots \cdot |\tilde{Z}_k|^\alpha$.

Proof: The proof can obtain easily in the line of Lemma 8 so its proof is omitted.

Lemma 10 [24]. Let $\tilde{Z}_1$, $\tilde{Z}_2$, $\widetilde{W}_1$, $\widetilde{W}_2$ be multi-fuzzy complex numbers with $\widetilde{W}_1 = \tilde{Z}_1 \oplus \tilde{Z}_2$ and $\widetilde{W}_2 = \tilde{Z}_1 \odot \tilde{Z}_2$. Then

1. $\widetilde{W}_1{}^\alpha = \{z_1 + z_2 : (z_1, z_2) \in \tilde{Z}_1^\alpha \times \tilde{Z}_2^\alpha\}$ for $\alpha \in I^\star$,
2. $\widetilde{W}_2{}^\alpha = \{z_1 \cdot z_2 : (z_1, z_2) \in \tilde{Z}_1^\alpha \times \tilde{Z}_2^\alpha\}$ for $\alpha \in I^\star$.

In the next, we generalize the results of Lemma 10.

Theorem 11. Let $\tilde{Z}_1 = (z, \mu^1_{\tilde{z}_1}(z), \mu^2_{\tilde{z}_1}(z), \dots, \mu^n_{\tilde{z}_1}(z), \dots)$, $\tilde{Z}_2 = (z, \mu^1_{\tilde{z}_2}(z), \mu^2_{\tilde{z}_2}(z), \dots, \mu^n_{\tilde{z}_2}(z), \dots)$, $\tilde{Z}_3 = (z, \mu^1_{\tilde{z}_3}(z), \mu^2_{\tilde{z}_3}(z), \dots, \mu^n_{\tilde{z}_3}(z), \dots)$, …, $\tilde{Z}_k = (z, \mu^1_{\tilde{z}_k}(z), \mu^2_{\tilde{z}_k}(z), \dots, \mu^n_{\tilde{z}_k}(z), \dots)$ be any k number of multi-fuzzy complex numbers. Then for $\alpha \in I^\star$

1. $(\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3 \oplus \dots \oplus \tilde{Z}_k)^\alpha = \tilde{Z}_1^\alpha + \tilde{Z}_2^\alpha + \tilde{Z}_3^\alpha + \dots + \tilde{Z}_k^\alpha$,
2. $(\tilde{Z}_1 \odot \tilde{Z}_2 \odot \tilde{Z}_3 \odot \dots \odot \tilde{Z}_k)^\alpha = \tilde{Z}_1^\alpha \cdot \tilde{Z}_2^\alpha \cdot \tilde{Z}_3^\alpha \cdot \dots \cdot \tilde{Z}_k^\alpha$.

Proof: We only prove the first part of the theorem with the aid of Lemma 10.

$$(\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3 \oplus \dots \oplus \tilde{Z}_k)^\alpha$$
$$= \tilde{Z}_1^\alpha + (\tilde{Z}_2 \oplus \tilde{Z}_3 \oplus \dots \oplus \tilde{Z}_k)^\alpha$$
$$= \tilde{Z}_1^\alpha + \tilde{Z}_2^\alpha + (\tilde{Z}_3 \oplus \tilde{Z}_4 \oplus \dots \oplus \tilde{Z}_k)^\alpha$$
$$= \tilde{Z}_1^\alpha + \tilde{Z}_2^\alpha + \tilde{Z}_3^\alpha + (\tilde{Z}_4 \oplus \tilde{Z}_5 \oplus \dots \oplus \tilde{Z}_k)^\alpha$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$= \tilde{Z}_1^\alpha + \tilde{Z}_2^\alpha + \tilde{Z}_3^\alpha + \tilde{Z}_4^\alpha + \tilde{Z}_5^\alpha + \dots + \tilde{Z}_k^\alpha.$$

Theorem 12. Let $\tilde{Z}_1 = (z, \mu^1_{\tilde{z}_1}(z), \mu^2_{\tilde{z}_1}(z), \dots, \mu^n_{\tilde{z}_1}(z), \dots)$, $\tilde{Z}_2 = (z, \mu^1_{\tilde{z}_2}(z), \mu^2_{\tilde{z}_2}(z), \dots, \mu^n_{\tilde{z}_2}(z), \dots)$, $\tilde{Z}_3 = (z, \mu^1_{\tilde{z}_3}(z), \mu^2_{\tilde{z}_3}(z), \dots, \mu^n_{\tilde{z}_3}(z), \dots)$, …, $\tilde{Z}_k = (z, \mu^1_{\tilde{z}_k}(z), \mu^2_{\tilde{z}_k}(z), \dots, \mu^n_{\tilde{z}_k}(z), \dots)$ be any k number of multi-fuzzy complex numbers. Then

1. $\overline{(\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3 \oplus \dots \oplus \tilde{Z}_k)} = \bar{\tilde{Z}}_1 \oplus \bar{\tilde{Z}}_2 \oplus \bar{\tilde{Z}}_3 \oplus \dots \oplus \bar{\tilde{Z}}_k$,

2. $\overline{(\tilde{Z}_1 \odot \tilde{Z}_2 \odot \tilde{Z}_3 \odot \dots \odot \tilde{Z}_k)} = \bar{\tilde{Z}}_1 \odot \bar{\tilde{Z}}_2 \odot \bar{\tilde{Z}}_3 \odot \dots \odot \bar{\tilde{Z}}_k$.

Proof: We only prove the first part of the theorem, the proof of the second part is similar. From Theorem 11, for $\alpha \in I^\star$ we obtain

$$(\bar{\tilde{Z}}_1 \oplus \bar{\tilde{Z}}_2 \oplus \bar{\tilde{Z}}_3 \oplus \dots \oplus \bar{\tilde{Z}}_k)^\alpha$$
$$= \bar{\tilde{Z}}_1^\alpha + \bar{\tilde{Z}}_2^\alpha + \bar{\tilde{Z}}_3^\alpha + \dots + \bar{\tilde{Z}}_k^\alpha$$
$$= \overline{\tilde{Z}_1^\alpha} + \overline{\tilde{Z}_2^\alpha} + \overline{\tilde{Z}_3^\alpha} + \dots + \overline{\tilde{Z}_k^\alpha}$$
$$= \{\bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \dots + \bar{z}_k \in \mathbb{C} : (z_1, z_2, z_3, \dots, z_k) \in (\tilde{Z}_1^\alpha \times \tilde{Z}_2^\alpha \times \tilde{Z}_3^\alpha \times \dots \times \tilde{Z}_k^\alpha)\}$$

Again, in view of Theorem 11 and for $\alpha \in I^\star$, we get

$$\overline{(\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3 \oplus \dots \oplus \tilde{Z}_k)}^\alpha$$
$$= \overline{(\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3 \oplus \dots \oplus \tilde{Z}_k)^\alpha}$$
$$= \overline{(\tilde{Z}_1^\alpha + \tilde{Z}_2^\alpha + \tilde{Z}_3^\alpha + \dots + \tilde{Z}_k^\alpha)}$$
$$= \{\overline{z_1 + z_2 + z_3 + \dots + z_k} \in \mathbb{C} : (z_1, z_2, z_3, \dots, z_k) \in (\tilde{Z}_1^\alpha \times \tilde{Z}_2^\alpha \times \tilde{Z}_3^\alpha \times \dots \times \tilde{Z}_k^\alpha)\}$$

This completes the proof.

3. Generalized Multi-fuzzy Complex Numbers

In this section, we define and study generalized multi-fuzzy complex numbers and the derivative of functions mapping complex numbers $\mathbb{C}$ into $\tilde{\mathbb{C}}$ as an extension of generalized fuzzy complex numbers based on the "star-like" [4] function $\tilde{f}(z) = \tilde{Z}(z)$ for $z \in \Omega \subset \mathbb{C}$. We suppose that $\tilde{Z}(z)^{1^+}$ is analytic single point belongs to the interior of $\tilde{Z}(z)^{\alpha^+}$, $\alpha \in I_1^\star$. For any strong α-cut of $\tilde{Z}(z)$, draw the ray $L(\gamma)$ from $\tilde{Z}(z)^{1^+}$ making angle $\gamma \in [0,2\pi)$ with the positive x-axis in the complex plane and suppose that $L(\gamma) \cap \left(\text{boundary of } \tilde{Z}(z)^{\alpha^+}\right) = w(z,\alpha,\gamma) = u(x,y,\alpha,\gamma) + i\, v(x,y,\alpha,\gamma)$ is analytic over Ω for all $\alpha \in I^\star$ and for all extend γ to be $[0,2\pi] \coloneqq I_\star^\star$.

Definition 13. A member $\tilde{Z} = (z, \mu_{\tilde{z}}^1(z), \mu_{\tilde{z}}^2(z), \ldots, \mu_{\tilde{z}}^n(z), \ldots)$ of $\tilde{\mathbb{C}}$ is a generalized multi-fuzzy complex number if and only if:

1. $\mu_{\tilde{z}}^n(z)$ is upper semi-continuous for all $n \in \mathbb{N}^0$;
2. $\tilde{Z}^{\alpha^+}$, $\alpha \in I^\star$, is compact and arcwise connected;
3. $\tilde{Z}^{1^+}$ is non-empty.

Theorem 14. Let $\tilde{W}$, $\tilde{Z}_1 = \left(z, \mu_{\tilde{z}_1}^1(z), \mu_{\tilde{z}_1}^2(z), \ldots, \mu_{\tilde{z}_1}^n(z), \ldots\right)$, $\tilde{Z}_2 = \left(z, \mu_{\tilde{z}_2}^1(z), \mu_{\tilde{z}_2}^2(z), \mu_{\tilde{z}_2}^3(z), \ldots, \mu_{\tilde{z}_2}^n(z), \ldots\right)$, $\ldots$, $\tilde{Z}_k = \left(z, \mu_{\tilde{z}_k}^1(z), \mu_{\tilde{z}_k}^2(z), \ldots, \mu_{\tilde{z}_k}^n(z), \ldots\right)$ be generalized multi-fuzzy complex numbers and $\circledast$ be the extended basic arithmetic operations with $\tilde{W} = \tilde{Z}_1 \circledast \tilde{Z}_2 \circledast \ldots \circledast \tilde{Z}_k$. Then, $\tilde{W}^{\alpha^+} = \left\{z_1 * z_2 * \ldots * z_k : (z_1, z_2, \ldots, z_k) \in \tilde{Z}_1^{\alpha^+} \times \tilde{Z}_2^{\alpha^+} \times \ldots \times \tilde{Z}_k^{\alpha^+}\right\}$ for all $\alpha \in I^\star$.

Proof: We only prove for extended addition, the proofs of the rest are similar.

If $y \in \tilde{W}^{\alpha^+}$, then for $n \in \mathbb{N}^0$

$$\mu_{\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \ldots \oplus \tilde{Z}_k}^n(y) = \bigvee_{z_1, z_2, \ldots, z_k \,|\, y = z_1 + z_2 + \cdots + z_k} \left(\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k)\right) \geq \alpha$$

For each $i = 1, 2, 3, \ldots$ we can find $z_{1_i} \in \tilde{Z}_1^{0^+}$, $z_{2_i} \in \tilde{Z}_2^{0^+}, \ldots, z_{k_i} \in \tilde{Z}_k^{0^+}$ so that $z_{1_i} + z_{2_i} + \cdots + z_{k_i} = y$ and $\mu_{\tilde{Z}_1}^n(z_{1_i}) \wedge \mu_{\tilde{Z}_2}^n(z_{2_i}) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_{k_i}) > \alpha - \alpha/i$ for all $n \in \mathbb{N}^0$.

Since $\tilde{Z}_1^{0^+}, \tilde{Z}_2^{0^+}, \ldots, \tilde{Z}_k^{0^+}$ are compact we may choose a subsequence $z_{1_{i_j}} \longrightarrow z_1$, $z_{2_{i_j}} \longrightarrow z_2$, $\ldots z_{k_{i_j}} \longrightarrow z_k$ with $y = z_1 + z_2 + \cdots + z_k$ and $\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k) \geq \alpha$ for all $n \in \mathbb{N}^0$ because $\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k)$ is upper semi-continuous for all $n \in \mathbb{N}^0$. This implies that $(z_1, z_2, \ldots, z_k) \in \tilde{Z}_1^{\alpha^+} \times \tilde{Z}_2^{\alpha^+} \times \ldots \times \tilde{Z}_k^{\alpha^+}$ and hence $y \in \left\{z_1 + z_2 + \cdots + z_k : (z_1, z_2, \ldots, z_k) \in \tilde{Z}_1^{\alpha^+} \times \tilde{Z}_2^{\alpha^+} \times \ldots \times \tilde{Z}_k^{\alpha^+}\right\}$ for $\alpha \in I_0^\star$. If $y \in \tilde{W}^{0^+}$, then we have two cases:

Case 1.

$$\mu_{\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \ldots \oplus \tilde{Z}_k}^n(y) = \bigvee_{z_1, z_2, \ldots, z_k \,|\, y = z_1 + z_2 + \cdots + z_k} \left(\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k)\right) = 0,$$

for all $n \in \mathbb{N}^0$. So there are $y_i \in \mathbb{C}$, $I_0^\star \ni \alpha_i \longrightarrow 0$ such that $\mu_{\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \ldots \oplus \tilde{Z}_k}^n(y_i) \geq \alpha_i$ and $y_i \longrightarrow y$. Since

$$\bigvee_{z_1, z_2, \ldots, z_k \,|\, y_i = z_1 + z_2 + \cdots + z_k} \left(\mu_{\tilde{Z}_1}^n(z_{1_i}) \wedge \mu_{\tilde{Z}_2}^n(z_{2_i}) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_{k_i})\right) \geq \alpha_i > 0,$$

for all $n \in \mathbb{N}^0$ we can find $(z_{1_i}, z_{2_i}, \ldots, z_{k_i}) \in \tilde{Z}_1^{0^+} \times \tilde{Z}_2^{0^+} \times \ldots \times \tilde{Z}_k^{0^+}$ such that $y_i = z_{1_i} + z_{2_i} + \ldots + z_{k_i}$ and $\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k) > \alpha_i - \alpha_i/i$ for all $n, i \in \mathbb{N}^0$. Since $\tilde{Z}_1^{0^+} \times \tilde{Z}_2^{0^+} \times \ldots \times \tilde{Z}_k^{0^+}$ are compact we can choose a subsequence $z_{1_{i_j}} \longrightarrow z_1$, $z_{2_{i_j}} \longrightarrow z_2$, $\ldots z_{k_{i_j}} \longrightarrow z_k$ with $y_{i_j} = z_{1_{i_j}} + z_{2_{i_j}} + \cdots + z_{k_{i_j}} \longrightarrow z_1 + z_2 + \cdots + z_k = y \in \tilde{Z}_1^{0^+} + \tilde{Z}_2^{0^+} + \cdots + \tilde{Z}_k^{0^+}$.

Case 2.

$$\mu_{\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \ldots \oplus \tilde{Z}_k}^n(y) = \bigvee_{z_1, z_2, \ldots, z_k \,|\, y = z_1 + z_2 + \cdots + z_k} \left(\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k)\right) = \alpha_0 > 0.$$

So there are $(z_{1_i}, z_{2_i}, \ldots, z_{k_i}) \in \tilde{Z}_1^{0^+} \times \tilde{Z}_2^{0^+} \times \ldots \times \tilde{Z}_k^{0^+}$ such that $y = z_{1_i} + z_{2_i} + \ldots + z_{k_i}$ and $\mu_{\tilde{Z}_1}^n(z_{1_i}) \wedge \mu_{\tilde{Z}_2}^n(z_{2_i}) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_{k_i}) > \alpha_0 - \alpha_0/i$ for all $n \in \mathbb{N}^0$ and $i = 1,2,3, \ldots$.

Since $\tilde{Z}_1^{0^+} \times \tilde{Z}_2^{0^+} \times \ldots \times \tilde{Z}_k^{0^+}$ are compact we may choose a subsequence $z_{1_{i_j}} \longrightarrow z_1$, $z_{2_{i_j}} \longrightarrow z_2$, $\ldots z_{k_{i_j}} \longrightarrow z_k$ with $y = z_{1_i} + z_{2_i} + \ldots + z_{k_i}$ and $\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k) \geq \alpha_0$ for all $n \in \mathbb{N}^0$. Because $\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k)$ is upper semi-continuous for all $n \in \mathbb{N}^0$. This implies that $z_1 + z_2 + \cdots + z_k \in \tilde{Z}_1^{\alpha_0^+} + \tilde{Z}_2^{\alpha_0^+} + \cdots + \tilde{Z}_k^{\alpha_0^+}$.

Hence, $y \in \left\{z_1 + z_2 + \cdots + z_k : (z_1, z_2, \ldots, z_k) \in \tilde{Z}_1^{0^+} \times \tilde{Z}_2^{0^+} \times \ldots \times \tilde{Z}_k^{0^+}\right\}$.

Now suppose $y \in \left\{z_1 + z_2 + \cdots + z_k : (z_1, z_2, \ldots, z_k) \in \tilde{Z}_1^{0^+} \times \tilde{Z}_2^{0^+} \times \ldots \times \tilde{Z}_k^{0^+}\right\}$. This implies that there are $z_1, z_2, \ldots, z_k$ so that $y = z_1 + z_2 + \cdots + z_k$ and $\mu_{\tilde{Z}_1}^n(z_1) \wedge \mu_{\tilde{Z}_2}^n(z_2) \wedge \ldots \wedge \mu_{\tilde{Z}_k}^n(z_k) \geq \alpha$ for all $n \in \mathbb{N}^0$. This means that $\mu_{\tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \ldots \oplus \tilde{Z}_k}^n(y)$ also exceeds α, and hence $y \in \tilde{W}^{\alpha^+}$ for all $\alpha \in I^\star$.

Definition 15. The multi-membership complex function $\mu_{\tilde{f}'(z)}^n(w)$ is defined by

$$\mu_{\tilde{f}'(z)}^n(w) = \bigvee\{\alpha : w = u_x(x,y,\alpha,\gamma) + i v_x(x,y,\alpha,\gamma), \alpha \in I^\star \text{ and } \gamma \in I_\star^\star\}$$

for all $n \in \mathbb{N}^0$ and $w \in \mathbb{C}$.

Theorem 16. If $u_x(x,y,\alpha,\gamma)+iv_x(x,y,\alpha,\gamma)$ is continuous of α and γ, then $\mu^n_{\tilde{f}'(z)}(w)$ is a generalized multi-fuzzy complex number for all $n \in \mathbb{N}^0$.

Proof: We prove the first condition of generalized multi-fuzzy complex numbers by way of contradiction. Let $w_k \longrightarrow w$, $\mu^n_{\tilde{f}'(z)}(w)=\alpha$ and $\mu^n_{\tilde{f}'(z)}(w_k)=\alpha_k$ for all $n \in \mathbb{N}^0$ suppose that $\lim \vee \alpha_k = \alpha^* > \alpha$. From the definition of $\mu^n_{\tilde{f}'(z)}(w)=\alpha$ there exist α_k, $\alpha \geq \alpha_k > \alpha - 1/k$, and $\gamma_k \in I^\star_\star$ such that $w = u_x(x,y,\alpha_k,\gamma_k) + i\,v_x(x,y,\alpha_k,\gamma_k)$. Take a subsequence $\alpha_{k_i} \longrightarrow \alpha^*$ and choose $\gamma_{k_i} \in I^\star_\star$ such that $w = u_x\left(x,y,\alpha_{k_i},\gamma_{k_i}\right) + i\,v_x\left(x,y,\alpha_{k_i},\gamma_{k_i}\right)$. Hence, we have $\gamma_{k_i} \longrightarrow \gamma^* \in I^\star_\star$. So by hypothesis, we get $w = u_x(x,y,\alpha^*,\gamma^*) + i\,v_x(x,y,\alpha^*,\gamma^*)$. This implies that $\mu^n_{\tilde{f}'(z)}(w) \geq \alpha^* > \alpha$ for all $n \in \mathbb{N}^0$. In the last, it is easy to view that $\tilde{f}'(z)^{\alpha^+}$, $\alpha \in I^\star$, is non-empty, compact and arcwise connected and this completes the proof.

Conclusion

In this paper, some important concepts and results related to fuzzy complex sets are modified, improved and generalized. We have shown that the extended basic arithmetic operations on multi-fuzzy complex numbers are not satisfied and the simply connected condition is inappropriate for the definition of multi-fuzzy complex numbers. Redefined multi-fuzzy complex number with restricted simply connected condition and have proved the modulus of the new version of multi-fuzzy complex numbers is a multi-fuzzy number and its α-cut sets are open. Also, proved that the modulus of addition of k number multi-fuzzy complex numbers may not be equal to the addition of its modulus but this property is preserved for conjugate on multi-fuzzy complex numbers. In addition, generalized multi-fuzzy complex numbers and the derivative of functions mapping complex numbers into generalized multi-fuzzy complex numbers are defined and proved that the generalized multi-fuzzy complex derivative is closed under some conditions. Lastly, generalized multi-fuzzy complex sets may be a foundation for researching fuzzy complex analytics and allowed us to define the concept of interval valued fuzzy derivative on a non-empty set $\Omega \subset \mathbb{C}$ as a mapping $\mu^{\tilde{f}'(z)}_{(w)}:\Omega \rightarrow \left[\mu^n_{\tilde{f}'(z)}(w), \mu_n^{\tilde{f}'(z)}(w)\right]$, where $\mu^n_{\tilde{f}'(z)}(w)$ and $\mu_n^{\tilde{f}'(z)}(w)$ are denotes for lower multi-fuzzy complex derivative and upper multi-fuzzy complex derivative about $\mu^{\tilde{f}'(z)}$, respectively and one can get some new results more easily.

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ملاحظات حول توسعة المجموعات المعقدة المضببة

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الملخص

الهدف من هذا البحث تحسين و تطوير تطابق النتائج الضعيفة للأعداد المعقدة الضبابية المتعددة كتوسيع للأعداد المركبة الضبابية، لاحقا نقدم دراسة عامة للأعداد المعقدة الضبابية المتعددة $\bar{\mathbb{C}}$ وحصلنا على بعض النتائج. و اخيرا، ناقشنا اشتقاق رسم دوال الاعداد المعقدة $\mathbb{C}$ الى $\bar{\mathbb{C}}$ كتوسعة لمشتقة الاعداد المعقدة الضبابية.