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A SINGLE MUTUAL FIXED POINT THEOREM USING Φ- CONTRACTION IN PARTIAL –b– METRIC SPACES

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Abstract
In this paper we proved a common fixed point theorem by using Φ- contraction condition and also provided an example which supports our main result.

Keywords: Partial b-metric space, weakly compatible mapping, Φ – contraction, partial metric space.

1.introduction

In 1989, Bakhtin [1] submit the connotation of quasi-Metric Space as a popularization of Metric Spaces (M.SP). in (1993),czerwik [2, 3] propagated many remarks referred to the b–metric spaces (b-M.SP). in (1994),matthews [4] admitted the connotation of partial metric space (P.M.SP) in which the self - distance of every point of space does not equal 0. in (1996), o'neill popularized the notion of Partial metric space (P.M.SP) by introduced non positive distances. in (2013),shukla [5] generalized both the concept of (b-M.SP) and (P.M.SP) by submitting the partial b-metric spaces (P.b-M.SP).For example, many of researchers recently studying this axiom and its popularization in various types of (M.SP) [6],[7],[8],[9],[10],[11], [12].

In this this paper we proved a common fixed point theorem for four maps in partial – b – metric space and also provided an example which supports our main result.

Definition 1.1 [13] Let M be a set and let $r \geq 1$ be a real no. A mapping $d: M \times M \to [0, \infty)$ is called a (b-M.SP) if ∀ u,v,w ∈ M the following conditions are holding:

i) d(u,v) = 0 iff u=v

ii) d(u,v) = d(v,u);

iii) $d(u,v) \leq r[d(u,w) + d(w,v)]$:

The pair (M, d) is called a b-Metric space (b-M.SP). $r \geq 1$ is called the factor of (M, d).

Definition 1.2 [4] Suppose M be a nonempty set. A mapping $p: M \times M \to [0, \infty)$ is called (P.M.SP) if ∀ u,v,w ∈ M the next terms are satisfied:

i) u=v iff p(u,u) = p(u,v) = p(v,v);

ii) $p(u,u) \leq p(u,v)$;

iii) p(u,v) = p(v,u);

iv) $p(u,v) \leq p(u,w) + p(w,v) - p(w,w)$:

The pair (M; P) is called (P.M.SP).

Remark 1.3 It is clear that the (P.M.SP) need not be a (b-M.SP), since in a (b-M.SP) if u = v , then d(u,u) = d(u,v) = d(v,v) = 0. But in a (P.M.SP) if u = v then p(u,u) = p(u,v) = p(v,v) maybe not equal to zero. Therefore the (P.M.SP) maybe not a (b-M.SP).

At the different side, Shukla [18] admit the concept of a (P.b-M.SP) as follows:

Definition 1.4 [5] Suppose M be a nonempty set and $r \geq 1$ be a real no. $P_b: M \times M \to [0, \infty)$ is called a (P.b-M.SP) if ∀ u,v,w ∈ M the next terms are hold:

i) u = v iff $P_b(u, u) = P_b(u, v) = P_b(v, v)$;

ii) $P_b(u, u) \leq P_b(u, v)$;

iii) $P_b(u, v) = P_b(v, u)$;

iv) $P_b(u, v) \leq r[P_b(u, w) + P_b(w, v)] - P_b(w; w)$:

the pair (M; P_b) is a (P.b-M.SP). $r \geq 1$ is called the factor of (M, P_b).

Remark 1.5 The class of (P.b-M.SP) (M, P_b) is surely greater than the grade of (P.M.SP) , because a (P.M.SP) partial metric space is a particular kind of a (P.b-M.SP) (M, P_b) when r = 1. Also, the grade of (P.b-M.SP) (M, P_b) is surely greater than the grade of (b-M.SP), because a (b-M.SP) is a particular kind of a (P.b-M.SP) (M, P_b) while the self - distance p(u; u) = 0.

the next examples discern that a (P.b-M.SP) on M need not be a (P.M.SP), nor a (b-M.SP) on M see also [14], [18].

Example 1.6 [5] Suppose M = [0,1). Let $P_b: M \times M \to [0, \infty)$ be a function whereas $P_b(u; v) = [\max\{u, v\}]^2 + |u - v|^2$, ∀ u, v ∈ M. Then (M, P_b) is a (P.b-M.SP) on M and the coefficient r = 2 > 1. But, P_b is not a (b-M.SP) nor a (P.M.SP) on M.

Proposition 1.7 [14] Every partial b-metric P_b defines a b - metric d_{P_b} , where

$d_{P_b}(u, v) = 2 P_b(u, v) - P_b(u, u) - P_b(v, v)$, ∀ u, v ∈ M.

Definition 1.8 [14] A sequence $\{u_n\}$ in a (P.b-M.SP) (M, P_b) is called:

i) P_b -convergent to a point u ∈ M if $\lim_{n\to\infty} P_b(u, u_n) = P_b(u, u)$

ii) a P_b -Cauchy sequence (C. Seq.) if $\lim_{n,m\to\infty} P_b(u_n, u_m)$ defined and is restricted;

iii) A (P.b-M.SP) (M, P_b) is called P_b-complete if any P_b-(C. Seq.) $\{u_n\}$ in M is P_b approaches to a point u ∈ M provided

$$\lim_{n,m\to\infty} P_b(u_n, u_m) = \lim_{n\to\infty} P_b(u_n, u) = P_b(u, u)$$

lemma:1.9 [14] A seq. $\{u_n\}$ is a P_b-(C. Seq.) in a (P.b-M.SP) (M, P_b) if and only if b--(C. Seq.) in the (b-M.SP) (M, d_{P_b}).

Lemma 1.10. [14] A (P.b-M.SP) (M, P_b) is P_b-Complete if and only if the (b-M.SP) (M, d_{P_b}) is b-Complete. Moreover, $\lim_{n,m\to\infty} d_{P_b}(u_n, u_m) = 0$ iff

$$\lim_{n,m\to\infty} P_b(u_m, u) = \lim_{n\to\infty} P_b(u_n, u) = P_b(u, u)$$

Definition 1.11 [15]: The pair of the self-mapping A and S of a (M.SP.) (M, d) are said to be weakened Compatible if they commute at coincidence points. i.e., if Au = Su ⟹ ASu = SAu for u in M.

2. Main Results

Theorem2.1: Suppose (M, P_b) be a (P.b-M.SP) with the factor $r \geq 1$. Suppose $A,B,C,D: M \to M$ be mappings satisfying the following

(2.1.1)

$$r.P_b(Au,Bv) \leq \Phi\left(\max\left\{\frac{P_b(Cu,Au)\cdot P_b(Dv,Bv)}{1+P_b(Cu,Dv)}, P_b(Cu,Dv)\right\}\right)$$

For all $u,v \in Z$ and $\Phi:[0,\infty)\to[0,\infty)$ be monotonically non-decreasing continuous function with $\Phi(t)<t$ for $t>0$.

(2.1.2) $A(M)\subseteq D(M)$, $B(M)\subseteq C(M)$

(2.1.3) either C(M) or D (M) is Complete subspace of M.

(2.1.4) One of (A, C) and (B, D) is weakened Compatible.

So the mappings A, B, C and D have a single mutual fixed point in M.

Proof:- Choose $u_0, v_0 \in u$. From (2.1.2), ∃ sequences $\{u_n\}$ and $\{v_n\}$ in u provided

$$Au_{2n} = Du_{2n+1} = v_{2n}$$

$$Bu_{2n+1} = Cu_{2n+2} = v_{2n+1} \ \forall n = 0,1,2, \ldots$$

Case – 1 : Let $v_{2n} = v_{2n+1}$ for some n.

Claim : $v_{2n+1} = v_{2n+2}$

Suppose $v_{2n+1} \neq v_{2n+2}$

From (2.1.1), we have that

$$r.P_b(v_{2n+1}, v_{2n+2})$$

$$= r.P_b(Au_{2n+2}, Bu_{2n+1})$$

$$\leq \Phi\left[\max\left\{\frac{P_b(Cu_{2n+2},Au_{2n+2})\cdot P_b(Du_{2n+1},Bu_{2n+1})}{1+P_b(Cu_{2n+2},Du_{2n+1})}, P_b(Cu_{2n+2},Du_{2n+1})\right\}\right]$$

$$= \Phi\left[\max\left\{\frac{P_b(v_{2n+1},v_{2n+2})\cdot P_b(v_{2n},v_{2n+1})}{1+P_b(v_{2n+1},v_{2n})}, P_b(v_{2n+1},v_{2n})\right\}\right]$$

$$\leq \Phi\left[\max\left\{\frac{P_b^{\,2}(v_{2n+1},v_{2n+2})}{1+P_b(v_{2n},v_{2n+1})}, P_b(v_{2n+1},v_{2n+2})\right\}\right]$$

$$= \Phi\left(P_b(v_{2n+1},v_{2n+2})\right)$$

$$< P_b(v_{2n+1},v_{2n+2})$$

Which is contradiction.

Hence $v_{2n+1} = v_{2n+2}$

Continuing in this way we can conclude that $v_{2n} = v_{2n+k}$

$\therefore \{v_{2n}\}$ is a Cauchy sequence in M.

Case – 2 : $v_n \neq v_{n+1}$ $\forall n$, put $P_n = P_d(v_n, v_{n+1})$

From (2.1.1), we have

$$r.P_b(Au_{2n},Bu_{2n+1}) \leq \Phi\left(\max\left\{\frac{P_b(u_{2n},Au_{2n})\cdot P_b(Du_{2n+1},Bu_{2n+1})}{1+P_b(u_{2n},Du_{2n+1})}, P_b(Cu_{2n},Du_{2n+1})\right\}\right)$$

$$= \Phi\left(\max\left\{\frac{P_b(v_{2n+1},v_{2n})\cdot P_b(v_{2n},v_{2n+1})}{1+P_b(v_{2n-1},v_{2n})}, P_b(v_{2n+1},v_{2n})\right\}\right)$$

If $\frac{P_b(v_{2n-1},v_{2n})\cdot P_b(v_{2n},v_{2n+1})}{1+P_b(v_{2n-1},v_{2n})}$ is maximum, then

$$r\cdot P_n(v_{2n},v_{2n+1})$$

$$\leq \Phi\left(\frac{P_b(v_{2n-1},v_{2n})\cdot P_b(v_{2n},v_{2n+1})}{1+P_b(v_{2n-1},v_{2n})}\right)$$

$$< \frac{P_b(v_{2n-1},v_{2n})\cdot P_b(v_{2n},v_{2n+1})}{1+P_b(v_{2n-1},v_{2n})}$$

It follows that

$$1+P_b(v_{2n-1},v_{2n}) < \frac{1}{3}P_b(v_{2n-1},v_{2n}) < P_b(v_{2n-1},v_{2n})$$

Which is a contradiction.

Hence $P_b(v_{2n-1},v_{2n})$ is a maximum.

Thus

$$r.P_b(v_{2n},v_{2n+1}) \leq \Phi\left(P_b(v_{2n-1},v_{2n})\right) \ldots(1)$$

$$< P_b(v_{2n-1},v_{2n})$$

It follows that

$$P_{2n} = P_b v_{2n}, v_{2n+1}\frac{1}{r}\cdot P_b(v_{2n-1},v_{2n}) \leq P_b(v_{2n-1},v_{2n})$$

$$= P_{2n-1} \ldots\ldots.(2)$$

$\therefore \{P_{2n}\}$ is non-increasing sequence of positive numbers. Hence it converges to some limit point $k \geq 0$.

Suppose k > 0.

Letting $n \to \infty$ in (1), we have that

$$r\cdot k \leq \Phi(k) < k$$

which is a contradiction

Hence *k*=0.

Thus $\lim_{n\to\infty} P_{2n} = \lim_{n\to\infty} P_b(v_{2n},v_{2n+1}) = 0 \cdot \ldots\ldots.(3)$

Now we prove that $\{v_{2n}\}$ is a (C. Seq.).

For n, m ∈ R with m > n. We have

$$P_b(v_{2n},v_{2m})$$

$$\leq r\left[P_b(v_{2n},v_{2n+1}) + P_b(v_{2n+1},v_{2n})\right] - P_b(v_{2n+1},v_{2n+1})$$

$$\leq r.P_b(v_{2n},v_{2n+1}) + r^2\cdot P_b(v_{2n+1},v_{2n+2}) + \ldots + r^{2m-2n}P_b(v_{2m-1},v_{2m}) \ (3)$$

Letting $n \to \infty$, we have that

$$\lim_{n\to\infty} P_b(v_{2n},v_{2m}) = 0 \cdot \ \ldots.(4)$$

Therefore $\{v_{2n}\}$ is a (C. Seq.) in M.

we can also prove $\{v_{2n+1}\}$ is a (C. Seq.) in M.

Therefore $\{v_n\}$ is a (C. Seq.) in M.

From Lemma (1.9), conclude that $\{v_n\}$ is a (C. Seq.) in (b-M. SP.) (M, d_{P_b}).

Suppose $D(M)$ is Complete subspace of M.

Since $\{v_{2n}\}$ is a (C. Seq.) in composition with (b-M. SP.) $(D(M), d_{P_b})$.

It follows that $\{v_{2n}\}$ approaches to x at D(M).

That is $\lim_{n\to\infty} d_{P_b}(v_{2n}, x) = 0$ for some $x \in D(M)$, there exist $t \in M$ such that $D(t) = x$.

Since $\{v_n\}$ is (C. Seq.) and $v_{2n} \to x$, it follows that $v_{2n+1} \to x$.

From Lemma (1.10) and (3), we have that

$$P_b(x,x) = \lim_{n\to\infty} P_b(v_{2n},x) = \lim_{n\to\infty} P_b(v_{2n+1},x) = 0 \quad \ldots(5)$$

Now we prove that $\lim_{n\to\infty} P_b(At, v_{2n}) = P_b(At, x)$

Since by definition of d_{P_b},

$$d_{P_b}(At, v_{2n}) = 2P_b(At, v_{2n}) - P_b(At, At) - P_b(v_{2n}, v_{2n})$$

By def. to d_{P_b}, and (3), (5), see that

$$\lim_{n\to\infty} d_{P_b}(Bt, v_{2n}) = \lim_{n\to\infty} P_b(Bt, v_{2n}) - P_b(At, At)$$

Implies that

$$\lim_{n\to\infty} P_b(Bt, v_{2n}) = P_b(Bt, x) \ldots\ldots (6)$$

From P_4, we have that

$$P_b(Bt,x) \le r\left[P_b(Bt, v_{2n}) + P_b(v_{2n}, x)\right] - P_b(v_{2n+1}, v_{2n+1})$$

Letting $n \to \infty$, we have that

$$P_b(Bt,x) \le \lim_{n\to\infty} P_b(Bt, v_{2n})$$

$$= s \cdot \lim_{n\to\infty} P_b(Au_{2n}), Bt$$

$$\le \lim_{n\to\infty} \Phi\left(\max\left\{\frac{P_b(u_{2n}, Au_{2n}) \cdot P_b(Dt, Bt)}{1 + P_b(Cu_{2n}, Dt)}, P_b(Cu_{2n}, Dt)\right\}\right)$$

$$= \lim_{n\to\infty} \Phi\left(\max\left\{\frac{P_b(v_{2n-1}, v_{2n}) \cdot P_b(x, Bt)}{1 + P_b(v_{2n-1}, x)}, P_b(v_{2n-1}, x)\right\}\right)$$

$$= \Phi\left(\max\{0, 0\}\right) = 0.$$

It is clear that $Bt = x = Dt$.

Since (B, D) is weakly compatible pair, we have $Bx = Dx$.

Now we claim that $Bx = x$.

Suppose $Bx \ne x$.

Consider

$$P_b(Bx, x) \le r\left[P_b(Bx, v_{2n}) + P_b(v_{2n}, x)\right] - P(v_{2n}, v_{2n})$$

Letting $n \to \infty$, we have that

$$P_b(B_x, x) \le \lim_{n\to\infty} r \cdot P_b(Bx, Au_{2n})$$

$$\le \lim_{n\to\infty} \Phi\left(\max\left\{\frac{P_b(Cu_{2n}, Au_{2n}) \cdot P_b(Dx, Bx)}{1 + P_b(Cu_{2n}, Du)}, P_b(Cu_{2n}, Dx)\right\}\right)$$

$$\le \lim_{n\to\infty} \Phi\left(\max\left\{\frac{P_b(v_{2n-1}, v_{2n}) \cdot P_b(x, Bx)}{1 + P_b(v_{2n-1}, Bx)}, P_b(v_{2n-1}, Bx)\right\}\right)$$

$$= \Phi\left(\max\{0, P_b(x, Bx)\}\right)$$

$$= \Phi\left(P_b(x, Bx)\right)$$

$$< P_b(Bx, x),$$

which is a contradiction.

Hence $Bx = x = Dx$ ……..(7)

Therefore, x is common fixed point of B and D.

Since $B(M) \le C(M)$ we have that $x = Bx = Cy$ for some $y \in M$.

From (2.1.1), we have that

$$r.P_b(Ay, Bx)$$

$$\le \Phi\left(\max\left\{\frac{P_b(Cy, Ay) \cdot P_b(Dx, Bx)}{1 + P_b(Cy, Dx)}, P_b(Cy, Dx)\right\}\right)$$

$$= \Phi\left(\max\left\{\frac{P_b(x, Ay) \cdot P_b(x, x)}{1 + P_b(x, x)}, P_b(x, x)\right\}\right)$$

$$= \Phi\left(\max\{0, 0\}\right) = 0.$$

It is clear that $Ay = x = Cy$

Since (A, C) is weakly compatible pair, we have $Ax = Cx$.

Now we prove that Ax = x.

Suppose that $Ax \ne Cx$.

From (2.1.1), we have that

$$r.P_b(Ax, x) = r \cdot P_b(Ax, Bx)$$

$$\le \Phi\left(\max\left\{\frac{P_b(Cx, Ax) \cdot P_b(Dx, Bx)}{1 + P_b(Cx, Dx)}, P_b(Cx, Dx)\right\}\right)$$

$$\le \Phi\left(\max\left\{\frac{P_b(x, Ax) \cdot P_b(x, x)}{1 + P_b(x, Ax)}, P_b(x, Ax)\right\}\right)$$

$$= \Phi\left(\max\{0, P_b(x, Ax)\}\right)$$

$$= \Phi\left(P_b(x, Ax)\right) < P_b(x, Ax),$$

which is contradiction.

Hence $Ax = x = Cx$ … (8)

From (7) and (8), we have

Therefore, x is mutual fixed point of A, B, C and D.

To prove singularity let z is another mutual fixed point of A, B, C and D. such that $x \ne z$.

From (2.1.1), we have that

$$r.P_b(x, z) = r \cdot P_b(Ax, Bz)$$

$$\le \Phi\left(\max\left\{\frac{P_b(x, Ax) \cdot P_b(Dz, Bz)}{1 + P_b(Cx, Dz)}, P_b(Cx, Dz)\right\}\right)$$

$$= \Phi\left(\max\left\{\frac{P_b(x, x) \cdot P_b(z, z)}{1 + P_b(x, z)}, P_b(x, z)\right\}\right)$$

$$= \Phi\left(\max\{0, P_b(x, z)\}\right)$$

$$= \Phi\left(P_b(x, z)\right)$$

$$< P_b(x, z),$$

And that represent a contradiction.

Hence x is single mutual fixed point of A, B, C and D.

The following example illustrates our main theorem

Example2.2: Suppose $M = [0,1)$ be (P.b-M.SP) with $P_b : M \times M \to [0, \infty)$ defined by $P_b(u,v) = \left[\max\{u, v\}\right]^2$ $\forall u, v \in M$ Clearly (M, P_b) be complete (P.b-M.SP) with r =2. Define that mapping $A, B, C, D : M \to M$ by

$$A(M) = \frac{u}{3\sqrt{1+u}},\quad B(M) = \frac{u}{6\sqrt{1+u}},\quad C(M) = \frac{u}{6},\quad D(M) = \frac{u}{3}$$

Also $\emptyset: [0, \infty) \to [0, \infty)$ by $\emptyset(t) = \frac{t}{2}$ Then A, B, C and D satisfies all conditions of Theorem 2.1 and 0 is unique common fixed point in M.

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مبرهنة النقطة الثابتة المشتركة المنفردة باستخدام انكماش $-\Phi$ في الفضاءات المترية الجزئية – $b-$

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الملخص

تم في هذا البحث دراسة مبرهنة النقطة الثابتة المشتركة الوحيدة واثباتها باستخدام شرط انكماش $-\Phi$ وتم في النهاية إعطاء مثال توضيحي يدعم النتيجة الرئيسية التي بني عليها هذا البحث.