ON PARETO SET FOR A BI-CRITERIA SINGLE MACHINE SCHEDULING PROBLEM

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ABSTRACT

This paper considers a bi-criteria planning problems on a single machine, with the goal of minimizing total square time duration and maximizing earliness. To solve this problem we have to find the Pareto set. We introduced a strong relation between lower bound, upper bound of the problem and the number of efficient solutions via a theorem which shows also that the lower bound is near to optimal solution if the number of efficient solutions is small.

1. Introduction

During recent years many approaches for solving multi-objective scheduling have been analyzed [7]. T’kindt and Billaut offered a comprehensive assessment of the over a hundred multi-objective scheduling problems, They concentrated on single-machine scheduling, parallel-machine scheduling, flow shop scheduling, and fuzzy scheduling issues [13]. The hierarchical problems and the simultaneous problems are two alternative structures of bi-criteria scheduling issues. Simultaneous difficulties lead to the discovery of a collection of non-dominated solutions (the Pareto set), which gives the decision maker more information about which solution to choose [11]. Several researchers paid attention on finding Pareto set for bi-criteria problems. Hoogeveen and Velde [5] simultaneously decreased maximum completion time and cost. Lazarev et al. the Pareto set was discovered for jobs with similar processing times according to the criteria Lmax and Cmax [9]. Nguyen and Bao used a genetic algorithm to tackle the mixed store scheduling issue [12]. For the scheduling problems which involving quadratic measure of performance, there are relatively little works done in this form. Townsend posed an issue using a quadratic optimization method for completion times. [14]. Bagga and Kalra modified in some sense the Townsend's algorithm [2]. Gupta and Sen improved branching procedure of quadratic penalty function of completion times [3]. Abdul-Razaq and Kawi combined their efforts to solve a function of square completion time and high tardiness [1]. The total square completion time and greatest earliest completion time were the two criteria we focused on this paper $\sum_{j=1}^{n} c_j^2$ and Emax. This problem was solved by Hoogeveen and Van de Velde simultaneously and they found all the efficient solutions of the problem [4]. We introduced a theorem which found a strong relation between optimal solution, lower bound and number of efficient solutions. This theorem is also can be
applied for all the problems that have the same structure.

2. Fundamental Ideas and Definitions

We describe the following in this section:

N: set of jobs \{1, ..., n\},
Pj: processing time for job j,
dj: due date for job j,
cj: completion time for job j.

MST: (minimal slack periods) Minimum slack times are sequenced in a non-descending order sj, where sj = dj - pj.


M: minimum rate of the job.

fmax: maximum cost function.

Problem in a polynomial time.

Vairaktarakis [10] reviewed computational leads to generate all the efficient solutions. Lee and Vairaktarakis [10] reviewed computational results of hierarchical minimization problems. Hoogeveen and Van de Velde [5] solved a simultaneous minimization problem in a polynomial time. It’s important to mention that the specific instance of the simultaneous scenario 1// F(f, g) is hierarchical scheduling problem 1// Lex (f, g). Where f is the fundamental criterion and the secondary criterion is g, and the simultaneous case is likewise NP-hard if the hierarchical issue is NP-hard.

4. Pareto Set and Optimal Solution

The Pareto set for the simultaneous situation was discovered by Hoogeveen and Van de Velde [6] // F (\sum_{j=1}^{n} c_j, E_{\text{max}} ). By employing a genetic algorithm, Kurz and Canterbury [8] discovered the Pareto set for the similar problem.

We will give a theorem that finds a relation between the Pareto set. The optimal solution, as well as the lowest bounds for total square completion time and greatest earliness. It is important to mention that this theorem can be applied to all the problems in this structure, this means the bi-criteria problems in simultaneous case with \( f_{\text{max}} \). Here in this paper \( f_{\text{max}} = E_{\text{max}} \).

Let the lower bound \( LB = \sum_{j=1}^{n} c_j^2 (SPT) + E_{\text{max}}(MST) \) and the upper bound \( UB = \sum_{j=1}^{n} c_j^2 (SPT) + E_{\text{max}}(SPT) \).

**Theorem 4.1:**

There exists an integer M and non-negative such that \( LB + M = \text{optimal value} \) and \( M \in [N_1-1, N_2 + 1] \) where \( N_1 \) = number of effective solutions and \( N_2 = E_{\text{max}}(SPT) - E_{\text{max}}(MST) \).

**Proof:**

Since \( LB \) is less the optimal value, so there exists an integer M and non-negative such that \( LB + M = \text{optimal value} \). The first section of the theorem is proved by this. It is still to demonstrate \( M \in [N_1-1, N_2 + 1] \) or to demonstrate \( N_1-1 \leq M \leq N_2 + 1 \). We have \( M = \text{optimal rate} - LB \leq UB - LB = \sum_{j=1}^{n} c_j^2 (SPT) + E_{\text{max}}(SPT) - \sum_{j=1}^{n} c_j^2 (MST) \).

Hence \( M \leq N_2 + 1 \).

To prove \( N_1-1 \leq M \) we will use mathematical induction on \( N_1 \). If \( N_1 = 1 \), that seems to be, there is just one effective solution, SPT. Then \( M = \sum_{j=1}^{n} c_j^2 (SPT) + E_{\text{max}}(SPT) - \sum_{j=1}^{n} c_j^2 (MST) = 0 \), because \( E_{\text{max}}(SPT) \leq E_{\text{max}}(MST) \).

This is a special case where the SPT sequence is the same as MST sequence and \( M \in [0, 1] \). Thus \( N_1-1 \leq M \leq N_2 + 1 \). As a result, the theorem is correct for \( N_1 = 1 \).

If \( N_1 = 2 \), that is, SPT and \( \sigma \) are the only two effective options, say. Since \( N_1 = 2 \), so \( N_1-1 = 1 \). The two options are as follows:

a- If \( SPT \) is optimal then \( \sum_{j=1}^{n} c_j^2 (SPT) + E_{\text{max}}(SPT) - \sum_{j=1}^{n} c_j^2 (MST) \) is true.

b- If \( \sigma \) is optimal then \( \sum_{j=1}^{n} c_j^2 (\sigma) + E_{\text{max}}(\sigma) - \sum_{j=1}^{n} c_j^2 (SPT) - E_{\text{max}}(MST) \geq 1 \), because \( \sum_{j=1}^{n} c_j^2 (\sigma) \geq \sum_{j=1}^{n} c_j^2 (SPT) \). Thus \( N_1-1 \leq M \leq N_2 + 1 \) and hence the theorem is true for \( N_1 = 2 \).
If \( N_1 = 3 \), that is, SPT, \( \sigma \) and \( \sigma_1 \) are the three most efficient solutions, say.

Since \( N_1 = 3 \), so \( N_1 - 1 = 2 \). The three cases are as follows:

a- If SPT is optimal then \( \sum_{j=1}^{N} c_j^2 (SPT) + E_{\max}(SPT) - \sum_{j=1}^{N} c_j^2 (SPT) - E_{\max}(MST) \), implies that \( E_{\max}(SPT) - E_{\max}(MST) \geq 2 = N_1 - 1 \).

b- If \( \sigma \) is optimal then \( M = \sum_{j=1}^{N} c_j^2 (\sigma) + E_{\max}(\sigma) - \sum_{j=1}^{N} c_j^2 (\sigma) + E_{\max}(MST) \), implies that \( \sum_{j=1}^{N} c_j^2 (\sigma) - \sum_{j=1}^{N} c_j^2 (\sigma) + E_{\max}(\sigma) \).

\( E_{\max}(MST) \geq 1 + 1 = 2 = N_1 - 1 \).

c- If \( \sigma_1 \) is optimal then \( M = \sum_{j=1}^{N} c_j^2 (\sigma_1) + E_{\max}(\sigma_1) - \sum_{j=1}^{N} c_j^2 (\sigma_1) + E_{\max}(MST) \), implies that \( \sum_{j=1}^{N} c_j^2 (\sigma_1) - \sum_{j=1}^{N} c_j^2 (\sigma_1) + E_{\max}(\sigma_1) \).

The three cases are \( \sigma_1 \), \( \sigma_2 \), \( \sigma_3 \). Let \( N_1 = k + 1 \) this implies, there are \( k + 1 \) solutions which are effective. If either of the first \( k \) efficient approaches is the optimal, as well as the theorem is correct, then for \( N_1 = k \) we obtain \( N_1 - 1 \leq M \) and hence \( N_1 - 1 \leq M \leq N_2 + 1 \). If \( \sigma_{k-1} \) final effective solution is optimal then \( M = \sum_{j=1}^{N} c_j^2 (\sigma_{k-1}) + E_{\max}(\sigma_{k-1}) - \sum_{j=1}^{N} c_j^2 (\sigma_{k-1}) + E_{\max}(MST) \).

Thus \( M \in [N_1-1,N_2+1] \) the theorem is correct for \( N_1 = k + 1 \).

### 4.1 Illustrated Example

Consider the following example with three jobs

<table>
<thead>
<tr>
<th>Job</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>( d )</td>
<td>1</td>
<td>6</td>
<td>12</td>
</tr>
</tbody>
</table>

At first we find all the possible sequences for 3 jobs which are 3! = 6. The results are as follows:

<table>
<thead>
<tr>
<th>Sequences</th>
<th>Note</th>
<th>( \sum_{j=1}^{N} c_j^2 )</th>
<th>( E_{\max} )</th>
<th>( \sum_{j=1}^{N} c_j^2 + E_{\max} )</th>
<th>Efficient solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2, 3)</td>
<td>MST-rule</td>
<td>77</td>
<td>4</td>
<td>81</td>
<td>Efficient</td>
</tr>
<tr>
<td>(1, 3, 2)</td>
<td></td>
<td>117</td>
<td>5</td>
<td>122</td>
<td></td>
</tr>
<tr>
<td>(2, 1, 3)</td>
<td>SPT-rule</td>
<td>74</td>
<td>6</td>
<td>80</td>
<td>Efficient and optimal</td>
</tr>
<tr>
<td>(2, 3, 1)</td>
<td></td>
<td>101</td>
<td>6</td>
<td>107</td>
<td></td>
</tr>
<tr>
<td>(3, 1, 2)</td>
<td></td>
<td>138</td>
<td>7</td>
<td>145</td>
<td></td>
</tr>
<tr>
<td>(1, 2, 3)</td>
<td></td>
<td>125</td>
<td>7</td>
<td>132</td>
<td></td>
</tr>
</tbody>
</table>

Here, the optimal solution is the sequence ((2, 1, 3) with the cost 80. To use the theorem we find

\[
\begin{align*}
\text{LB} &= \sum_{j=1}^{N} c_j^2 (SPT) + E_{\max}(MST) = 74 + 4 = 78 \\
\text{UB} &= \sum_{j=1}^{N} c_j^2 (SPT) + E_{\max}(MST) = 74 + 6 = 80
\end{align*}
\]

\( N_2 = E_{\max}(SPT) - E_{\max}(MST) = 6 - 4 = 2 \)

\( N_1 = \text{number of efficient solution} - 1 = 2 - 1 = 1 \), so \( M \in [N_1-1,N_2+1] = [0, 2] \). Therefore \( M = 2 \) because \( \text{LB} + 2 = 80 \).

### References


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الملخص

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