Let $R$ be a commutative ring with identity, and $M$ is a unitary left $R$-module. A proper submodule $E$ of an $R$-module $M$ is called a weakly quasi-prime if whenever $r, s \in R, m \in M$, with $0 \neq rs m \in E$, implies that $rm \in E$ or $sm \in E$. “We introduce the concept of a weakly quasi 2-absorbing submodule as a generalization of weakly quasi-prime submodule”, where a proper submodule $E$ of $M$ is called a weakly quasi 2-absorbing submodule if whenever $r, s, t \in R, m \in M$ with $0 \neq rs m \in E$, implies that $rs m \in E$ or $rm \in E$ or $sm \in E$. We study the basic properties of weakly quasi 2-absorbing. Furthermore, the relationships of weakly quasi 2-absorbing submodule with other classes of module are elustrated.

**Abstract**

In this work we generalize the concept of weakly quasi-prime submodule to a weakly quasi 2-absorbing submodule , a proper submodule $E$ of an $R$-module $M$ is called weakly quasi 2-absorbing if whenever $r, s, t \in R, m \in M$, with $0 \neq rs m \in E$ implies that either $sm \in E$ or $rm \in E$ or $st m \in E$. Furthermore, we prove that every “2-absorbing (weakly 2-absorbing) submodule is weakly” 2-absorbing submodule.

**2 – Weakly quasi 2-absorbing submodules**

In this section we introduce, the definition of weakly quasi 2-absorbing submodule, as a generalization of weakly quasi prime submodule.

**Definition 2.1**

A Proper submodule $E$ of an $R$-module $M$ is called a weakly quasi 2-absorbing, if whenever $r, s, t \in R$, $m \in M$, with $0 \neq st m \in E$, then $sm \in E$ or $rm \in E$ or $st m \in E$.

And an ideal of a ring $R$ is weakly quasi 2-absorbing if it is a weakly quasi 2-absorbing submodule of an $R$-module $R$.

**Remark 2.2**

Every weakly quasi-prime “submodule of an $R$-module $M$ is a weakly” quasi 2-absorbing submodule of $M”$ but the converse need not to be true.

**Proof**

Assume $E$ is a weakly quasi-prime submodule of an $R$-module $M$, and let $0 \neq st m \in E$, where $r,s,t \in R$, $m \in M$, and suppose that $rm \in E$. Then $0 \neq rs(tm) \in E$. By hypothesis we have $r(tm) \in E$ or $s(tm) \in E$. That is $tm \in E$ or $st m \in E$. Hence $E$ is weakly quasi 2-absorbing submodule of $M$.

For the converse consider the following example :- let $M = Z$, $R = Z$ and $E = 4Z$. $E$ is a weakly quasi 2-absorbing submodule, but not weakly quasi-prime submodule of $Z$ since $2, 2, 3 \in 4Z$ where $2, 3 \in Z$, but $2 \cdot 3 = 6 \notin 4Z$, and $4Z$ is weakly quasi 2-absorbing since $0 \neq 1 \cdot 2 \cdot 2 \cdot 1 \in 4Z$, we have $2 \cdot 2 \cdot 1 \in 4Z$.

**PROPOSITION 2.3**

“Let $E$ and $D$ be a submodules of an $R$-module $M$ with $E \subseteq D”$. If $E$ is a weakly quasi 2-absorbing submodule in $M$, then $E$ is a weakly quasi 2-absorbing submodule in $D$. 
Assume that \(0 \neq rstm \in E\) with \(r, s, t \in R, m \in M\), hence \(m \in M\). Since \(E\) is weakly quasi 2-absorbing submodule in \(M\), then \(rsm \in E\) or \(rmt \in E\) or \(stm \in E\). Hence \(E\) is weakly quasi 2-absorbing in \(D\).

**REMARK 2.4**

A submodule of a weakly quasi 2-absorbing submodule need not to be a weakly quasi 2-absorbing. The following example explain that.

Let \(M = Z\), \(R = Z\), and \(E = 4Z\), \(D = 36Z\). \(4Z\) is a weakly quasi 2-absorbing in \(Z\), and we have \(36Z \subseteq 4Z\), \(36Z\) is not weakly quasi 2-absorbing in \(Z\), since \(0 \neq 2 \times 3 \times 3 \in 36Z\), but \(2 \times 2 \times 2 \times 3 = 12 \notin 36Z\).

**PROPOSITION 2.5**

Let \(E\) and \(D\) are submodules of a module \(M\) with \(D \in E\). Then \(E\) is a weakly quasi 2-absorbing in \(M\) if and only if \(\frac{E}{D}\) is weakly quasi 2-absorbing in \(\frac{M}{D}\).

**Proof**

Let \(E\) weakly quasi 2-absorbing in \(M\), and let \(0 \neq \frac{rstm + D}{D} \in \frac{E}{D}\), where \(r, s, t \in R\), \(m \in M\). It follows that \(0 \neq \frac{rstm}{E} \in E\). Since \(E\) is a weakly quasi 2-absorbing submodule in \(M\), then either \(rsm \in E\) or \(rmt \in E\) or \(stm \in E\). It follows that either \(rsm + D \in E + D\) or \(rmt + D \in E + D\) or \(stm + D \in E + D\). That is either \(\frac{rsm + D}{D} \in E\) or \(\frac{rmt + D}{D} \in E\) or \(\frac{stm + D}{D} \in E\). Hence \(\frac{E}{D}\) is weakly quasi 2-absorbing submodule in \(\frac{M}{D}\).

Conversely: suppose that \(\frac{E}{D}\) is weakly quasi “2-absorbing submodule” in \(\frac{M}{D}\), with \(0 \neq \frac{rstm}{E} \in E\), where \(r, s, t \in R\), \(m \in M\). Hence \(0 \neq \frac{rstm + D}{D} \in \frac{E}{D}\). That is \(0 \neq \frac{rstm}{E} \in \frac{E}{D}\). Since \(\frac{E}{D}\) is a weakly quasi 2-absorbing in \(\frac{M}{D}\), then either \(\frac{rsm}{D} \in \frac{E}{D}\) or \(\frac{rmt}{D} \in \frac{E}{D}\) or \(\frac{stm}{D} \in \frac{E}{D}\). It follows that either \(\frac{rsm + D}{D} \in \frac{E}{D}\) or \(\frac{rmt + D}{D} \in \frac{E}{D}\) or \(\frac{stm + D}{D} \in \frac{E}{D}\). Thus \(\frac{E}{D}\) is weakly quasi 2-absorbing submodule.

**REMARK 2.6**

The intersection of two weakly quasi 2-absorbing “submodules of an R-module M” need not to be weakly quasi 2-absorbing submodule in \(M\) as the following example explain that:

Let \(M = Z\), \(R = Z\), \(E = 4Z\), \(D = 9Z\), we are quasi 2-absorbing submodules in \(Z\). But \(E \cap D = 36Z\) is not weakly quasi 2-absorbing submodule in \(Z\).

**PROPOSITION 2.7**

The intersection of two quasi-prime submodules of an R-module \(M\) is weakly quasi 2-absorbing submodule.

**Proof**

Let \(E, D\) be two quasi-prime “submodules of \(M\)”, with \(0 \neq rstm \in E\) or \(0 \neq rstm \in D\), where \(r, s, t \in R, m \in M\), since \(E\) is a quasi-prime submodule in \(M\) we assume that \(rm \in E\), also since \(D\) is quasi-prime in \(M\), we assume that \(sm \in D\), it follows that \(rsm \in E\) or \(rstm \in D\). Hence \(E \cap D\) is weakly quasi 2-absorbing submodule in \(M\).

“Since every” weakly “prime submodule is a weakly quasi-prime submodule[1]”. Hence we get the following result.

**COROLLARY 2.8**

The intersection of two weakly prime “submodule of an R-module \(M\)” is weakly quasi 2-absorbing.

**PROPOSITION 2.9**

The inverse image of weakly quasi 2-absorbing submodule is weakly quasi 2-absorbing submodule.

**Proof**

We assume that \(f\) is an R-epimorphism from \(M\) to \(E\) and \(E\) is weakly quasi 2-absorbing of \(M\). Let \(0 \neq rstm \in f^{-1}(E)\), where \(r, s, t \in R, m \in M\). It follows that \(0 \neq rstm \in E\), but \(E\) is weakly quasi 2-absorbing in \(M\), the \(0 \neq rstm \in E\) or \(rstm \in E\). Hence \(rstm \in f^{-1}(E)\) or \(rstm \in f^{-1}(E)\). Thus \(f^{-1}(E)\) is a weakly quasi 2-absorbing in \(M\).

**PROPOSITION 2.10**

Let \(f: M \to M\) be an R-epimorphism, and \(E\) be a proper submodule of \(M\) with \(\text{Ker}(f) \subseteq E\). Then \(E\) is a weakly quasi 2-absorbing submodule in \(M\) iif \(f(E)\) is a weakly quasi 2-absorbing submodule in \(M\).

**Proof**

Assume that \(0 \neq rstm \in f(E)\), where \(r, s, t \in R, m \in M\), since \(f\) is onto, then \(m = f(m)\) for some \(m \in M\). Hence \(0 \neq rstm \in f(E)\), implies that \(0 \neq rstm \in f(E)\), implies that \(0 \neq rstm \in f(E)\), implies that \(0 \neq rstm \in f(E)\). Hence \(f(rstm - c) = 0\), implies that \(0 \neq rstm \in f(E)\). That is \(0 \neq rstm \in E\), since \(E\) is a weakly quasi 2-absorbing in \(M\), then \(0 \neq rstm \in E\), implies that \(0 \neq rstm \in E\), implies that \(0 \neq rstm \in E\). Hence \(f(rstm - c) = 0\), implies that \(0 \neq rstm \in E\). Hence \(f(E)\) is a weakly quasi 2-absorbing.

\(\Leftarrow\) Let \(0 \neq rstm \in E\), where \(r, s, t \in R, m \in M\), then \(0 \neq f(rstm) \in f(E)\), it follows that \(0 \neq rstm \in E\). But \(f(E)\) is a weakly quasi 2-absorbing in \(M\), then \(0 \neq rstm \in E\) or \(rstm \in f(E)\). If \(rstm \in f(E)\), implies that \(0 \neq rstm \in f(E)\). Hence \(f(rstm - c) = 0\), implies that \(0 \neq rstm \in E\), implies that \(0 \neq rstm \in E\). Similarly we get \(rstm \in E\) or \(rstm \in E\). Hence \(E\) is a weakly quasi 2-absorbing in \(M\).

**REMARK 2.11**

If \(K, L\) are submodules of \(M\) with \(K\) isomorphic to \(L\) and \(K\) is a weakly quasi 2-absorbing submodule, then \(L\) is not weakly quasi 2-absorbing submodule. The following example shows that;

Let \(M = Z, R = Z, K = Z, L = 8Z\), are submodule of \(M\), \(2Z \cong 8Z\), we have \(Z\) weakly quasi 2-absorbing submodule, but \(8Z\) is not weakly quasi 2-absorbing in \(Z\), since \(0 \neq 2 \times 2 \times 1 \in 8Z\), \(2 \times 2 \times 1 \notin 8Z\).

**PROPOSITION 2.12**

Every weakly quasi 2-absorbing submodules of \(M\) is weakly quasi 2-absorbing submodule.
Proof
Assume that $E$ is a weakly 2-absorbing in $M$, and let $0 \neq rstm \in E$ with $r, s, t \in R, m \in M$. Therefore either $r(sm) \in E$ or $s(tm) \in E$ or $rs \in \{E : M\}$. The first two leads us to that $E$ is a weakly quasi 2-absorbing in $M$.

PROPOSITION 2.13
Assume that $M$ is cyclic module, and $E$ be a proper submodule of $M$. Then $E$ is a weakly 2-absorbing in $M$ if and only if $E$ is a weakly quasi 2-absorbing in $M$.

Proof
The first part follows by proposition (2.12).

The second part: suppose that $M$ is a weakly quasi 2-absorbing in $M$, then $E$ is cyclic, mean that $M = R x$ for some $x \in M$. Let $0 \neq rsm \in E$, with $r, s \in R, m \in M, m = tx$, where $t \in R$. Thus $0 \neq rstm \in E$, since $E$ is a weakly quasi 2-absorbing in $M$, then either $rsx \in E$ or $rxt \in E$ or $stm \in E$ and hence either $rs \in \{E : M\}$ or $tm \in E$ or $sm \in E$.

Since “every 2-absorbing submodule is weakly 2-absorbing [5]”, we get the following corollary.

Corollary 2.14
Let $E$ be 2-absorbing “submodule of an $R$-module $M$”. Then $E$ is a weakly quasi 2-absorbing.

PROPOSITION 2.15
"Let $M$ be an $R$-module, and $E$ be a proper submodule of $M$. Then $E$ is a weakly quasi 2-absorbing in $M$ if and only if $[E : M]$ is a weakly quasi 2-absorbing ideal of $R$ for every $m \in E$.

Proof $(\Rightarrow)$ We have $[E : m]$ proper ideal in $R$, since $m \in M, m \in E$. Let $0 \neq rsm \in E$, where $r, s, t \in R$ then $0 \neq rstm \in E$, but $E$ is a weakly quasi 2-absorbing submodule in $M$, then either $rsm \in E$ or $rstm \in E$ or $tm \in E$ hence $rs \in \{E : M\}$ or $rt \in \{E : M\}$ or $st \in \{E : M\}$, hence $\{E : M\}$ is a weakly quasi 2-absorbing.

$(\Leftarrow)$ Let $0 \neq rstm \in E$, where $r, s, t \in R, m \in M$, with $m$ does not belong to $0$, then $0 \neq rstm \in E$. But $E$ is a weakly quasi 2-absorbing ideal in $R$, then $rs \in \{E : m\}$ or $rt \in \{E : m\}$ or $st \in \{E : m\}$, and hence $rsm \in E$ or $rstm \in E$ or $tm \in E$.

PROPOSITION 2.16
Let $E$ be proper quasi 2-absorbing submodule of an $R$-module $M$ then $S^{-1}E$ is weakly quasi 2-absorbing submodule of $S^{-1}M$ as $S^{-1}R$-module.

Proof
Let $0 \neq \overline{abc} \overline{m} \in S^{-1}E$, where $a = \frac{a_1}{s_1}, \overline{b} = \frac{b_2}{s_2}, \overline{c} = \frac{c_2}{s_2}$ are elements in $S^{-1}E$, where $a_1, b_1, c_1 \in R$, and $\overline{m} = \frac{m_1}{s_4} \in S^{-1}M$, where $m_1 \in M, s_1, s_2, s_3, s_4 \in S$. Hence $0 \neq \frac{a_1}{s_1} \frac{b_1}{s_2} \frac{c_1}{s_3} \frac{m_1}{s_4} \in S^{-1}E$, that is $0 \neq \frac{a_1}{s_1} \frac{b_1}{s_2} \frac{c_1}{s_3} \frac{m_1}{s_4} \in S^{-1}E$. Thus $\overline{m} = \frac{m_1}{s_4} \in S^{-1}M$, where $m_1 \in M$. Therefore $\overline{r} \overline{m} \in S^{-1}E$, which is $0 \neq \frac{a_1}{s_1} \frac{b_1}{s_2} \frac{c_1}{s_3} \frac{m_1}{s_4} \in S^{-1}E$. Hence $\overline{r} \overline{m} \in S^{-1}E$. Thus either $\overline{a} \overline{b} \overline{m} \in S^{-1}E$ or $\overline{a} \overline{c} \overline{m} \in S^{-1}E$ or $\overline{b} \overline{c} \overline{m} \in S^{-1}E$.

Hence $S^{-1}E$ is weakly quasi 2-absorbing in $S^{-1}M$.

PROPOSITION 2.17
Let $E$ be a proper submodule of an $R$-module $M_1$. Then $E$ is a weakly quasi 2-absorbing submodule in $M_1$ if $E \oplus M_2$ is a weakly quasi 2-absorbing submodule of an $R$-module $M_1 \oplus M_2$, where $M_2$ is an $R$-module.

Proof
Let $(0, 0) \neq rstm \in E \oplus M_2$, where $r, s, t \in R, (m_1, m_2) \in M_1 \oplus M_2$ with $m_1$ is a nonzero element in $M_1$ and $m_2$ is a nonzero element in $M_2$, then $rstm \in E \oplus M_2$, since $E \oplus M_2$ is a weakly quasi 2-absorbing submodule in $M_1 \oplus M_2$, then either $rs \in E$ or $rt \in E$ or $st \in E$, then either $0 \neq rstm \in E \oplus M_2$ or $rstm \in E \oplus M_2$ or $stm \in E \oplus M_2$. Hence $E \oplus M_2$ is a weakly quasi 2-absorbing submodule in $M_1 \oplus M_2$.

Conversely: suppose that $E \oplus M_2$ is a weakly quasi 2-absorbing in $M_1 \oplus M_2$, and let $0 \neq rstm \in E$, where $r, s, t \in R, m_1$ is a nonzero element in $M_1$. Then for each $m_2 \in M_2$, we have $0 \neq rstm \in E \oplus M_2$, since $E \oplus M_2$ is a weakly quasi 2-absorbing in $M_1 \oplus M_2$, then either $rs \in E \oplus M_2$ or $rt \in E \oplus M_2$ or $st \in E \oplus M_2$. It follows that either $rs \in E$ or $rt \in E$ or $st \in E$. Hence $E$ is a weakly quasi 2-absorbing submodule in $M_1$.

PROPOSITION 2.18
Let $E$ be a proper submodule of an $R$-module $M_2$, then $E$ is a weakly quasi 2-absorbing in $M_2$ if and only if in $M_1 \oplus E$ is a weakly quasi 2-absorbing in $M_1 \oplus M_2$.

Proof
Similarly as in proposition (2.17).

"PROPOSITION 2.19
Let $M$ be a $R$-module, and $E$ be a proper submodule of $M$. Then the statements are equivalents.

1. $E$ is a weakly quasi 2-absorbing submodule in $M$
2. For each $r, s \in R, m \in M$ if $0 \neq r \in E$, then $[E : rsm] = [E : rm] \cup \{E : sm\}$
3. For each $r, s \in R, m \in M$ if $0 \neq r \in E$, then \[E : rsm\} = \{E : rm\} \cup \{E : sm\} = [E : sm].

Proof
1 $\Rightarrow$ 2
Let $t \in \{E : rsm\}$, then $0 \neq rstm \in E$. Since $E$ is a weakly quasi 2-absorbing in $M$ and $0 \neq rsm \in E$, then either $stm \in E$ or $rtm \in E$, then $t \in \{E : s m\}$ or $t \in \{E : rsm\}$. 103
Recall that a proper submodule $E$ of an $R$-module is called quasi prime if $rsm \in E$, where $r, s \in R, m \in M$ implies that either $rm \in E$ or $sm \in E$ [4].

It is well known that every quasi prime submodule is a weakly quasi-prime [1], we get the following result.

**PROPOSITION 2.20**

Quasi prime submodule is weakly quasi 2-absorbing.

**Proof**

Follows by Remark (2, 2).

"Recall that a proper submodule $N$ of an $R$-module $M$ is a prime if $rm \in N$, with $r \in R, m \in M$, implies that either $m \in N$ or $r \in \{N:M\}$ [6]."

"It is well known prime submodule is quasi-prime [4]", we have the following corollary.

**COROLLARY 2.21**

Every prime submodule is weakly quasi 2-absorbing.

**Proof:** Follows by proposition (2.20).

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The equations (2) and (3) of the previous page are referenced as follows: