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A Priori and a Posteriori Error Analysis for Generic Linear Elliptic Problems

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ABSTRACT

In this paper, a priori error analysis has been examined for the continuous Galerkin finite element method which is used for solving a generic scalar and a generic system of linear elliptic equations. We derived optimal order a priori error bounds in H_0^1 (energy) norm utilising standard a priori error analysis techniques and tools. Also, a posteriori error analysis is investigated for a generic scalar linear elliptic equation and for a generic system of linear elliptic equations. We derived optimal residual-based a posteriori error estimates energy technique in H_0^1 norm.

1 Introduction

The finite element methods (FEMs) are broad family of numerical and approximate methods which used for solving ordinary differential equations (ODEs) and partial differential equations (PDEs) and also it is used for solving integro-differential equations (IDEs). The FEMs have many excellent numerical features that make them popular and widely used in scientific computing. The main advantage of the FEMs is its ability for solving a wide variety of problems on different computational domains with different shapes. For example, finite difference methods (FDMs) can solve problems on rectangular and triangular meshes while FEMs can handle geometries of any shapes. The beginning of the FEMs is dated back to the 1940s in the works on using variational methods for solving engineering problems in particular in Courant's work [1]. Engineers utilised the FEMs for solving and approximating a wide range of engineering application problems in 1950s and 1960s. The rigorous mathematical foundation of the FEMs started in the late 1970s. From the 1980s and onwards a huge number of research papers, monographs and books appeared in the literature about the FEMs and their applications [2].

Elliptic PDEs have been studied extensively during the last three decades from different numerical points of view and a plethora of references about FEM solutions of elliptic problems have been appeared in the literature, just to name a few $[3-16]$. In [17] the authors solved Poisson equation using FEM and derived a posteriori error bounds for the numerical method and then they designed an adaptive finite element method (AFEM) utilising these a posteriori error bounds. The a priori error estimates for a coupled semilinear PDE-ODE system (where an elliptic PDE coupled with a semilinear ODE) are obtained in $H_0^1(0,T;L_2(\Omega))$ norm in [18]. Ern and Meunier [19] in (2007) derived a posterirori error estimates for EulerGalerkin FEM used for solving coupled elliptic-parabolic problems. In [20] Kim et al investigated the numerical solution of elliptic problems using staggered discontinuous Galerkin (SDG) method on rectangular meshes. They obtained optimal convergence results in L_2 and H_1 norms. In (2003) Georgoulis [21] studied and investigated the hp-version interior penalty (hp-DGFEM) for linear elliptic and parabolic equations. Virtanen [22] considered and derived adaptive DGFEM for linear fourth order elliptic and parabolic equations. Guignard in [23] examined the error analysis for low regularity elliptic problems with random input data. Sabawi [24] examined and derived a posteriori and a priori error estimates for elliptic and parabolic interface problems using discontinuous Galerkin DGFEM. Also, Cangiani and coworkers studied and investigated the adaptivity and convergence of the DGFEM for the elliptic and parabolic interface problems in [25] and [26], respectively.

In [27] the authors considered and presented a class of post-processing operator in the context of studying the a posteriori error analysis for post-processed solutions of elliptic problems. Yang [28] in (2020) examined and studied the error analysis for elliptic problems with low regularity. Ye and Zhang in [29] analysed and studied the error estimates for continuous and discontinuous weak Galerkin (WG) FEMs for elliptic problems with low regularity solutions in energy and L_2 norms. Casas and coworkers [30] examined the numerical solution of semilinear elliptic equations. They proved the existence and uniqueness of a sequence of bounded solutions in $L_\infty(\Omega)$. The a posteriori error analysis for elliptic obstacle problem is investigated in [31].

In this paper, we considered deriving the a posteriori and a priori error estimates of the FEM solution of generic linear elliptic equations and also for generic linear systems of elliptic equations using conforming Galerkin finite element method. The main contribution of this paper is deriving optimal order residual based a posteriori error estimates in H_0^1 norm for generic scalar linear elliptic equation and also for a generic linear system of elliptic equations using energy techniques. Additionally, optimal order a priori error bounds in H_0^1 norm for generic scalar linear elliptic problem and for a generic linear system of elliptic equtions are obtained using energy arguments and standard interpolation error estimates.

This paper is organised as follows. In section 2 we give the necessary and relevant definitions and preliminaries of the problem. The a posteriori error bounds for a general scalar linear elliptic equation and for a general linear system of elliptic equations are derived in section 3. Section 4 is devoted for the a priori error analysis for the general scalar linear elliptic equation and for a general linear system of elliptic equations. The conclusions are given in section 5.

2 Problem Setting and Notation

Consider the following generic scalar elliptic boundary value problem as a mathematical model

$Au + \kappa u = f \; on \; \Omega,$ (1)

$u = 0 \; on \; \partial\Omega,$

where $A: V \to V$ is a second order self-adjoint linear elliptic operator, $\kappa \geq 0$ is a parameter and Ω is a bounded domain in $R^n, n \geq 1$ with sufficiently smooth boundary $\partial\Omega$. The solution function $u \in H^2(\Omega) \cap H_0^1(\Omega)$and the source function $f \in L_2(\Omega)$. For simplicity of notations, we use $V = H_0^1(\Omega)$ unless otherwise stated. Testing (1) with a test function $v \in V$, and then integrating the resulting equation over the domain Ω, yields

$a(u,v) = \ell(v), \quad \forall v \in V.$ (2)

$a(u,v) = \int_\Omega (Au + ku)v \, dx, \; \forall \, v \in V,$ (3)

where a is the bilinear form associated with the linear elliptic operator A defined by

$(Au,v) = a(u,v), \forall v \in V,$ (4)

and is $\ell(v)$ a linear functional defined by

$\ell(v) = (f,v) = \int_\Omega fv \, dx, \forall \, v \in V.$ (5)

Also, the bilinear form $a(.,.)$ satisfies the continuity (boundedness) and coercivity (V-ellipticity) conditions as follows

$a(u,w) \geq C_{cont} \|u\|_V \|w\|_V, \quad \forall u,w \in V,$ (6)

$a(u,u) \geq C_{coer} ||u||_v^2, \quad \forall u \in V,$ (7)

where C_{cont} and C_{coer} are positive constants. Now, we seek to find a finite element approximate solution of u which satisfies

$a(u_h,v) = (f,v), \forall v \in V,$ (8)

picking $v = \varphi \in V_h \subset V$ in the weak form (8), then the problem becomes: find $u_h \in V_h$ such that

$a(u_h,\varphi) = (f,\varphi), \forall \varphi \in V_h.$ (9)

The right-hand side function f can be approximated using its L_2 projection f_h which is defined by

$(f,\varphi) = (f_h,\varphi), \forall \varphi \in V_h,$ (10)

where $f_h = P_0 f$ is the L_2 projection of f and $P_0 : L_2 \to V_h$ is the L_2 projection operator. Also, define the discrete elliptic operator $A_h : V_h \to V_h$ as

$(A_h v,\varphi) = a(v,\varphi), \forall \varphi \in V_h,$ (11)

using (10) in the variational form (9), we obtain

$a(u_h,\varphi) = (f_h,\varphi), \forall \varphi \in V_h,$ (12)

which can be written as

$(A_h u_h + \kappa u_h - f_h, \varphi) = 0, \forall \varphi \in V_h,$ (13)

which can be expressed in the pointwise form as

$A_h u_h + \kappa u_h - f_h = 0,$ (14)

since $A_h u_h + k u_h - f_h \in V_h$ and its projection with respect to every element in V_h is zero. We can conclude from (14), that the approximate finite element solution u_h of the original elliptic PDE problem in (1) is the true solution of the elliptic PDE with discrete elliptic operator A_h and the right-hand side function f_h. The pointwise form (14) is the discrete version of the original elliptic PDE in (1).

3 A Priori Error Analysis of Linear Elliptic Problems

The a priori error analysis is very important topic in the study of error analysis and convergence analysis of differential equations using FEMs and other methods. In a priori error analysis we are interested in finding an error estimator of the form

$||e||_v = ||u - u_h||_v \leq (u,f,V).$ (15)

Notice that in general, the bound in the a priori error analysis depends upon the data of the problem, the forcing term f, the exact solution u of the problem and the space V. The a priori error bounds in general are not computable since they depend on the exact solution of the problem u which in most cases is unknown. While the a posteriori error estimators are computable and can be computed since they depend on the approximate solution u_h which is known. For this reason, we use a posteriori error bounds in designing adaptive numerical methods. While the a priori error analysis is used in the study of convergence of the exact solution of the original problem. The a priori error analysis is used in finding the order of convergence of the exact solution and it tells us the required information about how the convergence is fast or how it is slow. In our problem, the a priori error bound depends on the data of the problem, the right-hand side function f, and the exact solution u of the original problem (1). In this section, we consider deriving a priori error bounds for a generic scalar linear elliptic equation and for a generic linear system of elliptic equations.

3.1 A Priori Error Analysis for a Generic Scalar Linear Elliptic Problems

In this section we derive a priori error estimates for a generic scalar linear elliptic PDE in (1). Now we start the error analysis by subtracting (9) from (2), we obtain

$a(u - u_h, \psi) = (f - f_h, \psi) = 0, \forall \psi \in V_h$. (16)

Now, we splitting the error in the following form

$e = u - u_h = (u - \pi u) + (\pi u - u_h) = \rho + \theta$, (17)

where $\pi u \in V_h$ is the interpolant of the exact solution $u \in V, \rho = u - \pi u$ represents the interpolation error which is available in the literature. The idea here is to bound the quantity $\theta = \pi u - u_h \in V_h$ for which we do not have a bound by the quantity in terms of ρ for which we have a bound, consequently, the whole error e can then be bounded in terms of ρ, i.e.,

$||e||_v = ||u - u_h||_v = ||(u - \pi u) + (\pi u - u_h)||_v = ||\rho + \theta||_v \leq ||\rho||_v + ||\theta||_v$, (18)

then, we need to bound θ by a bound depends upon ρ i.e.,

$||\theta||_v \leq E(\rho)$. (19)

Finally, the whole error is bounded by a bound in terms of ρ

$||e||_v = ||\rho + \theta||_v \leq E(\rho) + ||\rho||_v = F(\rho)$. (20)

Note that from now on we use the following notation for the energy norm $||.||_{H_0^1(\Omega)} = ||.||_0$.

Theorem 3.1 (H_0^1 A Priori Error Bound for a Generic Scalar Linear Elliptic Equation)

The finite element approximate solution u_h of the problem (1), satisfies the following a priori energy (H_0^1) error estimate

$||e||_0 = ||u - u_h||_0 \leq C_3 \tilde{h} ||Du||_{L^2(\Omega)}$. (21)

Proof. Substituting $e = \rho + \theta$ in (16), and testing with $\psi = \theta$, we have

$a(\theta, \theta) = -a(\rho, \theta) = a(-\rho, \theta)$, (22)

using the continuity and ellipticity of $a(.,.)$, we have

$||\theta||_0 \leq C_1 ||\rho||_0$, (23)

where $C_1 = C_{cont}/C_{coer}$, and

$||\rho||_0 = ||\nabla \rho||_{L_2(\Omega)} = ||\nabla (u - \pi u)||_{L_2(\Omega)} \leq C \sum_{k \in T} h_k^2 ||Du||_{L_2(\Omega)}$,

which represents the L_2 norm of the gradient of the interpolation error and D is the total derivative of the function u, and

$||\rho||_{L_2(\Omega)}^2 = ||u - \pi u||_{L_2(\Omega)}^2 \leq C \sum_{k \in T} h_k^4 ||Du||_{L_2(\Omega)}^2$. (24)

Now, let $\tilde{h} = max_{k \in T} h_K$ hence, we get

$||\rho||_{L_2(\Omega)} \leq C \tilde{h}^2 ||Du||_{L_2(\Omega)}$, (25)

$||\nabla \rho||_{L_2(\Omega)} \leq C \tilde{h} ||Du||_{L_2(\Omega)}$. (26)

From (26), we have

$||\rho||_0 = ||\nabla \rho||_{L_2(\Omega)} \leq C \tilde{h} ||Du||_{L_2(\Omega)}$. (27)

Hence,

$||\theta||_0 \leq C_2 \tilde{h} ||Du||_{L_2(\Omega)}$, (28)

where $C_2 = CC_1$. Finally, combining both bounds in (27) and (28) yields the required estimate

$||e||_0 \leq ||\theta||_0 + ||\rho||_0 \leq C_2 \tilde{h} ||Du||_{L_2(\Omega)} + C \tilde{h} ||Du||_{L_2(\Omega)} = C_3 \tilde{h} ||Du||_{L_2(\Omega)}$, (29)

where $C_3 = C + C_2$. Note that since u_h is a piecewise linear then Du is a piecewise constant function and $D_2 u_h = 0$, where

$D_{u_h} = \frac{\partial u_h}{\partial x} + \frac{\partial u_h}{\partial y}$,

$D_{u_h}^2 = \frac{\partial_{u_h}^2}{\partial x^2} + 2 \frac{\partial_{u_h}^2}{\partial x \partial y} + \frac{\partial_{u_h}^2}{\partial y^2}$.

3.2 A Priori Error Analysis of Generic Systems of Linear Elliptic PDEs

The techniques and results of a priori and a posteriori error analysis for a generic scalar elliptic PDE can be extended and generalised to a generic system of any size of elliptic PDEs. For simplicity, we consider a generic linear elliptic system of two equations, noting that the case of a system of n equations follows similarly

$$-\epsilon_{11} \Delta u - \epsilon_{21} \Delta v + k_{11} u + k_{21} v = f_1,$$
$$-\epsilon_{21} \Delta u - \epsilon_{22} \Delta v + k_{21} u + k_{22} v = f_2, \quad (30)$$
$$u = v = 0 \ on \ \partial\Omega,$$

where $\epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{22}$ are diffusion parameters, $k_{11}, k_{12}, k_{21}, k_{22}$ are non-negative parameters and f_1, f_2 are source functions of x, y. For convenience, we introduce a vector function

$w: L_2(\Omega) \times L_2(\Omega) \to R$, where, $w = \binom{u}{v}$,

using this notation, we can express the system as a generic scalar vector elliptic equation

$-\epsilon \Delta w + kw = f$, (31)

where $\epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix}$, Δw is the Laplacian operator defined elementwise $\Delta w = \binom{\Delta u}{\Delta v}$ and the function $f = \binom{f_1}{f_2}$. To write (31) in the weak form, we first multiply it by a vector function $\psi \in H = H_0^1(\Omega) \cap H_0^1(\Omega)$ with $\psi = \binom{\psi_1}{\psi_2}$, where $\psi_1, \psi_2 \in V = H_0^1(\Omega)$, integrating over the domain Ω, we get

$\int_\Omega (-\epsilon \Delta w + kw) \psi \, dx = \int_\Omega f \psi \, dx$. (32)

Integrating the first term on the right-hand side of (32) using Green's formula to obtain

$\int_\Omega (-\Delta w) \psi \, dx = \int_\Omega \nabla w \nabla \psi \, dx - \nabla w \psi|_{\partial\Omega} = \int_\Omega \nabla w \nabla \psi \, dx$, (33)

since $\psi = 0$ on $\partial\Omega$ because $\psi = \binom{\psi_1}{\psi_2}$ and $\psi_1 = \psi_2 = 0$ on $\partial\Omega$. Substituting (33) in (32), we get

$\int_\Omega (\epsilon \nabla w \nabla \psi + kw\psi) dx = \int_\Omega f \psi \, dx, \forall \psi \in H$.

Then, the variational formulation becomes

$a(w, \psi) = (\psi), \forall \psi \in H$, (34)

where
$a(w,\psi) = \int_\Omega (\epsilon \nabla w \nabla \psi + w\psi)dx \,\forall\, w,\psi \in H,$
and $a(.,.)$ is the bilinear form defined as $a: H \times H \to R$ and ℓ is linear functional $\ell : H \to R$ defined by
$l(\psi) = \int_\Omega f\psi\, dx, \forall\, \psi \in H,$
where
$a(w,\psi) = \int_\Omega (\epsilon \nabla w \nabla \psi + kw\psi)dx = \int_\Omega (\epsilon \nabla u \nabla \psi_1 + ku\psi_1)dx + \int_\Omega (\epsilon \nabla u \nabla \psi_1 + ku\psi_2)\, dx = a(u,\psi_1) + a(v,\psi_2),$
which represents the bilinear form on $V \times V$. The right-hand side is defined by
$\ell(\psi) = \int_\Omega f\psi\, dx = \int_\Omega f_1\psi_1 + \int_\Omega f_2\psi_2\, dx = \ell(\psi_1) + \ell(\psi_2) = (f_1,\psi_1) + (f_2,\psi_2),$
which represents the $L_2(\Omega)$ inner product. The H norm is defined as
$||w||_H^2 = \int_\Omega (\epsilon(\nabla w)^2 + kw^2)dx = ||u||_H^2 + ||v||_H^2.$
To solve this problem numerically, we seek an approximation $w_h = \binom{u_h}{v_h} \in V_h = (V_h \times V_h) \subset H$, which is a vector of finite element approximations of the functions u and v. Hence the problem becomes: find $w_h \in V_h$ such that
$a(w_h,\varphi) = (\varphi), \forall \varphi \in V_h.$ (35)

Theorem 3.2 (H_0^1 A Priori Error Bound for a Generic Linear Elliptic System) The finite element approximate solution w_h of the problem (30), satisfies the following a priori energy (H_0^1) error estimate
$||e||_0 = ||w - w_h||_0 \le \tilde{C}\tilde{h}\left(||u||_{L2(\Omega)} + ||v||_{L2(\Omega)}\right) = \tilde{C}\tilde{h}||Dw||_{L2(\Omega)}.$ (36)

Proof. The numerical error $e = w - w_h$ can be split up in the following form
$e = w - w_h = (w - \pi w) + (\pi w - w_h) = \rho + \theta,$
where $\rho = w - \pi w \in H$ represents the interpolation error of w and ρ also can be split up as $\rho = w - \pi w = \rho_1 + \rho_2 = (u - \pi u) + (v - \pi v)$, where
$||\rho||_0 \le ||\rho_1||_0 + ||\rho_2||_0 = ||u - \pi u||_0 + ||v - \pi v||_0,$
$\le Ch||Du||_{L2(\Omega)} + Ch||Dv||_{L2(\Omega)}$
$Ch(||Du||_{L2(\Omega)} + ||Dv||_{L2(\Omega)} = Ch||Dw||_{L2(\Omega)},$
where ρ_1 represents the interpolation error of u and ρ_2 represents the interpolation error of v. Consequently, we have
$||\rho||_0 \le Ch||Dw||_{L2(\Omega)},$
while $\theta = \pi w - w_h \in V_h$, we do not have a bound of θ and we use the known bound of ρ to bound θ and consequently obtaining a bound of the total error e as follows
$||e||_0 = ||\rho + \theta||_0 \le ||\rho||_0 + ||\theta||_0,$
we want to have a bound such that
$||\theta||_0 \le C||\rho||_0,$
and hence, we get
$||e||_0 \le ||\rho||_0 + ||\theta||_0 \le C^{\wedge}||\rho||_0.$
Subtracting (35) from (34) results in
$a(e,\varphi) = a(w - wh,\varphi) = a(\rho + \theta,\varphi) = 0, \forall \varphi \in V_h.$
Now, testing by $\varphi = \theta$, and using some mathematical manipulations leads to
$a(\theta,\theta) = -a(\rho,\theta) = a(-\rho,\theta),$
using coercivity and continuity of the bilinear form, we have
$||\theta||_0 \le C_1||\rho||_0,$
where $C_1 = C_{cont}/C_{coer}$. Finally, we get
$||e||_0 \le ||\rho||_0 + ||\theta||_0 \le ||\rho||_0 + C_1||\rho||_0 = C_2||\rho||_0,$
where $C_2 = 1 + C_1$. Consequently, we have,
$||e||_0 \le C_2||\rho||_0 = C_2Ch||Dw||_{L2(\Omega)} = C_2Ch\left(||u||_{L2(\Omega)} + ||v||_{L2(\Omega)}\right) = C^{`}h||Dw||_{L2(\Omega)},$
where $\tilde{C} = C_2C$.

4 A Posteriori Error Analysis of Linear Elliptic PDEs

A posteriori error analysis is a very important and efficient technique in devising robust, efficient and effective adaptive methods. It is used for finding a bound or estimate for the error $e = u - u_h$ in terms of the approximate solution u_h, data of the problem and the right-hand side function f. We need to find an a posteriori estimator function $F = F(u_h, f; V)$ which depends upon the functions u_h, f and the space V, such that f satisfies the following relation $||e||_V = ||u - u_h||_V \le F(u_h, f; V)$. The a posteriori error estimators help in reducing the computational cost of solving a problem using the numerical method and this is the crucial aspect for any effective and reliable adaptive method. In this section, we derive the a posteriori residual based error bounds for a generic scalar linear elliptic equation and a generic linear system of elliptic equations.

4.1 A Posteriori Error Analysis for a Generic Scalar Linear Elliptic Problems

We consider the BVP (1) when $A = -\epsilon\,\Delta$ which becomes
$-\epsilon\Delta u + ku = f \text{ on } \Omega,$ (37)
$u = 0 \text{ on } \partial\Omega,$
where κ, u, f as in (1) and $\epsilon > 0$ is the diffusion parameter.

Theorem 4.1 (H_0^1 *A* Posteriori Error Bound for a Generic Scalar Linear Elliptic Equation)
The finite element approximate solution u_h of the problem (37), satisfies the following a posteriori energy (H_0^1) error estimate
$||e||_0^2 + ||u - u_h||_0^2 \le \sum_{k\in T} \xi_k^2(u_h),$ (38)
where $\xi_K(u_h)$ is the element wise residual which is defined as
$\xi_k(u_h) = h_k||R(u_h)||_{L2(\Omega)} + \frac{1}{2} h_k^{1/2} ||[n.\nabla u_h]||_{L2(\partial k/_{\partial\Omega})},$ (39)
where $R(u_h) = f + \epsilon\Delta u_h + ku_h$ is the residual which expresses the amount the approximate solution u_h misses to satisfy the weak form and $R \in V^0$ is the residual operator and V^0 is the dual space of V =

$H_0^1(\Omega)$, and $[n.\nabla u_h]$ is the jump in the normal derivative of the approximate solution u_h on the interior edges of element K.

Proof. Subtracting the finite element approximation weak form (9) from (2), this yields

$$a(u-u\_h,v) = \int_\Omega (\epsilon\nabla(u-u\_h).\nabla v + k(u-u\_h)v)dx = 0, \quad \forall v \in V. (40)$$

Let $e = u - u_h$, then we get

$$a(e,v) = \int_\Omega (\epsilon\nabla e.\nabla v + kev)dx = 0, \quad \forall v \in V. (41)$$

Now, test by $v = e \in V$, we obtain

$$a(e,e) = ||e||_0^2 = \int_\Omega (\epsilon(\nabla e)^2 + ke^2)dx = 0. \quad (42)$$

Note thate $\in H_0^1(\Omega)$ since $e = u - u_h = 0$ on $\partial\Omega$ and $||e||_0 = ||e||_1$ when $\epsilon = k = 1$. Using Galerkin orthogonality, we have

$$||e||_0^2 = \epsilon\int_\Omega \nabla e.\nabla(e-\pi e)dx + k\int_\Omega e(e-\pi e)dx, \quad (43)$$

where πe is the interpolant of e. Integrating elementwise and using Green's formula on the first integral on the left-hand side, we obtain

$$||e||_0^2 = \epsilon\int_\Omega \nabla e.\nabla(e-\pi e)dx + k\int_\Omega e(e-\pi e)dx = \sum_{k\in T} -\epsilon\int_\Omega \Delta e(e-\pi e)dx + \epsilon\int_{\partial\Omega} n.\nabla e(e-\pi e)ds + \sum_{k\in T} k\int_\Omega e(e-\pi e)dx,$$

where T is the triangulation of the domain Ω. Notice that e and its interpolant πe both vanish on the boundary $(\partial\Omega)$, this yields

$$||e||_0^2 = \sum_{k\in T} -\epsilon\int_K \nabla e.\nabla(e-\pi e)dx + k\int_K e(e-\pi e)\, dx = \sum_{k\in T} -\epsilon\int_{\partial k/\partial\Omega} \Delta e(e-\pi e)dx + \epsilon\int_{\partial\Omega} n.\nabla e(e-\pi e)ds + \sum_{k\in T} k\int_K e(e-\pi e)dx. \quad (44)$$

Upon observing that $(-\epsilon\Delta e + ke)|_k = (f + \epsilon\Delta u_h - ku_h)|_k$, we obtain

$$||e||_0^2 = \sum_{k\in T}\int_K (f+\epsilon\Delta u_h - ku_h)(e-\pi e)dx + \sum_{k\in T}\epsilon\int_{\partial k/\partial\Omega} n.\nabla e(e-\pi e)ds, \quad (45)$$

since we have two contributions from each edge E (because the edge E is a common edge between two elements (triangles) K^+ and K^-, considering these contributions, we arrive at

$$\int_{\partial k+/\partial k-} n.\nabla e(e-\pi e)ds = \int_E (n^+.\nabla e^+(e^+-\pi e^+) + n^-.\nabla e^-(e^- - \pi e^-))\, ds, \quad (46)$$

since the error function is continuous, so we have

$(e+ \; -\pi e+)|E = (e- \; -\pi e-)|E$.

Therefore,

$$\int_{\partial k+/\partial k-} n.\nabla e(e-\pi e)ds = \int_E (n^+.\nabla e^+(e^+-\pi e^+) + n^-.\nabla e^-(e^- - \pi e^-))\, ds \quad (47)$$

$$= \int_E (n^+.\nabla e^+ + n^-\nabla e^-)(e-\pi e)\, ds.$$

Also, since u_h is a piecewise linear function then its $\int_{\partial k/\partial\Omega} n.\nabla e(e-\pi e)ds$ gradient $\nabla u_h|_E$ is a piecewise constant function and in general is not continuous. Hence, we should take into consideration that the jump in normal derivative $n.\nabla u_h$ may be different on neighbouring elements K^+ and K^-. In addition, the gradient $\nabla u|_E$ is continuous, so the jump term $(n^+.\nabla u^+ + n^-.\nabla u^-)|_E = 0$. Consequently, we obtain

$$\int_E (n^+.\nabla e^+ + n^-.\nabla e^-)(e-\pi e)\, ds = -\int_E (n^+.\nabla u_h^+ + n^-.\nabla u_h^-)(e-\pi e)\, ds = -\int_E [n.\nabla u_h](e-\pi e)\, ds. \quad (48)$$

From (48), we conclude that

$$\sum_{k\in T}\int_K n.\nabla e(e-\pi e)ds = -\sum_{E\in E_1}\int_E [n.\nabla u_h]\,(e-\pi e)ds. \quad (49)$$

Using the fact that each element contributes by half amount of the jump, we finally have

$$||e||_0^2 = \sum_{k\in T}\int_K (f+\Delta u_h - ku_h)(e-\pi e)dx - \frac{1}{2}\int_{\partial k/\partial\Omega} [n.\nabla u_h\,(e-\pi e)ds. \quad (50)$$

The equation in (50) is called the error representation formula. Now, returning back to the first term on the right-hand side of (50), we can bound it using the standard interpolation error bounds and Cauchy - Schwarz inequality to obtain

$$\int_K (f+\epsilon\Delta u_h - ku_h)(e-\pi e)dx \le ||f+\epsilon\Delta u_h - ku_h||_{L2(k)}\, ||e-\pi e||_{L2(k)} \quad (51)$$

$$\le ||f+\epsilon\Delta u_h - ku_h||_{L2(k)}\, Ch_k\, ||De||_{L2(k)}.$$

Using the scaled trace inequality for the edge contribution, we get

$$||e-\pi e||_{L2(\partial k)} \le C(h_k^{-1}\,||e-\pi e||_{L2(k)}^2 + h_k\,||\nabla(e-\pi e)||_{L2(k)}^2). \quad (52)$$

Inserting (52) in (51) with the aid of the Cauchy-Schwarz inequality, we get

$$\int_{\partial k} [n.\nabla u_h](e-\pi e)ds \le ||[n.\nabla u_h]||_{L2(\partial k)} ||e-\pi e||_{L2(\partial k)} \quad (53)$$

$$\le ||[n.\nabla u_h]||_{L2(\partial k)} C(h_k^{-1}\,||e-\pi e||_{L2(k)}^2 + h_k\,||D(e-\pi e)||_{L2(k)}^2)$$

$$||[n.\nabla u_h]||_{L2(\partial k)} Ch_k^{\frac{1}{2}}\,||De||_{L2(k)}.$$

Notice that, we used in (53), the standard interpolation error estimates. Combining (51) and (53) in (54), we finally have

$$||e||_0^2 \le C\sum_{k\in T}(h_k^2\,||f+\epsilon\Delta u_h - ku_h||_0^2 + \frac{1}{2}h_k^{1/2}\,||[n.\nabla u_h]||_{L2(\partial k/\partial\Omega)}^2 = C\sum_{k\in T}\xi_k^2(u_h). \quad (54)$$

4.2 A Posteriori Error Analysis for a Generic System of Linear Elliptic Equations

In this section, we consider deriving a posteriori error estimate for the general linear system of elliptic equations in (30).

Theorem 4.2 (H_0^1 A Posteriori Error Bound for a Generic Linear Elliptic System)

The finite element approximate solution w_h of the problem (30), satisfies the following a posteriori energy (H_0^1) error estimate

$$||e||_0^2 = ||w-w_h||_0^2 \le C\sum_{k\in T}\xi_k^2(w_h), \quad (55)$$

where $\xi_K(w_h)$ is the elementwise residual which is defined as

$$\xi_k(w_h) = h_k||R(w_h)_{L2(\Omega)} + \frac{1}{2}h_k^{1/2}||[n.\nabla w_h]||_{L2(\partial k/\partial\Omega)}, \quad (56)$$

where $R(w_h) = f + \epsilon\Delta w_h + kw_h$ is the residual which expresses the amount the approximate solution

w_h misses to satisfy the weak form and $R \in V_0$ is the residual operator, and $[n.\nabla w_h]$ is the jump in the normal derivative of the approximate solution w_h on the interior edges of element K.

Proof. Following the same steps as before, we arrive at the following weak forms

$a(w,\varphi) = \ell(\varphi), \forall\varphi \in V_h$, (57)

And

$a(w_h,\varphi) = \ell(\varphi), \forall\varphi \in V_h$, (58)

subtracting (58) from (57), we get

$a(e,\phi) = a(w - w_h,\phi) = 0, \quad \forall \in v_h$,

where

$e = w - w_h = \binom{u}{v} - \binom{u_h}{v_h} = \binom{u-u_h}{v-v_h} = \binom{e_u}{e_v}$.

Now, testing by $\varphi = e \in H$, we have

$a(e,e) = ||e||_0^2 = \int_\Omega (\epsilon(\nabla e)^2 + ke^2)dx$.

Using Galerkin orthogonality, we get

$||e||_0^2 = \epsilon \int_\Omega \nabla e \nabla(e - \pi e)dx + k\int_\Omega e(e - \pi e)\,dx = \sum_{k\in T} -\epsilon\int_K \Delta e(e - \pi e)dx$

$+\epsilon\int_{\partial k} n.\nabla e(e - \pi e)ds + \sum_{k\in T}\int_K e(e - \pi e)dx$,

where

$\pi e = \binom{\pi e_u}{\pi e_v} = \binom{u-\pi u}{v-\pi v}. \nabla e = \binom{\nabla e_u}{\nabla e_v} = \binom{\nabla(u-u_h)}{\nabla(v-v_h)}$,

where πe_u is the interpolation error for the function u and πe_v is the interpolation error for the function $v, \nabla e_u$ is the gradient of the interpolation error of u and ∇e_v is the gradient of the interpolation error of v.

Noting that $(-\epsilon\Delta e + ke)|_k = (f + \epsilon\Delta w_h - kw_h)|_k$, where

$(-\epsilon\Delta e_u + ke_u)|_k = (f + \epsilon\Delta u_h - ku_h)$,

and

$(-\epsilon\Delta e_v + ke_v)|_k = (f + \epsilon\Delta e_v - kv_h)$.

Considering the contributions of the internal edge E between the two elements (triangles) K^+ and K^-, we obtain

$\int_{\partial k^+\cap\partial k^-} n.\nabla e(e - \pi e)ds = \int_E \left(n^+.\nabla n^+(e^+ - \pi e^+) + \nabla e^-(e^- - \pi e^-)\right)ds$,

where

$e^+ = \binom{e_u^+}{e_v^+} = \binom{(u-u_h)^+}{(v-v_h)^+}$, $e^- = \binom{e_u^-}{e_v^-} = \binom{(u-u_h)^-}{(v-v_h)^-}$,

since the error function is continuous, we have

$(e^+ - \pi e^+)|E = (e^- - \pi e^-)|_E$,

where $(e_u^+ - \pi e_u^+)|_E = (e_u^- - \pi e_u^-)|_E =$ and $(e_v^+ - \pi e_v^+)|_E = (e_v^- - \pi e_v^-)|_E$.

Therefore,

$\int_{\partial K^+\cap\partial K^-} n.\nabla e(e - \pi e)ds = \int_E (n^+\nabla e^+ + n^-\nabla e^-)(e - \pi e)ds$.

Also, since the approximate functions u_h and v_h are piecewise linears then their gradients $\nabla u_h|_E$ and $\nabla v_h|_E$ are piecewise constants. So, we have to take into account that the jumps in normal derivatives $n.\nabla u_h$ and $n.\nabla v_h$ may be not the same on the adjacent elements K^+ and K^-. Moreover, the gradients $\nabla u|_E$ and $\nabla v|_E$ are continuous. Hence, the jump terms are

$(n^+.\nabla u^+ + n^-.\nabla u^-)|_E = 0$ and $(n^+.\nabla v^+ + n^-.\nabla v^-)|_E = 0$.

So, we have

$\int_E(n^+\nabla e^+ + n^-\nabla e^-)(e - \pi e)ds = -\int_E (n^+\nabla w_h^+ + n^-\nabla w_h^-)(e - \pi e)ds =$

$-\int_E(n^+\nabla u_h^+ + n^-\nabla u_h^-)(e - \pi e)ds - \int_E (n^+\nabla v_h^+ + n^-\nabla v_h^-)(e - \pi e)ds$

$= -\int_E[n\nabla u_h](e - \pi e)ds - \int_E[n\nabla v_h](e - \pi e)dx = -\int_E \left(n^+\nabla(u_h^+ + v_h^+) + n^-\nabla(u_h^- + v_h^-)\right)(e - \pi e)ds = -\int_E[n\nabla w_h](e - \pi e)ds$.

Upon observing that every element contributes by half amount of the jump, so we get

$||e||_0^2 = \sum_{k\in T}\int_K(f + \epsilon\Delta w_h - kw_h)(e - \pi e)dx - \frac{1}{2}\int_{\partial k/\partial\Omega} [n\nabla w_h](e - \pi e)ds$

$= \sum_{K\in T}\int_K (f + \epsilon_{11}\Delta u_h + \epsilon_{12}\Delta v_h - k_{11}u_h - k_{12}v_h)(e - \pi e)ds$

$+\sum_{K\in T}\int_K (f_2 + \epsilon_{21}\Delta u_h + \epsilon_{22}\Delta v_h - k_{21}u_h - k_{22}v_h)(e - \pi e)ds$

$-\frac{1}{2}\int_{\partial k/\partial\Omega} [n\nabla u_h + n\nabla v_h](e - \pi e)$.

Using standard interpolation error estimates and some mathematical techniques, we arrive at

$\int_K (f + \epsilon\Delta w_h - k\,w_h)(e - \pi e)ds \le \left|\left|f + \epsilon w_h - kw_h\right|\right|_{L2(\Omega)} \left|\left|e - \pi e\right|\right|_{L2(\Omega)}$.

$\le \left|\left|f_1 + \epsilon_{11}\Delta u_h + \epsilon_{12}\Delta v_h - k_{11}u_h - k_{12}v_h\right|\right|_{L2(\Omega)} Ch_k\,||De_u||_{L2(\Omega)}$

$+\left|\left|f_2 + \epsilon_{21}\Delta u_h + \epsilon_{22}\Delta v_h - k_{21}u_h - k_{22}v_h\right|\right|_{L2(\Omega)} Ch_k\,||De_v||_{L2(\Omega)}$.

Using the scaled version of the trace-inequality, we get

$||e - \pi e||_{L2(\partial\Omega)} = ||e_u - \pi e||_{L2(\partial k)} + ||e_v - \pi e_v||_{L2(\partial k)}$.

$\le C\left(h_k^{-1}\left|\left|e_u - \pi e_u\right|\right|_{L2(k)}^2 + h_k\left|\left|\nabla(e_u - \pi e_u)\right|\right|_{L2(k)}^2\right)$

$+C\left(h_k^{-1}\left|\left|e_v - \pi e_v\right|\right|_{L2(k)}^2 + h_k\left|\left|\nabla(e_v - \pi e_v)\right|\right|_{L2(k)}^2\right)$.

Using this inequality with the Cauchy-Schwarz inequality, we obtain

$\int_{\partial\Omega} [n\nabla w_h](e - \pi e)ds \le ||n\nabla w_h||_{L2(\partial\Omega)}||e - \pi e||_{L2(k)}$

$\le \left|\left|[n\nabla w_h]\right|\right|_{L2(\partial\Omega)} C\left(h_k^{-1}\left|\left|e - \pi e\right|\right|_{L2(k)}^2 + h_k\left|\left|D(e - \pi)\right|\right|_{L2(k)}^2\right)$

$\le \left|\left|[n\nabla w_h]\right|\right|_{L2(\partial\Omega)} Ch^{1/2}\,||De||_{L2(k)}$

$= \left(\left|\left|[n\nabla u_h]\right|\right|_{L2(\partial k)} + \left|\left|[n\nabla v_h]\right|\right|_{L2(\partial\Omega)}\right) Ch^{1/2}\,||De||_{L2(\partial k)}$.

Finally, we get

$||e||_0^2 = ||e_u||_0^2 + ||e_v||_0^2 \le C\sum_{k\in T}(h_k^2\left|\left|f + \epsilon\Delta w_h - kw_h\right|\right|_{L2(k)}^2 + \frac{1}{2}h_k^{1/2}\left|\left|[n\nabla w_h]\right|\right|_{L2(\partial k/\partial\Omega)}^2)$

$$\leq C \sum_{k\in T}(h_k^2(\left|\left|f_1 + \epsilon_{11}\Delta u_h + \epsilon_{21}\Delta v_h - k_{11}u_h - k_{12}v_h\right|\right|^2_{L2(k)})$$
$$+\left|\left|f_2 + \epsilon_{21}\Delta u_h + \epsilon_{22}\Delta v_h - k_{21}u_h - k_{22}v_h\right|\right|^2_{L2(k))})$$
$$+\frac{1}{2}\, h_k^{1/2}(\left|\left|[n\nabla u_h]\right|\right|^2_{L2(\partial k/_{\partial\Omega})} + \left|\left|[n\nabla v_h]\right|\right|^2_{L2(\partial k/_{\partial\Omega})})$$
$$= C\left(\sum_{k\in T}(\xi_k^2\,(u_h)\xi_k^2(v_{h)}\,)\right) = \sum_{k\in T}\xi_k^2\,(\,w_h).$$

5 Conclusions

We studied the error analysis of the finite element solution of generic scalar linear elliptic BVP and also, we considered the error analysis of the finite element solution of a generic system of linear elliptic equations in $2D$. Continuous Galerkin finite element method (CGFEM) is used on a mesh of triangular elements. Piecewise linear polynomials are used for the space discretisation. A posteriori error analysis for a general scalar linear elliptic problem and for a general system of linear elliptic equations is investigated. Optimal order a posteriori error bounds in H_0^1 norm are obtained. These error estimates are residual based and derived using energy techniques. In addition, a priori error analysis for a generic scalar linear elliptic problem and for a generic system of linear elliptic equations is examined. Optimal order a priori error bounds are obtained using energy arguments and standard interpolation error estimators.

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تحليل الخطأ القبلي والبعدي لمعادلات القطع الناقص الخطية العامة

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الملخص

تم في هذا البحث دراسة الخطأ القبلي لطريقة العناصر المنتهية من النوع *Galerkin* المتوافقة المستخدمة لحل المعادلات التفاضلية الجزئية من نوع القطع الناقص في حالة معادلة ونظام من المعادلات. اذ تم الحصول على قيد خطأ قبلي مثالي الرتبة حسب مقياس H_0^1 باستخدام الأساليب القياسية المستخدمة في تحليل الخطأ البعدي. كذلك تم دراسة الخطأ البعدي لمعادلة ولنظام من المعادلات. اذ تم الحصول على قيد خطأ بعدي مثالي الرتبة حسب مقياس H_0^1 باستخدام أسلوب مقياس الطاقة.