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# On Supra – Separation Axioms for Supra Topological Spaces

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## Abstract
Through the concepts of supra open sets and supra $\alpha -$ open sets, we introduce a new class of separation axioms and study some of their properties. We could comparative between these two items. At last we investigate the hereditary and other properties of them.

## 1- Introduction
The supra topological spaces had been introduced by A. S. Mashhour at [2] in 1983. So the supra open sets are defined where the supra topological spaces are presented. We have known that every topological space is a supra topological space, so as every open set is a supra open set, but the converse is not always true. Consideration the intersection condition is not necessary to have a supra topological space. Njastad at [6] in 1965 introduced$\alpha -$ open set. In 2008, R. Devi, S. Sampathkumar and M. Caldas [5] introduced the supra $- \alpha -$ open.
Some topological spaces we applied the properties of the separation axioms. These properties were studied by many researchers as Sze - Tsen, Hu at [3] in 1964 and S. Lipschutz at [1] in 1965. Also D. Sreerja and C. Janaki at[4] has discussed a new type of separation axioms in topological spaces in 2012.
In this paper we study the relationships between supra separation axioms and supra $\alpha -$ separation axioms and study some of characterizations of them.

## 2- Preliminaries and Basic Definitions
The spaces considered in this paper are supra topological spaces. $(X,\tau)$ is said to be a supra topological space if it is satisfing these conditions:

1- $\emptyset , X \in \tau$ .

2$-$ The union of any number of sets in $\tau$ belongs to $\tau$.

Each element $A \in \tau$ is called a supra open set in$(X,\tau)$ , and $A^c$ is called a supra closed set in $(X,\tau)$ [2].
The supra closure of a set A is denoted by supra cl(A) are defined as $cl(A) = \cap \{B: B \text{ is a supra closed and } A \subseteq B\}$. The supra interior of a set A is denoted by supra $int(A) = \cup \{B: B \text{is a supra open and } A \supseteq B\}$.[2]
The set A of X is called a supra $\alpha -$ open set, if $A \subseteq$ supra int(supra cl(supra int(A))). The complement of a supra $\alpha -$ open set is a supra $\alpha -$ closed set[5].

## 3- A New Classes of Supra Separations Axioms
In this section we introduce a new classes of supra separation axioms and we study it's characterizations.
**Definition1:** If $(X,\tau)$ is a supra topological space, for all $x, y \in X, x \neq y$, and there exist a supra open set G such that $x \in G$ and $y \notin G$. Then $(X,\tau)$ is called a supra $T_0$- space.

**Definitino2**: If $(X,\tau)$ is a supra topological space, $A \subseteq X$ , $A \neq \emptyset$ , $\tau_A$ is the class of all intersection of A with each element in $\tau$ , then $(A, \tau_A)$ is called a supra topological subspace of $(X,\tau)$.
The hereditary property of a supra $T_0$- axiom will be proved in the following theorem.
**Theorem1:** If $(X,\tau)$ is a supra $T_0$- space and $(Y,\tau)$ is a supra topological subspace of $(X,\tau)$ then $(Y,\tau_y)$ is a supra $T_0$- space.
Proof:
Suppose that $x, y \in Y, x \neq y$ , since $Y \subseteq X$ then $x, y \in X$
Since $(X,\tau)$ is a supra $T_0$- space means that there exist a supra open set $G \subseteq X$ such that $x \in G$ and $y \notin G$ .
We have that $G_y = Y \cap G$ , $G_y$ is a supra open set in Yand $x \in G_y$ but $y \notin G_y$, so we found a supra open set $G_y \subseteq Y$ which it contained x and not contained y.
Hence; $(Y,\tau_y)$ is a supra $T_0$- space. □
The following theorem needs to define a supra open function that means ((the image of any supra open set in $(X,\tau)$ is a supra open set in $(X^*,\tau^*)$ where these are two supra topological spaces))
**Theorem2:** If $(X,\tau)$ , $(X^*,\tau^*)$ are two supra topological spaces, $(X,\tau)$ is a supra $T_0$- space and f is a supra open function and bijective then $(X^*,\tau^*)$ is a supra $T_0$- space.
Proof:
Suppose that $(X,\tau)$ is a supra $T_0$- space.
Now we have to prove that $(X^*,\tau^*)$ is a supra $T_0$-space.
Let $x^*, y^* \in X^*, x^* \neq y^*$, since f is a bijective function then there exist $x, y \in X$ such that : $x^* = f(x)$ , $y^* = f(y)$ and $x \neq y$
Since $(X,\tau)$ is a supra $T_0$- space then there exists $G \subseteq X$ is a supra open set such that $x \in G$ and $y \notin G$
We obtain that $f(G) \subseteq X^*$ is a supra open sets in $X^*$ because f is a supra open function.
So $x^* \in f(G)$ and $y^* \notin f(G)$.
Then $(X^*,\tau^*)$ is a supra $T_0$- space. □
We can transferee the supra $T_0$- property, by using a continuous, bijective, function what it will be proved in the following theorem.
**Theorem3:** If $(X,\tau)$ , $(X^*,\tau^*)$ are two supra topological spaces, where $(X^*,\tau^*)$ is a supra $T_0$-space and $f: X \to X^*$ is a bijective continuous function then $(X,\tau)$ is a supra $T_0$- space.
Proof:

Suppose that $x, y \in X, x \neq y$, since f is a bijective function then there exist $x^*, y^* \in X^*$, $x^* \neq y^*$ such that $x^* = f(x)$, $y^* = f(y)$.
Since $X^*$ is a supra $T_0$- space, then there exist a supra open set $G \subseteq X^*$ such that $f(x) \in G, f(y) \notin G$.
Since f is a continuous function then $f^{-1}(G)$ is a supra open set in X contains x, but not contains y.
Hence; $(X, \tau)$ is a supra $T_0$- space. □

**Remark1:** Every $T_0$- space is a supra $T_0$- space, but the converse is not true as in the next example.

**Example1:** Let $X = \{x, y, z\}$, $= \{\emptyset, X, \{x\}, \{y\}, \{x, z\}, \{y, z\}, \{x, y\}\}$.

**Definition3:** If $(X, \tau)$ is a supra topological space, $G, H \subseteq X$ are supra open sets and if $x \in G$, $x \notin H$ and $y \notin G$, $y \in H$ then $(X, \tau)$ is called a supra $T_1$- space.

**Theorem4:** If $(X, \tau)$ is a supra topological space then $(X, \tau)$ is a supra $T_1$-space if and only if for every $x \in X$, $\{x\}$ is a supra closed set.

Proof: → First
Let $(X, \tau)$ be a supra topological space, we show that $\{x\}^c$ is a supra open set in X.
Suppose that $a \in \{x\}^c$, $a \neq x$ then by (def. 3) there exist $G_a$ is a supra open set in X where $G_a$ does not contain x. Hence; $a \in G_a \subseteq \{x\}^c$ and $\{x\}^c = \{G_a : a \in \{x\}^c\}$.
This means $\{x\}^c$ is a union of all supra open sets and by (axiom two from the def. of supra topological space).
$\{x\}^c$ is a supra open sets. Then $\{x\}$ is a supra closed set.
← Conversely
Suppose that $\{x\}$ is a supra closed set in X and let a, b∈ X where $a \neq b$ then $a \in \{b\}^c$, $b \in \{a\}^c$ and $\{b\}^c$, $\{a\}^c$ are supra open sets in X.
Hence $(X, \tau)$ is a supra $T_1$- space. □

The next theorem will use the definition of a supra $T_1$- space with the idea of the proof of (theorem1).

**Theorem5:** If $(X, \tau)$ is a supra $T_1$- space and $(Y, \tau)$ is a supra topological subspace of $(X, \tau)$ then $(Y, \tau_y)$ is a supra $T_1$- space.

The next theorem will use the definition of a supra $T_1$- space with the idea of the proof of (theorem2).

**Theoerm6:** If $(X, \tau)$, $(X^*, \tau^*)$ are two supra topological spaces, $(X, \tau)$ is a supra $T_1$- space and f is a supra open function and bijective then $(X^*, \tau^*)$ is a supra $T_1$- space.

We want to show that the role of the finite set in a supra $T_1$- space.

**Definition4:** If $(X, \tau)$ is a supra topological space, $A \subseteq X$ and $x \in X$, then x is said to be a supra limit point of A if for every supra open set G contains x, $A \cap G/\{x\} \neq \emptyset$.

The set of all supra limit points is called a supra derivative set and denoted by $\acute{A}$.

**Remark2:** A subset A of a supra topological space is a supra closed set if and only if A contains each of its supra limit points ($\acute{A} \subseteq A$).

**Theorem7:** If $(X, \tau)$ is a supra $T_1$- space and A is a finite sub set of X then A has no supra limit points.

Proof:
Suppose that $A = \{a_1, a_2, \ldots, a_n\}$ then A is a supra closed set and contains all its supra limit points, but $\{a_2, a_3, \ldots, a_n\}$ is also a finite supra – closed set, which means that $\{a_2, a_3, \ldots, a_n\}^c$ is a supra open set contains $a_1$.
Hence $a_1$ is not a supra limit point of A.
Similarly, no other points of A is a supra limit point of A. □

**Remark3:** We know that every finite $T_1$- space is a discrete space from [1], but in a supra $T_1$- space that is not always true, as an example (1).

The next theorem shows that a supra $T_1$- axiom is a hereditary property.

**Remark4:** Every supra $T_1$- space is a supra $T_0$- space, but the converse is not true as in the next example.

**Example2:** Let $X = \{x, y\}$, $\tau = \{\emptyset, X, \{x\}\}$. This space is a supra $T_0$- space but it is not a supra $T_1$-space.

**Definition5:** If $(X, \tau)$ is a supra topological space, $x, y \in X, x \neq y$, then there exist $G, H \subseteq X$ are supra open sets such that $x \in G, y \in H, G \cap H = \emptyset$, then $(X, \tau)$ is called a supra $T_2$- space.

As we proved in the supra $T_0$- space, here also the hereditary property of a supra $T_2$- space will be proved.

**Theorem8:** If $(X, \tau)$ is a supra $T_2$- space and $(Y, \tau)$ is a supra topological subspace of $(X, \tau)$ then $(Y, \tau_y)$ is a supra $T_2$- space.

Proof:
Suppose that $, y \in Y, x \neq y$, since $Y \subseteq X$ then $x, y \in X$ which means that there exist two supra open sets $G, H \subseteq X$ such that $x \in G$ and $y \in H$, $G \cap H = \emptyset$. Now $G_y = G \cap Y, H_y = H \cap Y$ are two supra open sets in Y such that $x \in G_y$ and $y \in H_y$.
Since $G \cap H = \emptyset$, then $G_y \cap H_y = \emptyset$.
So $(Y, \tau_y)$ is a supra $T_2$- space. □

**Theorem9:** If $(X, \tau)$, $(X^*, \tau^*)$ are two supra topological spaces, $(X, \tau)$ is a supra $T_2$- space and f is a supra open function and bijective then $(X^*, \tau^*)$ is a supra $T_2$- space.**:**

Proof:
Suppose $(X, \tau)$ is a supra $T_2$- space.
Now we have to prove that $(X^*, \tau^*)$ is a supra $T_2$-space.
Let $x^*, y^* \in X^*$ where $x^* \neq y^*$, since f is a bijective function then there exist $x, y \in X$ such that $x^* = f(x)$, $y^* = f(y)$ and so $x \neq y$.
Since $(X, \tau)$ is a supra $T_2$- space then there exist $G, H \subseteq X$ two supra open sets such that $x \in G$, $y \in H$ and $G \cap H = \emptyset$.
We obtain that $f(G), f(H) \subseteq X^*$ are two supra open sets in $X^*$ because f is supra open function.
So $x^* \in f(G)$ and $y^* \in f(H)$ and $f(G) \cap f(H) = \emptyset$.
Then $(X^*, \tau^*)$ is a supra $T_2$- space. □

**Remark4:** Every supra $T_2$- space is a supra $T_1$- space, but the converse is not true such as in the next example.

**Example3:** Let $X = \{x, y, z\}$ , $\tau = \{\emptyset, X, \{x,y\}, \{y,z\}, \{x,z\}\}$

This space is a supra $T_1$- space but it is not a supra $T_2$- space.

**Definition6:** If $(X,\tau)$ is a supra topological space, for all $x, y \in X, x \neq y$, and there exist a supra $\alpha -$ open set G such that $x \in G$ and $y \notin G$. Then $(X,\tau)$ is called a supra $\alpha - T_0$- space.

**Definitino7:** If $(X,\tau)$ is a supra topological space, $A \subseteq X$ , $A \neq \emptyset$ , $\tau_A$ is the class of all intersection of A with each element in $\tau$ , then $(A, \tau_A)$ is called a supra topological subspace of $(X,\tau)$.

The hereditary property of a supra $\alpha - T_0$- axiom will be proved in the following theorem.

**Theorem10:** If $(X,\tau)$ is a supra $\alpha - T_0$- space and $(Y,\tau)$ is a supra topological subspace of $(X,\tau)$ then $(Y, \tau_y)$ is a supra $\alpha - T_0$- space.

Proof:

Suppose that $x, y \in Y$, $x \neq y$, since $Y \subseteq X$ then $x, y \in X$

Since $(X,\tau)$ is a supra $\alpha - T_0$- space means that there exist a supra $\alpha -$open set $G \subseteq X$ such that $x \in G$ and $y \notin G$ .

We have that $G_y = Y \cap G$ , $G_y$ is a supra $\alpha -$open set in Yand $x \in G_y$ but $y \notin G_y$, so we found a supra $\alpha -$open set $G_y \subseteq Y$ which it contained x and not contained y.

Hence; $(Y, \tau_y)$ is a supra $\alpha - T_0$- space. □

**Theorem11:** If $(X,\tau)$ , $(X^*, \tau^*)$ are two supra topological spaces, $(X,\tau)$ is a supra $\alpha - T_0$- space and f is a supra open function and bijective then $(X^*, \tau^*)$ is a supra $\alpha - T_0$- space.

Proof:

Suppose that $(X,\tau)$ is a supra $\alpha - T_0$- space.

Now we have to prove that $(X^*, \tau^*)$ is a supra $\alpha - T_0$- space.

Let $x^*, y^* \in X^*, x^* \neq y^*$, since f is a bijective function then there exist $x, y \in X$ such that, $x^* = f(x)$ , $y^* = f(y)$ and $x \neq y$

Since $(X,\tau)$ is a supra $\alpha - T_0$- space then there exists $G \subseteq X$ is a supra $\alpha -$open set such that $x \in G$ and $y \notin G$.

We obtain that $f(G) \subseteq X^*$ is a supra $\alpha -$open sets in $X^*$ because f is a supra open function.

So $x^* \in f(G)$ and $y^* \notin f(G)$.

Then $(X^*, \tau^*)$ is a supra $\alpha - T_0$- space. □

**Definition8:** If $(X,\tau)$ is a supra topological space, $G, H \subseteq X$ are supra $\alpha -$open sets and if $x \in G$ , $x \notin H$ and $y \notin G, y \in H$ then $(X,\tau)$ is called a supra $\alpha - T_1$- space.

The next theorem show us the hereditary property of a supra $\alpha - T_1$- axiom which will be proved by using the definition of a supra $\alpha - T_1$- space with the idea of the proof of (theorem10).

**Theorem12:** If $(X,\tau)$ is a supra $\alpha - T_1$- space and $(Y,\tau)$ is a supra topological subspace of $(X,\tau)$ then $(Y, \tau_y)$ is a supra $\alpha - T_1$- space.

The next theorem will use the definition of a supra $\alpha - T_1$- space with the idea of the proof of (theorem11).

**Theoerm13:** If $(X,\tau)$ , $(X^*, \tau^*)$ are two supra topological spaces, $(X,\tau)$ is a supra $\alpha - T_1$- space and f is a supra open function and bijective then $(X^*, \tau^*)$ is a supra $\alpha - T_1$- space.

**Definition9:** If $(X,\tau)$ is a supra topological space,for all $x, y \in X, x \neq y$, then there exist $G, H \subseteq X$ are supra $\alpha -$open sets such that $x \in G$, $y \in H$ , $G \cap H = \emptyset$, then $(X,\tau)$ is called a supra $\alpha - T_2$- space.

As we proved in the supra $\alpha - T_0$- space, and supra $\alpha - T_1$- space, here also the hereditary property of a supra $\alpha - T_2$- space will be proved.

**Theorem14:** If $(X,\tau)$ is a supra $\alpha - T_2$- space and $(Y,\tau)$ is a supra topological subspace of $(X,\tau)$ then $(Y, \tau_y)$ is a supra $\alpha - T_2$- space.

Proof:

Suppose that $x, y \in Y, x \neq y$, since $Y \subseteq X$ then $x, y \in X$ which means that there exist two supra $\alpha -$open sets $G, H \subseteq X$ such that $x \in G$ and $y \in H$, $G \cap H = \emptyset$ .

Now $G_y = G \cap Y$ , $H_y = H \cap Y$ are two supra $\alpha -$open sets in Y such that $x \in G_y$ and $y \in H_y$ .

Since $G \cap H = \emptyset$ , then $G_y \cap H_y = \emptyset$.

So $(Y, \tau_y)$ is a supra $\alpha - T_2$- space. □

**Theorem15:** If $(X,\tau)$ , $(X^*, \tau^*)$ are two supra topological spaces, $(X,\tau)$ is a supra$\alpha - T_2$- space and f is a supra open function and bijective then $(X^*, \tau^*)$ is a supra $\alpha - T_2$- space.

Proof:

Suppose $(X,\tau)$ is a supra $\alpha - T_2$- space.

Now we have to prove that $(X^*, \tau^*)$ is a supra $\alpha - T_2$- space.

Let $x^*, y^* \in X^*$ where $x^* \neq y^*$ , since f is a bijective function then there exist $x, y \in X$ such that $x^* = f(x)$ , $y^* = f(y)$ and so $x \neq y$ .

Since $(X,\tau)$ is a supra $\alpha - T_2$- space then there exist $G, H \subseteq X$ two supra $\alpha -$open sets such that $x \in G$ , $y \in H$ and $G \cap H = \emptyset$ .

We obtain that $f(G), f(H) \subseteq X^*$ are two supra $\alpha -$open sets in $X^*$ because f is a supra open function.

So $x^* \in f(G)$and $y^* \in f(H)$and$f(G) \cap f(H) = \emptyset$.

Then $(X^*, \tau^*)$ is a supra $\alpha - T_2$- space. □

## 4- Some Relationships Between Two Classes (Supra Separation Axioms And Supra $\alpha -$ Separation Axioms)

The following diagram introduced the relations between the two classes.

$$\begin{array}{ccc}
\text{supra } T_2 - \text{space} & \Longrightarrow & \text{supra } \alpha - T_2 - \text{space} \\
\Downarrow & & \Downarrow \\
\text{supra } T_1 - \text{space} & \Longrightarrow & \text{supra } \alpha - T_1 - \text{space} \\
\Downarrow & & \Downarrow \\
\text{supra } T_0 - \text{space} & \Longrightarrow & \text{supra } \alpha - T_0 - \text{space}
\end{array}$$

We have to prove the above relationships and give an example to show that the converse is not always true.

**Lemma1:** Every supra open set is a supra $\alpha-$ open set.[5]

**Theorem16:** Every supra $T_2$- space is a supra $\alpha-T_2$- space.

Proof:

Let $(X,\tau)$ be a supra $T_2-$ space, and let $x,y\in X, x\neq y$ then there exist two supra open sets $G,H\subseteq X$ such that $x\in G, y\in H, G\cap H=\emptyset$.

Since every supra open set is a supra $\alpha-$ open set (by lemma).

Then $G,H\subseteq X$ are two supra $\alpha-$ open sets such that $x\in G, y\in H, G\cap H=\emptyset$.

Hence $(X,\tau)$ is a supra $\alpha-T_2$- space. □

**Theorem17:** Every supra $T_1$- space is a supra $\alpha-T_1$-

Proof:

Let $(X,\tau)$ be a supra $T_1-$ space, and let $x,y\in X, x\neq y$ then there exist two supra open sets $G,H\subseteq X$ such that $x\in G$ and $x\notin H, y\notin G, y\in H$

Since every supra open set is a supra $\alpha-$ open set (by lemma1).

Then $G,H\subseteq X$ are two supra $\alpha-$ open sets such that $x\in G$ and $x\notin H, y\notin G,, y\in H$.

Hence $(X,\tau)$ is a supra $\alpha-T_1$- space. □

**Theorem18:** Every supra $T_0$- space is a supra $\alpha-T_0$- space.

Proof:

Let $(X,\tau)$ be a supra $T_0-$ space, and let $x,y\in X, x\neq y$ then there exist a supra open set $G\subseteq X$ such that $x\in G$ and $y\notin G$.

Since every supra open set is a supra $\alpha-$ open set (by lemma1).

Then $G\subseteq X$ is a supra $\alpha-$ open set such that $x\in G$ and $y\notin G$

Hence $(X,\tau)$ is a supra $\alpha-T_0$- space. □

**Theorem19:** Every supra $\alpha-T_2$- space is a supra $\alpha-T_1$- space.

Proof:

Let $(X,\tau)$ be a supra $\alpha-T_2-$ space, and let $x,y\in X, x\neq y$ then there exist two supra $\alpha-$open sets $G,H\subseteq X$ such that $x\in G, y\in H, G\cap H=\emptyset$.

Since $G\cap H=\emptyset$, that is mean $x\in G$ and $x\notin H, y\notin G, y\in H$.

Hence $(X,\tau)$ is a supra $\alpha-T_1$- space. □

**Theorem20:** Every supra $\alpha-T_1$- space is a supra $\alpha-T_0$- space.

Proof:

Let $(X,\tau)$ be a supra $\alpha-T_1-$ space, and let $x,y\in X, x\neq y$ then there exist two supra $\alpha-$open sets $G,H\subseteq X$ such that $x\in G$ and $x\notin H, y\notin G, y\in H$.

That is mean there exists a supra $\alpha-$open set $G\subseteq X$ such that $x\in G$ and $y\notin G$.

Hence $(X,\tau)$ is a supra $\alpha-T_0$- space. □

**Theorem21:** Every supra $\alpha-T_2$- space is a supra $\alpha-T_0$- space.

Proof:

Let $(X,\tau)$ be a supra $\alpha-T_2-$ space, and let $x,y\in X, x\neq y$ then there exist two supra $\alpha-$open sets $G,H\subseteq X$ such that $x\in G, y\in H, G\cap H=\emptyset$.

Since $G\cap H=\emptyset$, that is mean $x\in G$ and $x\notin H, y\notin G, y\in H$.

Then there exists a supra $\alpha-$open set $G\subseteq X$ such that $x\in G$ and $y\notin G$.

Hence $(X,\tau)$ is a supra $\alpha-T_0$- space. □

In section 3 we introduced some examples such as 2, 3 that is enough to show the converse of the above relationships is not true.

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# حول بديهيات الفصل الفوقية للفضاءات التبولوجية الفوقية

بلقيس خليل محمود الأمين

*مديرية تربية صلاح الدين ، تكريت ، العراق*

**الملخص**

من خلال مفاهيم المجموعات المفتوحة الفوقية والمجموعات المفتوحة الفوقية من النمط α ، قدمنا صف جديد من بديهيات الفصل وقمنا بدراسة بعض من خصائصها، والمقارنة بينها. واخيرا" تحرينا خصائصها الوراثية بالاضافة الى خصائص اخرى.