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STUDY ABOUT CYCLIC MAP ON PARTIAL b – METRIC SPACES AND FIXED-POINT THEOREM

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Abstract

In this paper, we study the existence and uniqueness of fixed point of cyclic map α-admissible on partial b – metric space

1. Introduction and premilinaries

The fixed point theory has been rapidly field since the pioneering work of Banach, became one of the most interesting area of research in the last sixty years. A great number of studies about fixed point theorem of contractions on several spaces has been reported. It is one of very useful instrument in non – linear analysis. A lot of popularizations for metric fixed point theorem ordinarily begin from banach contractive criterion. In the development of nonlinear analysis. Fixed point theorem takes a very useful role also it has been widely used in various branches of engineering and sciences, and solving existence problems in a lot of branches of real-analysis, it is no surprise that there is a great number of popularizations of this standard theorem. the idea of b-metric space and partial metric space were introduced by [1-4] respectively.[5] introduced another generalization that is called a partial b-metric space. The purpose of this study is to investigate existence and uniqueness of fixed point for cyclic map (α – admissible) on partial b-metric space.

Definition (1-1) [6] Let D be a set and let h ≥ 1 be a real number and a mapping $d: D \times D \to [0,\infty)$, then (D,d) is called a b-metric space (b-M.SP) and h ≥ 1 is called the factor of (D,d) if $\forall a,b,c \in D$ then following conditions satisfied:

i) $d(a,b) = o \; iff \; a = b$;

ii) $d(a,b) = d(b,a)$;

iii) $d(a,b) \le h[d(a,c) + d(c,b)]$.

Definition (1-2) [4] Let D be a non-empty set, and a mapping $p: D \times D \to [0,\infty)$ then (D,p) is called partial metric space (P.M.SP) if $\forall a,b,c \in D$ then the following is satisfied:

i) $a = b \; iff \; p(a,a) = p(a,b) = p(b,b)$;

ii) $p(a,a) \le p(a,b)$;

iii) $p(a,b) = p(b,a)$;

iv) $p(a,b) \le p(a,c) + p(c,b) - p(c,c)$.

Remark (1-3) Clearly the (P.M.SP) may not to be a metric space (M.SP), because in a (b-M.SP) if $a = b$, then $d(a,a) = d(a,b) = d(b,b) = 0$. but in a (P.M.SP) if $a = b$ then $p(a,a) = p(a,b) = p(b,b)$ need not to equal zero. therefore (P.M.SP) does not need to be a (b-M.SP).

On the other hand, Shukla [5] admit the notion of a (P.b-M.SP) as follows:

Definition (1-4) [5] Let $D \ne \emptyset$ and h ≥ 1 be a real number, and $P_b: D \times D \to [0,\infty)$ be a mapping then (D,P_b) is called a partial b-metric space (P.b-M.SP) and h ≥ 1 is called The Factor of (D,P_b) if $\forall a,b,c \in D$ then the following satisfied

i) $a = b \; iff \; P_b(a,a) = P_b(a,b) = P_b(b,b)$

ii) $P_b(a,a) \le P_b(a,b)$;

iii) $P_b(a,b) = P_b(b,a)$;

iv) $P_b(a,b) \le h[P_b(a,c) + P_b(c,b) - P_b(c,c)]$

Remark (1-5) The class of (P.b-M.SP) (D,P_b) is surely more than the class of (P.M.SP) , Since a (P.M.SP) is a particular type of a (P.b-M.SP) (D,P_b) when $h = 1$. Also, the class of (P.b-M.SP) (D,P_b) is more than the class of (b-M.SP), Because a (b-M.SP) is a particular type of a (P.b-M.SP) (D,P_b) while the self-distance $p(a,a) = 0$.The next examples explain that a (P.b-M.SP) on D need not be a (P.M.SP), nor a (b-M.SP) on D.

Example (1-6) [5] Let $D = [0,1)$ and let $P_b: D \times D \to [0,\infty)$ be a function such that $P_b(a,b) = [max\{a,b\}]^2 + |a-b|^2, \forall a,b \in D$.then (D,P_b) is a (P.b-M.SP) on D and the factor h = 2 > 1. But P_b is not a (b-M.SP) nor a (P.M.SP) on D.

Definition (1-7) [7] Every partial b-metric P_b defines a b-metric d_{P_b} , where

$d_{P_b}(a,b) = 2P_b(a,b) - P_b(a,a) - P_b(b,b), \forall a,b \in D$.

Definition (1-8) [7] A sequence $\{a_n\}$ in a (P.b-M.SP) (D,P_b) is called:

i) P_b -convergent to a point $a \in D$ if $\lim_{n\to\infty} P_b(a,a_n) = P_b(a,a)$

ii) A P_b-Cauchy sequence (C.Seq.) if $\lim_{n,m\to\infty} P_b(a_n,a_m)$ defined and limited;

iii) A (P.b-M.SP) (D,P_b) is called P_b-complete if any P_b - (C.Seq.) $\{a_n\}$ in D is P_b converges to a Point $a \in D$ Such that $lim_{n,m\to\infty} P_b(a_n,a_m) = lim_{n\to\infty} P_b(a_n,a) = P_b(a,a)$

Lemma (1-9) [7] A sequence $\{a_n\}$ is a P_b - (C. Seq.) in a (P.b-M.SP) (D, P_b) if and only if b-(C. Seq.) in the (b-M.SP) (D, d_{P_b}).

Lemma (1-10) [7] A (P.b-M.SP) (D, P_b) is P_b - Complete if and only if the (b - M.SP) (D ,d_{P_b}) is b-complete. Moreover, $\lim_{n,m\to\infty} d_{P_b}(a_n,a_m) = 0$ if and only if $lim_{m\to\infty} P_b(a_m,a) = lim_{n\to\infty} P_b(a_n,a) = P_b(a,a)$

Definition (1-11) [8] The pair of the self-mapping A and S of a (M.SP.) (D, d) are said to be weakened compatible if they subrogate at fortuity points that is if $Aa = Sa \Longrightarrow ASa = SAa$ for a in D.

Definition (1-12) [9] suppose that (D,P_b) be a (P.b-M.SP) and $M: D \to D$ be a given mapping. We say that M is $\alpha-$admissible if $a,b \in D, \alpha(a,b) \ge 1$ implied $\alpha(Ta,Tb) \ge 1$. In addition, we called M is L_α-admissible (R_α-admissible) if $a,b \in D, \alpha(a,b) \ge 1$ and hence $\alpha(Ma,b) \ge 1$ or $\alpha(a,Mb) \ge 1$.

Example (1-13) [9] Let $D = [0,\infty)$,define $M: D \to D$ and $\alpha: DaD \to [0,\infty)$ by $M_a = \ln a$ for all $a \in D$ and $\alpha(a,b) = \begin{cases} 2, if\ a \ge b \\ 0, if\ a < b \end{cases}$ Then M is α _admissible.

Example (1-14) [9] Let $D = [0,\infty)$, define $M: D \to D$ and $\alpha: DaD \to [0,\infty)$ by $M_a = \sqrt{a}$ for all $a \in D$ and $\alpha(a,b) = \begin{cases} e^{a-b}, if\ a \ge b \\ 0, if\ a < b \end{cases}$ Then M is α _admissible.

Definition (1-15) Let we have non-empty subsets A and B of (P.b-M.SP)(D,P_b). a cyclic contraction $M: AUB \to AUB$ is called P_b-cyclic – Banach Contraction Mapping (P_b-c-BCM) if $\exists\, \lambda \in [0,1)$ provided that $h \ge 1, h.\lambda < 1$, then $P_b(Ma,Mb) \le \lambda P_b(a,b)$ Holds both for $a \in A, b \in B$ and for $a \in B, b \in A$

2. Main Result

Theorem (2-1) Let (D,P_b) be a P_b – complete (P.b-M.SP) with a coefficient $h \ge 1$. moreover, suppose two subsets of (D,P_b), A and B and let $M: A \cup B \to A \cup B$ (P_b-c-BCM). If M is L_α-admissible map then $A \cap B$ is not equal $\emptyset$ and M has a fixed point in $A \cap B$.

Proof: Let $a_1 \in M$ provided $\alpha(a,Ma_1) \ge 1$. define sequence $\{a_n\}$in D by $a_{n+1} = Ma_n = M^n a$ for all $n \ge 1$.

If $a_n = a_{n+1}$ for some $n \in N$,Then $a = a_n$ is a fixed point of M and the proof is completed.

Hence, we may suppose that $a_n \ne a_{n+1}$ for all n

Since M is α -admissible, we see that

$$\alpha(a_0, Ta_0) = \alpha(a_0,a_1) \ge 1 \Rightarrow \alpha(Ta_0, Ta_1) = \alpha(a_1,a_2) \ge 1$$

by induction on n we have $\alpha(a_n,a_{n+1}) \ge 1 \forall n \in N$ ....(2.1)

now, by applying α -admissible P_b – cyclic contraction and using (2.1) we see that

$$\begin{aligned} P_b(M^n a, M^{n+1}a) &\le \alpha(a_n,a_{n+1}).h.P_b(M^n a, M^{n+1}a) \\ &= \alpha(a_n,a_{n+1}).h.P_b(Ma_n, Ma_{n+1}) \\ &\le \lambda P_b(a_n,a_{n+1}) \end{aligned}$$

As the same way we can conclude that

$$\begin{aligned} P_b(M^{n+1}a, M^{n+2}a) &\le \lambda P_b(a_{n+1},a_{n+2}) \\ &\le \lambda^2 P_b(a_n,a_{n+1}) \end{aligned}$$

Continuing this way, we see that

$$P_b(M^n a, M^{n+1}a) \le \lambda^n P_b(a_0,a_1)$$

Now for $m > n$

$$\begin{aligned} P_b(M^n a, M^m a) &\le P_b(M^n a, M^m a) \\ &= h[P_b(M^n a, M^{n+1}a) + P_b(M^{n+1}a, M^m a)] - P_b(M^{n+1}a, M^{m+1}a) \\ &\le hP_b(M^n a, M^{n+1}a) + h^2 P_b(M^{n+1}a, M^{n+2}a) + ... + h^m P_b(M^{m-1}a, M^m a) \\ &\le h\lambda^n P_b(a_0,a_1) + h^2 \lambda^{n+1} P_b(a_0,a_1) + ... + h^m \lambda^{m-1} P_b(a_0,a_1) \\ &= \lambda^n [h + h^2\lambda + h^3\lambda^2 + ... + h^m \lambda^{m-n-1}] P_b(a_0,a_1) \\ &\to 0 \text{ as } m,n \to \infty \end{aligned}$$

$\therefore \{M^m a\}$ is a (C.Seq.) in (D,P_b)

since (D,P_b) is ($P_b -$c P.b-M.SP), we have $\{M^n a\}$ is approach to b in a.

that is meaning $P_b(b,b)=\lim_{n\to\infty} P_b(M^n a,b)=\lim_{n\to\infty} P_b(M^n a,M^m b)=0$.......(2.2)

Now observe that $\{M^{2n}a\}$ be a sequence in A and $\{M^{2n-1}a\}$ be a sequence in B and both converges to b. Also note that A and B are closed, we have $b\in A\cap B$ on the other hand since $\alpha(a_{n+1},b)\geq 1$

$$\begin{aligned} P_b(b,Mb) &\leq hP_b(b,M^{n+1}a)+hP_b(M^{n+1}a,Mb)\\ &\leq hP_b(b,M^{n+1}a)+h\alpha(a_{n+1},b)P_b(M^{n+1}a,Mb)\\ &\leq hP_b(b,M^{n+1}a)+\lambda P_b(a_n,b)\\ &\leq hP_b(b,M^{n+1}a)+\lambda P_b(M^{n-1}a,b)\\ &\to 0 \text{ as m, n} \to\infty. \end{aligned}$$

Cleary b is a fixed point of the mapping M.

Let c be another common fixed point of M such that $c\neq b$

Since $\alpha(c,b)\geq 1$,we have that

$$\begin{aligned} P_c(a,b) &\leq h^2\alpha(c,b).P_b(Ma,Mb)\\ &\leq h\lambda P_b(c,b)\\ &< P_b(c,b) \end{aligned}$$

Which is a contradiction.

Hence, b is a unique fixed point of M.

Theorem (2-2) Let (D,P_b) be a $P_b -$complete (P.b-M.SP) with a coefficient $h\geq 1$. Let A and B be two non-empty subsets of (D,P_b), such that $F = A\cup B$. Let $f: A\to B$ and $g: A\to B$ be two maps provided $f(a)=g(a)$ for all $a\in A\cap B$ and $P_b(f(a),g(a))\leq \alpha P_b(a,b), \forall a\in A, b\in B$, where $0<\alpha<1$, then there exist a unique $a_0\in A\cap B$ such that $f(a_0)=g(a_0)=a_0$.

Proof: by applying theorem (2-1) to the map $M: A\cup B\to A\cup B$ defined by the setting $M(a)=\begin{cases} f(a), & a\in A\\ g(a), & b\in B\end{cases}$. See that if we assume $f(a)=g(a)$ for all $a\in A\cap B$ which implies that M is well defined. Note that in the metric space case, the condition 4 implies that the map M is well defined.

Theorem (2-3) let $(A_i)_{i=1}^k\neq\emptyset$ and closed subsets of a $P_b -$ complete (P_b metric space) and suppose that $M:\cup_{i=1}^k A_i\to\cup_{i=1}^k A_i$ satisfying the following properties (where $A_{k+1}=A_1$)

(i) $M(A_i)\subseteq A_{i+1}$ for $1\leq i\leq$ k

(ii) There exists $\alpha\in(0,1)$ such that $P_b(M(a),M(b))\leq\alpha P_b(a,b), \forall a\in A_i, b\in A_{i+1}$, for $1\leq i\leq$ k. then M has a unique fixed point.

Proof: we need to see that given $a\in\cup_{i=1}^k A_i$, infinitely many conditions of the Cauchy sequence $\{M^n(a)\}$ belongs to each A_i. Hence $\cap_{i=1}^k A_i\neq\emptyset$, and the limitation of M to this intersection is a cyclic mapping.

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دراسة التطبيق الدوري حول الفضاءات المترية الجزئية b ومبرهنة النقطة الثابتة

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قسم الرياضيات ، كلية التربية للعلوم الصرفة ، جامعة تكريت ، تكريت ، العراق

الملخص

درسنا في هذا البحث وجود ووحدانية النقطة الثابتة للتطبيق الدوري (تطبيق α المقبول) في الفضاءات المترية الجزئية من النوع b .