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# L-Hollow modules

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## ABSTRACT

To consider R is a commutative ring with unity, M be a nonzero unitary left R-module, M is known hollow module if each proper submodule of M is small. L-hollow module is a strong form of hollow module, where an R-module M is known L-hollow module if M has a unique maximal submodule which contains each small submodules. The current study deals with this class of modules and give several fundamental properties related with this concept.

## Introduction

Throughout the following paper R represents a commutative ring with identity, and each R-module are left until. A proper submodule A of an R-module M is known a small if $A + B \neq M$ for each proper submodule B of M [1]. A non –zero module M is known hollow module if each proper submodule of M is small [2]. A proper submodule N of an R-module M is known a maximal submodule in M, if K is a submodule of M with $N < K$, so $K = M$ [3]. An R-module M is known local if M has a unique maximal submodule which contains each proper submodules of M [4]. In this paper, we give a strong form of hollow module, we call it L-hollow module which is a module has a unique maximal submodule which contains each small submodule of M. this work contains three sections. In section one, we give the definition of L-hollow modules a strong form of hollow modules we investigate the properties of this class of modules . In section two we investigate some conditions under which hollow modules and L-hollow modules are equivalent. The third section investigate the relation between the L-hollow modules and other modules such as amply supplemented, indecomposable modules and lifting modules.

**1. L-Hollow Modules:** In this part the study present the concept of L-hollow modules, and study the basic properties of this kind of modules

**Definition(1.1):** An R-module M is called L-hollow module if M has unique maximal submodules which contains each small submodules of M.

**Example:** The Z - module $Z_4$ is L-hollow module, while the Z-module $Z_6$ is not L-hollow module.

**Remarks with Examples (1.2):**

1. Each L-hollow module is hollow module.

**Proof:** Assume that M is L-hollow module, then there exists a unique maximal Submodule contains every small submodule say N in M. And since N is a submodule of M. Then each small is contains in M. By definition hollow module so, N is a small submodule of M, implies that M is hollow module.while the converse remark (1,2)(1) is not true in general, for example, $Z_p^\infty$ is hollow module, while is not L-hollow modules.

2. Each local module is L-hollow module, while the converse is not true in general. For example $Z_2 \oplus Q$ is L-hollow module. While is not local module since $\{0\} \oplus Q$ is a unique maximal submodule of $Z_2 \oplus Q$ and $\{0\} \oplus \{0\}$ is a small submodule of $Z_2 \oplus Q$ and contained in $\{0\} \oplus Q$, but $Z_2 \oplus \{0\}$ is a proper

submodule of $Z_2 \oplus Q$, but $Z_2 \oplus \{0\}$ is not contained in $\{0\} \oplus Q$.

3. Each simple module is not L-hollow module, for example the Z-module $Z_5$ is simple module, while is not L-hollow module, and each L-hollow module is not simple module, for example the Z-mod. $Z_8$ is L-hollow module, while is not simple module.

Throughout the following proposition the study present some of the basic properties of L-hollow. Modules.

**Proposition(1.3):** Epimorphic image of L-hollow module is L-hollow module.

**Proof :** Suppose that $M_1$ L-hollow module, let $f : M_1 \rightarrow M_2$ be an epimorphism with $M_2$ is R-module. Assume that N is a unique maximal submodule of $M_2$ and $N + K = M_2$ where K is a proper submodule of $M_2$. Now, $f^{-1}$ (N) is a unique maximal submodule of $M_1$ since otherwise $f^{-1}(N) = M_1$ hence $f(f^{-1}(N)) = f(M_1) = M_2$ implies that $N = M_2$ which is contradiction. With N is a unique maximal submodule of $M_2$, thus $f^{-1}(N)$ is a unique maximal submodule of $M_1$. Since $M_1$ is L-hollow module, therefore $f^{-1}$ (N ) contains each small submodule of $M_1$ hence $f(f^{-1}(N))$ is a small submodule of $f(M_1)$, that is to say that N is a small submodule of $M_2$. Therefor $M_2$ is L-hollow module.

**Proposition(1.4):** To consider K small submodule of module M, if $M/K$ is L-hollow module, then M is L-hollow module.

**Proof :** Assume that $M/K$ is L-hollow module, with K is a small submodule of M then there exists a unique maximal submodule $N/K$ of $M/K$ with $A + L = M$ where L is a submodule of M and A is a proper submodule of M then $(A + L)/K = M/K$, implies that $((A + K)/K) + ((L + K)/K) = M/K$ since $(A + K)/K$ is proper submodule of $N/K$ and $M/K$ is L-hollow module, then $(A + K)/K$ is small submodule of $M/K$. Thus $(L + K)/K = M/K$, so $L + K = M$, since K is a small submodule of M, then $L = M$. Therefore M is L-hollow module.

**Corollary (1.5):** To consider M an R-module, if M is L-hollow module, then $M/N$ is L-hollow module for each proper submodule N of M.

**Proof:** clear by(prop. 1.3).

**Definition(1.6): [3]** A pair (P, f) is a projective cover of the module M in case P is a projective module and $f : P \rightarrow M$ where f is an epimorphism and kerf is a small submodule of P (we call P it self a projective cover of M).

**Proposition(1.7):** Let $f: M_1 \rightarrow M_2$ is projective cover of $M_2$, if $M_2$ is L-hollow module, then $M_1$ is L-hollow module.

**Proof** : Suppose $M_2$ L-hollow module. Since $f: M_1 \rightarrow M_2$ is an epimorphism therefore $M_1/\ker f$ is isomorphism to $M_2$, hence it is L-hollow module and kerf is a small submodule of $M_1$. Thus by (prop. 1.4) we get $M_1$ is L-hollow module.

**Proposition(1.8):** Let M R-module, so M is L-hollow module, and finitely generated module if and only if M is a cyclic module, and has a unique maximal submodule.

**Proof** : To consider M finitely generated L-hollow module therefore $M = R_{x_1} + R_{x_2} + \cdots + R_{x_n}$. If $M \neq R_{x_1}$ then $R_{x_1}$ is proper submodule of M. Implies that $R_{x_1}$ is small submodule of M. Hence $M = R_{x_2} + R_{x_3} + \cdots + R_{x_n}$. Therefore we cancel the summand one by one until we have $M = R_{x_i}$ for some i. Thus M is a cyclic module and since M is L-hollow module. So, M has a unique maximal submodule by (def., 1.1).

Conversely, to consider M is a cyclic module having unique maximal submodule say N, so M finitely generated. To consider L is proper submodule of M with $L + K = M$ where K is a submodule of M. Now, when L is not small submodule of M implies that $K \neq M$. So K is a proper submodule of M , K is submodule of N and since M is finitely generated, then K is contained in a maximal submodule. But by assumption M has a unique maximal submodule N. Thus L is submodule of N (L is contained in N). Therefore $L + N = N = M$ which is a contradiction. Hence $K = M$, L is submodule of N and L is a small submodule of M. So M is L-hollow module.

**Proposition(1.9):** Let N maximal submodule of a module M. when M is L-hollow module and $M/N$ is finitely generated then M is finitely generated.

**Proof:** To consider N maximal submodule of L-hollow module M with $M/N$ is finitely generated. Then $M/N = R(x_1 + N) + R(x_2 + N) + \cdots + R(x_n + N)$ where $x_i \in M$ for all $i = 1,2,\cdots,n$ we claim that $M = Rx_1 + Rx_2 + \cdots + Rx_n$. Let $m \in M$, so $m + N \in M/N$, implies that, $m + N = r_1(x_1 + N) + r_2(x_2 + N) + \cdots + r_n(x_n + N) = r_1x_1 + r_2x_2 + \cdots + r_nx_n + N$. This implies that $m = r_1x_1 + r_2x_2 + \cdots + r_nx_n + n$ for some $n \in N$. Thus $M = r_1x_1 + r_2x_2 + \cdots + r_nx_n + N$ and since M L-hollow module, so N is a small submodule of M which implies that $M = r_1x_1 + r_2x_2 + \cdots + r_nx_n$. Thus M is finitely generated.

## 2. L-hollow modules and hollow modules

The first section suggests that each L-hollow module is hollow module, and we give an example shows that the converse is not true. In this section we investigate conditions under which hollow modules can be L-hollow modules.

**Proposition(2.1):** Let M be an R-module, M is a L-hollow module if and only if M is a hollow and cyclic module.

**Proof** : Assume that M L-hollow module, so it has a unique maximal submodule N such that N contains each small submodule of M. To consider $x \in M$ with $x \notin N$ so $R_x$ is a submodule of M. We claim that $R_x = M$. If $R_x \neq M$ then $R_x$ is a proper small submodule of M , hence $R_x$ is a submodule of N which implies that $x \in N$ which is a contradiction. Thus $R_x = M$, so M is a cyclic module . Now, since M is L-hollow module Therefore M is hollow module by (Remark. 1,2) (1).

Conversely, Assume that M is hollow module and cyclic module, so it is a finitely generated module and hence M has a maximal submodule contains each proper small submodule say N. Let L be a proper small submodule of M. If L is not contained in N then $L + N = M$, while M is L-hollow module, so $N = M$ which is a contradiction. This implies that every proper small submodule of M is contained in N, thus M is a L-hollow module.

**Proposition(2.2):** Let M be an R-module, M is L-hollow module if and only if M is a hollow module and has a unique maximal submodule.

**Proof:** Assume that M is L-hollow module, so M is a hollow module, by (Remark. 1,2) (1). And by (definition. 1,1), so M has a unique maximal submodule.

Conversely, to consider M is hollow module. Such that has a unique maximal submodule, say N, we only have to show that M is a cyclic module . To consider $x \in M$ and $x \notin N$, so $R_x + N = M$ and since M is a hollow module then N is a small submodule of M and so, $M = R_x$ Therefore M is a cyclic module, and by (Proposition. 2.1). Then M is L-hollow module.

**Proposition(2.3):** To consider M be an R-module. M is L-hollow module if and only if it is a cyclic module and every non-zero factor module of M is indecomposable.

**Proof** ; Suppose that M is L-hollow module, so by (Proposition. 2.1). M is a hollow and cyclic module and by [4,Proposition. (41.4)]. Then every non-zero factor module of M is indecomposable.

Conversely, let M be cyclic module and every non-zero factor module of M is indecomposable, then by [4,Proposition.(41.4)]. M is a hollow module and by (proposition. 2.1). Thus M is L-hollow module.

**Proposition(2.4):** Let M be a module, M is L-hollow module if and only if M is a hollow module and $RadM \neq M$.

**Proof:** Assume that M L-hollow module, then M is hollow and cyclic module by (prop. 2,1). And since M is cyclic module, so M is finitely generated , hence $RadM \neq M$.

Conversely, let M is a hollow module and $RadM \neq M$, then RadM is a small submodule of M. Also by [3,Proposition.(1.3.13),P.36]. RadM is the a unique maximal submodule of M and thus $M/ \text{Rad}\, M$ simple module and hence cyclic. Implies that $M/\text{Rad}\, M = < m + \text{Rad} >$ for some $m \in M$. We prove that $M = Rm$. To consider $w \in M$ so, $w + \text{Rad}\, M \in M/\text{Rad}\, M$, and therefore there is, $r \in R$ such as $w + \text{Rad}\, M = r(m + \text{Rad}\, M) = rm + \text{Rad}\, M$. Implies that $w - rm \in \text{Rad}\, M$ which implies that $w - rm = y$ for same $y \in \text{Rad}\, M$. So $w = rm + y \in Rm + RadM$, hence $M = Rm + RadM$. But RadM is small submodule of M implies $M = Rm$. Thus M is a cyclic module and by (proposition. 2.1). We get M is L-hollow module.

**Proposition(2.5):** Let M L-hollow module if and only if Rad M is a small and maximal in M.

**Proof:** Suppose that Rad M is a small and maximal submodule. To prove that M is L-hollow module, first we want to show that RadM is a unique maximal submodule in M. Suppose that L is another maximal submodule in M, then $M = L + \text{Rad}\, M$, while Rad M is a small submodule which implies that $L = M$, which is a contradiction. Thus Rad M is aunique maximal submodule in M. We claim every small submodule of M is contained in Rad M. Let N be a small submodule of M, if N is not contained in RadM, then $N + RadM = M$. while RadM is a small submodule of M which implies that $N = M$ so, have a contradiction. Therefore M is L-hollow module.

Conversely, suppose that M is L-hollow module so, by (Remark 1.2) (1), therefore M is hollow module and by ([3],Lemma 1.3.13,P.36). Then Rad M is a maximal submodule. Since M is L-hollow module. Thus RadM is a unique maximal submodule of M, hence $RadM + N = M$ for seme proper submodule N of M. If RadM is not small submodule of M then N is a small submodule of M. thus $RadM = M$ which is contradiction by [4,Prop. (41.4)]. Hence RadM is small submodule of M.

## 3. L-hollow modules and some other modules

This section tackles the relation between L-hollow module and other modules such that amply supplemented , indecomposable and lifting modules.

**Definition(3.1):[4]** A module M is called amply supplemented, if for every two submodules U,V of M such that $M = U + V$, there exists a supplement $V_1$of U in M, such that $V_1 \leq V$ ".

**Example:** The Z-module $Z_4$ is amply supplemented. while the Z-module $Z_{12}$ is not amply supplemented.

**Proposition(3.2):** Every L-hollow module is amply supplemented.

**Proof:** Let M L-hollow module and to consider U is a unique maximal submodule of M. Since M is L-hollow module, so we have $U + M = M$ and $U \cap M = U$ is a small submodule of M. Therefore M is amply supplemented.

**Remark(3.3):** The converse of (Prop. 3,2) is not true in general, as given in this example, the Z-module $Z_6$ is amply supplemented, while not L-hollow module.

**Definition(3.4):[1]** An R-module M is indecomposable if $M \neq 0$ and the only a direct summands of M are $<\bar{0}>$ and M. Implies that M has no a direct sum of two non-zero submodule.

**Example:** The simple module is indecomposable, while the Z-module $Z_6$ is not indecomposable.

**Proposition (3.5):** Every L-hollow module is indecomposable.

**Proof** ; Let M L-hollow module then there exists a unique maximal submodule N such as contains each small submodule of M, suppose that M is decomposable, so there are a proper submodules K and L such that K,L are submodule of N and $M = K \oplus L$. But M is L-hollow module then either L is a small submodule of M with L is submodule of N implies that $K = M$ or $K$ is small submodule of $M$ with $K$ is submodule of $N$ implies that $L = M$ which is a contradiction. Then $M$ is indecomposable.

**Proposition(3.6):** Let $M$ a cyclic module, $M$ is L-hollow module if and only if every non-zero factor module of $M$ is indecomposable.

**Proof:** Suppose $M/A$ is non-zero factor mod. of $M$. Since $M$ is L-hollow module therefore $M/A$ is L-hollow module by(corollary 1.5). And by (prop.3.5) we get $M/A$ is indecomposable.

Conversely, to consider $N$ maximal submodule of $M$ and to consider $L$ is a submodule of $N$. Suppose that $M = L + K$, where $K$ is a submodule of $M$ by[3,Lemma(1.3.10), P. 34], we get $M/(L \cap K) \cong (M/L) \oplus (M/K)$. While $M/(L \cap K)$ is indecomposable then either $M/L = 0$ or $M/K = 0$. Since $L$ is a submodule of $N$, and $N$ is a submodule of $M$. Hence $L$ is a proper submodule of $M$. Then $M/L \neq 0$ therefore $M/K = 0$. Hence $M = K$. Therefore $L$ is small submodule of $M$. Thus $M$ is hollow module and since $M$ is acyclic module so by (prop.2.1). Thus $M$ is L-hollow module.

**Definition(3.7):** **[5]** Let $M$ be a module, $M$ is said to be lifting module (or satisfieds $D1$) if for each submodule $N$ of $M$ there are submodule $K$ and $L$ of $M$ where $M = K \oplus L$, $K$ is a submodule of $N$ and $N \cap K$ is a small submodule of $K$.

**Example:** The Z -module $Z_6$ is lifting module. While the Z-module $Z_{12}$ is not lifting module.

**Proposition (3.8):** Every L-hollow module is lifting module.

**Proof:** Let $M$ be L-hollow module, then there exists a unique maximal $N$ of $M$ contains all small submodule, then $M = M \oplus \{0\}$ where $\{0\}$ is a submodule of $N$, $N \cap M = N$ and since $M$ is L-hollow module .Therefore $N \cap M = N$ is a small submodule of $M$. Thus $M$ is lifting module.

**Remark(3.9):** The converse of proposition (3.8) is not true in general, as given in this example . The Z-module $Z_{10}$ is lifting module. While is not L-hollow module.

**Proposition(3.10):** Assume that $M$ a cyclic indecomposable module, if $M$ is lifting module, so $M$ is L-hollow module.

**Proof:** Suppose that $N$ is a proper submodule of $M$, since $M$ is lifting module so, $M = A + B$, where $A$ is a submodule and $N \cap A$ is small submodule of $A$. While $M$ is an indecomposable, thus $B = 0$ and hence $A = M$. Which $N \cap M = N$, so $N$ is a small submodule of $M$. Hence $M$ is hollow module and since $M$ is cyclic module. So $M$ is L-hollow module by (prop. 2,1).

**Definition(3.11):[7]** Let $N$ and $L$ be submodules of $M$. $N$ is said to be a supplement of $L$ in $M$ if it is minimal with respect to the property $M = N + L$".

**Proposition(3.12):** Let $K$ be a maximal submodule of mod. $M$. If $L$ is a supplement of $K$ in $M$, then $L$ is L-hollow module.

**Proof:** Suppose that $L$ a supplement of $K$ and to consider $L_1$ is proper sub module of $L$ with $L_1 + L_2 = L$ for some submodule $L_2$ of $L$. Now,$k + L = M = K + L_1 + L_2 = M$ and $L_1$is a submodule of $K$, since otherwise $K$. $L_1 = M$ and $K$ is maximal submodule of $M$ we get $L_1 = L$, which is a contradiction. Thus $K + L_2 = M$ and since $K$ is maximal submodule of $M$ we get $L_2 = L$. Implies that $L$ is a hollow module.To show that $L$ is a cyclic module, let $x \in M$ and $x \notin K$ then $R_x + K = M$ and this implies that $R_x = L$ by minimality of $L$. And by(prop. 2,1), thus $L$ is L-hollow module

**Definition(3.13):[6]** A submodule $N$ of an R-module $M$ is said to be coclosed in $M$ if $N/K$ is a small submodule of $M/K$ implies that $N = K$ for each submodule $K$ of $M$ contained in $N$.

**Example:** $<\bar{2}> = \{\bar{0}, \bar{2}, \bar{4}\}$ is coclosed submodule in the Z-module $Z_6$.

**Proposition(3.14):** If $M$ is L-hollow module then each non-zero coclosed submodule of maximal submodule of $M$ is L-hollow module.

**Proof:** Assume that $M$ L-hollow module and to consider $N$ be a unique maximal submodule of $M$. Let $A$ be a non-zero coclosed submodule of $N$, suppose that $L$ is a proper submodule of $A$. Since $M$ is L-hollow module thus $L$ is a small submodule of $M$ contained in $N$. And hence $A$ is coclosed submodule of $M$. Thus $L$ is a small submodule of $A$. Hence $A$ is L-hollow module.

**Proposition(3.15):** Suppose that $A$ a submodule of an R-module $M$. If $A$ is L-hollow module, so either $A$ is a small submodule of $M$ or coclosed submodule of $M$, while not both.

**Proof:** Assume that $A$ is not coclosed submodule of $M$. To prove that $A$ is a small submodule of $M$, then there is a proper submodule of $M/B$. While $A$ is L-hollow module so $A$ is hollow module by(Remark 1.2) (1). Then by[4,prop(19. 3)] we get $B$ is a small submodule of $A$ and hence $A$ is a small submodule of $M$ by [4,prop(19. 3)]. Now, we want to prove $A$ is not coclosed and $A$ is a small submodule of $M$ we must show that $A$ is zero submodule of $M$. Since $A$ is L-hollow module then $A$ is not zero submodule.

**Proposition(3.16):** Let $M$ be a cyclic module, and let $f{:}P \to M$ be a projective cover of $M$ and then the following statements are equivalent.

**(1)** $M$ is L-hollow module.

**(2)** $M$ is hollow module.

**(3)** $P$ is hollow module.

**(4)** $P$ is indecomposable and supplemented.

**(5)** End $(P)$ is local ring.

**Proof:** (1) $\Rightarrow$ (2) clear by(Remark 1,2)(1)

(2) $\Rightarrow$ (3) To consider $M$ hollow module and since $f{:}\, P \to M$ is an epimorphism, so $P/\ker f$ is isomorphism to $M$ and therefore a hollow module and since $\ker f$ is small submodule of $P$, so $P$ is a hollow module by[3,prop(1.3.3)P.31].

(3) $\Rightarrow$ (4) clear by [3,prop(1.3.5)P.32] and [3,prop(1.3.9)P.34].

$(4) \Rightarrow (5)$

To consider $g : P \to P$ is a homomorphism then we have two cases.

Case 1: $g$ is onto. Since $P$ is a projective module consider this diagram:

$$\begin{tikzcd} & P \arrow[dl, "\exists h"'] \arrow[d, "I"] \\ P \arrow[r, "g"'] & P \end{tikzcd}$$

Where $I : P \to P$ is the identity homomorphism and there is a homomorphism $h : P \to P$ where $f \circ h = I$, implies that $g$ has a right inverse, this implies that $P = \ker g \oplus h(P)$, but $P$ is indecomposable by (4). Then $\ker g = 0$, thus $P = h(P)$. Then $g$ is one to one. Hence $g$ is an isomorphism.

Case 2: $g$ is not onto.We know that $P = g(P) + (I - g)(P)$, $P$ is amply supplemented by[3, prop (1.2.12)P.25], then there is a supplement $K$ of $g(P)$ in $(I - g)(P)$. implies that $P = g(P) + K$ and $g(P) \cap K$ is a small submodule of $K$, and there exists a supplement $L$ of $K$ in $g(P)$. Implies that $P = L + K$ and $L \cap K$ is a small submodule of $K$. Now, $L$ and $K$ are matual supplements and hence( $L \cap K$)=0. so $P = L \oplus K$, but $P$ is indecomposable and $K \neq 0$ then $K = P$. Now, $K$ is a submodule of $(I - g)(P)$ this implies that $(I - g)(P) = P$. Implies that $(I - g)(P)$ is onto and by the previous argument $I - g$ is an isomorphism. $(5) \Rightarrow (1)$

To explain that $M$ is a hollow module we need only to show that $P$ is hollow by [3,prop(1.3.3)P.31]. Define $g : P \to P/(L \cap K)$ as follows. For $x \in P$, $x = s + t$ for some $s \in L$ and $t \in K$. Set $g(x) = s + L \cap K$, $g$ is a well defined and homomorphism and since $P$ is a projective module, there is a homomorphism $\psi: P \to P$ where this diagram is commutative:

$$\begin{tikzcd} & P \arrow[dl, "\exists\psi"'] \arrow[d, "g"] \\ P \arrow[r, "\pi"'] & P/(P \cap K) \end{tikzcd}$$

Where $\pi : P \to P/(L \cap K)$ is the natual epimorphism. To prove that $\psi(P)$ is a submodule of $L$. To see this, let $y \in \psi(P)$ then there exists $w \in P$ such that $y = \psi(w)$. Now, $(\pi \circ \psi)(w) = g(w)$ where $w = s + t$ for some $s \in L$ and $t \in K$. Implies that $\psi(w) + L \cap K = s + L \cap K$ implies that $\psi(w) - s \in L \cap K$ is a submodule of $L$. Then $\psi(w) \in L$, and hence $\psi(P)$ is a submodule of $L$. Similarly one can show that $(I - \psi)(P)$ is a submodule of $K$. Now, $\psi \in End(P)$ and by (5) $End(P)$ is a local ring then $\psi$ or $(I - \psi)$ is onto, but $\psi$ is not onto since otherwise $\psi(P)$ is a submodule of $L$ which implies that $L = P$ which is a contradiction. Therefore $(I - \psi)$ is onto. Implies that $K = P$ thus $P$ is a hollow module. Since $P$ is a hollow module implies that $M$ is a hollow module and since $M$ is cyclic module. Therefore $M$ is L-hollow Module by (prop.2,1 ).

## Conclusion

The main results are as follows.

Each L-hollow module is hollow module, while the converse is not true in general (see Remark with Example) (1.2)(1). and the converse is true under certain conditions (cyclic, unique maximal submodule, $RadM \neq M$), every local modules is L-hollow modules, but the converse is not true in general (see Remark with Example) (1.2)(2), and the converse is true under certain conditions, every L-hollow module is amply supplemented, while the converse is not true in general (see proposition 3.2), and the converse is true under certain conditions, every L-hollow module is indecomposable module, while the converse is not true in general (see proposition 3.5), and the converse is true under certain conditions (cyclic module see proposition 3.6), and we get every L-hollow module is lifting module, while the converse is not true in general (see proposition 3.8), and the converse is true under certain conditions (cyclic indecomposable see proposition 3.10)

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# المقاسات المحلية المجوفة

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**الملخص**

يقال للمقاس الغير صفري $M$ انه مجوف, اذا كان كل مقاس جزئي فعلي فيه صغيرا. في هذا البحث سنعطي اعماما لهذا النوع من المقاسات نطلق عليها اسم المقاسات المحلية المجوفة. ندرس بعض الخواص الاساسية لهذا الصنف من المقاسات مع دراسة العلاقة بينها وبين المقاسات المجوفة من جهة و علاقتها باصناف اخرى من المقاسات من جهة اخرى مثل المقاسات التكميلية الواسعة, المقاسات الغير قابلة للتحلل ومقاسات الرفع.