Approximaitly Primary Submodules
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1. Introduction
The study deals with the notion of an approximaitly primary submodules of unitary left R-module M over a commutative ring R with identity as a generalization of a primary submodules and approximaitly prime submodules, where a proper submodule N of an R-module M is called an approximaitly primary submodule of M, if whenever aM ⊆ N + soc(M) or aM + N ⊆ M, for a ∈ M, implies that either y ∈ N + soc(M) or aM ⊆ N + soc(M) for some positive integer k of Z. Several characterizations, basic properties of this concept are given. On the other hand the relationships of this concept with some classes of modules are studied. Furthermore, the behavior of approximaitly primary submodule under R-homomorphism are discussed.

ABSTRACT
The study deals with the notion of an approximaitly primary submodules of unitary left R-module M over a commutative ring R with identity as a generalization of a primary submodules and approximaitly prime submodules, where a proper submodule N of an R-module M is called an approximaitly primary submodule of M, if whenever aM ⊆ N + soc(M) or aM + N ⊆ M, for a ∈ M, implies that either y ∈ N + soc(M) or aM ⊆ N + soc(M) for some positive integer k of Z. Several characterizations, basic properties of this concept are given. On the other hand the relationships of this concept with some classes of modules are studied. Furthermore, the behavior of approximaitly primary submodule under R-homomorphism are discussed.

2. Approximaitly Primary Submodules
This section intrudes the definition of the notion of approximaitly primary submodule and discuss some of it is basic properties, and some characterizations of this are given.
Definition 2.1: A proper submodule $N$ of an $R$-module $M$ is called an approximately primary (for short app-primary) submodule of $M$, if whenever $ay \in L$, where $a \in R$, $y \in M$, implies that either $y \in N + soc(M)$ or $a^nM \subseteq N + soc(M)$ for some positive integer $n$ of $Z$. And an ideal $J$ of a ring $R$ is called an app-primary ideal of $R$ if $J$ is an app-primary submodule of an $R$-module $R$.

Remarks and Examples 2.2:

1) It is clear that every primary submodule of an $R$-module $M$ is an app-primary submodule, but not conversely. The following example explains that:

Consider the $Z$-module $Z_{12}$, the submodule $N = (\bar{6})$ is not primary submodule of $Z_{12}$. Since $2 \in Z_{12}$, $3 \in Z_{12}$, but $2 \notin N$ and $2 \notin \sqrt{([6]:Z_{12})} = \sqrt{6Z} = 6Z$. While $N = (\bar{6})$ is an app-primary submodule of $Z_{12}$ since $soc(Z_{12}) = (\bar{2})$ and for all $a \in Z$, $y \in Z_{12}$ such that $ay \in (\bar{6})$ implies that either $y \in (\bar{6}) + soc(Z_{12}) = (\bar{6})$ or $a \in \sqrt{([6]:soc(Z_{12}))Z_{12}} = \sqrt{([6]:soc(Z_{12}))Z_{12}} = \sqrt{2Z} = 2Z$. That is if $2.3 \in (\bar{6})$, implies that $2 \in \sqrt{([6]:soc(Z_{12}))Z_{12}} = 2Z$.

2) It is clear that every approximately prime submodule of an $R$-module $M$ is an app-primary submodule, but the convers is not true in general. The following example shows that:

Consider the $Z$-module $Z$, the submodule $N = (\bar{8})$ is not approximately-prime submodule of $Z$, since $2.4 \in (\bar{8})$ but $2 \notin (\bar{8}) + soc(Z) = (\bar{8})$ and $4 \notin (\bar{8}) + soc(Z) = (\bar{8})$, while $N = (\bar{8})$ is an app-primary of the $Z$-module $Z$, since for all $a \in Z$, $y \in M$ such that $ay \in N = (\bar{8})$, implies that either $y \in N + soc(Z) = (\bar{8}) + (0) = (\bar{8})$ or $a \in \sqrt{[N + soc(Z):Z]} = \sqrt{[\bar{8} + (0):Z]} = \sqrt{[\bar{8}]} = (\bar{2})$. That is if $2.4 \in (\bar{8})$, implies that $2 \in \sqrt{([8]:soc(Z):Z)} = (\bar{2})$.

3) It is clear that every approximately quasi-prime submodule of an $R$-module $M$ is an app-primary submodule of $M$, but the convers is not true in general for the convers consider the following example.

Let $M = Z$, $R = Z$, the submodule $N = (\bar{4})$ is not approximately quasi-prime submodule of $M$, since $2.2.1 = 4 \in (\bar{4})$, but $2.1 \notin (\bar{4}) + soc(Z) = (\bar{4})$. But $N = (\bar{4})$ is an app-primary submodule of $M$, since for all $a \in Z$, and $y \in Z$ such that $ay \in (\bar{4})$, implies that either $y \in (\bar{4}) + soc(Z) = (\bar{4})$ or $a \in \sqrt{([4]:soc(Z):Z]} = \sqrt{([4]:Z]} = \sqrt{[\bar{4}]} = (\bar{2})$. That is if $2.2 \in (\bar{4})$, then $2 \in \sqrt{([4]:soc(Z):Z]} = (\bar{2})$.

4) It is clear that every prime submodule of an $R$-module $M$ is an app-primary submodule, but not conversely. Consider the following example for the converse:

Let $M = Z_{4}$, $R = Z$, the submodule $N = (\bar{0})$ is not prime submodule of $Z_{4}$, since $2.\bar{2} = \bar{0} \in N$, for $2 \in Z_{2}, \bar{2} \in Z_{4}$, but $\bar{2} \notin (\bar{0})$ and $2 \notin ([0]:Z_{4}) = (\bar{4})$.

But $N = (\bar{0})$ is an app-primary submodule of $Z_{4}$, since $soc(Z_{4}) = (\bar{2})$ and for all $a \in Z$, $y \in Z_{4}$ such that $ay \in (\bar{0})$, implies that either $y \in (\bar{0}) + soc(Z_{4}) = (\bar{2})$ or $a \in \sqrt{([0]:soc(Z_{4}):Z_{4}) = \sqrt{[\bar{2}]:Z_{4}] = \sqrt{[\bar{0}]} = (\bar{2})$. That is if $2.2 \in (\bar{0}) + soc(Z_{4}) = (\bar{2})$ or $2 \in \sqrt{([0]:soc(Z_{4}):Z_{4}) = (\bar{2})}$. The following results are characterizations of app-primary submodules.

Proposition 2.3: Let $K$ be a proper submodule of an $R$-module $M$. Then $K$ is an app-primary submodule of $M$ if and only if whenever $JL \subseteq K$, for $L$ is a submodule of $M$, $J$ is an ideal of $R$, implies that either $L \subseteq K + soc(M)$ or $J \subseteq \sqrt{[K + soc(M):R]}$.

Proof:

($\Rightarrow$) Assume that $K$ is an app-primary submodule of an $R$-module $M$ and $JL \subseteq K$, where $J$ is an ideal of $R$, is a submodule of $M$, with $L \not\subseteq K + soc(M)$, then there exists $l \in L$ such that $l \not\subseteq K + soc(M)$. Now we have $JL \subseteq K$, then for any $b \in JL \subseteq K$. But $K$ is an app-primary submodule of $M$, and $l \not\subseteq K + soc(M)$, it follows that $b^n \in [K + soc(M):R]$ for some $n \in Z^+$, that is $J^n \subseteq [K + soc(M):R]$ for some $n \in Z^+$. Hence $J \subseteq \sqrt{[K + soc(M):R]}$.

($\Leftarrow$) Assume that $ay \in K$, for $a \in R$, $y \in M$, then $ay = (a)(y)$, that is $JL \subseteq K$ where $J = (a)$, $L = (y)$, then by hypothesis either $L \subseteq K + soc(M)$ or $J \subseteq \sqrt{[K + soc(M):R]}$. That is either $a \in \sqrt{[K + soc(M):R]}$ and $y \in K + soc(M)$, then $K$ is an app-primary submodule of an $R$-module $M$.

As a direct consequence of proposition (2.3) we get the following corollary.

Corollary 2.4: Let $K$ be a proper submodule of an $R$-module $M$. Then $K$ is an app-primary submodule of $M$ if and only if whenever $al \subseteq K$, for $a \in R$, $L$ is a submodule of $M$, implies that either $L \subseteq K + soc(M)$ or $a^n \in [K + soc(M):R]$.

Proposition 2.5: A zero submodule of a non-zero $R$-module $M$ is an app-primary submodule of $M$ if and only if $ann_R(L) \subseteq \sqrt{[soc(M):R]}$ for all non-zero submodule $L$ of $M$, with $L \not\subseteq soc(M)$.

Proof:

($\Rightarrow$) Let $L$ be a non-zero submodule of $M$, such that $L \not\subseteq soc(M)$, and let $a \in ann_R(L)$, that is $aL = (0)$ but $(0)$ is an app-primary submodule of $M$ and $L \not\subseteq soc(M) = (0) + soc(M)$, it follows by corollary (2.4) that $a \in \sqrt{((0) + soc(M):R]} = \sqrt{[soc(M):R]}$. That is $ann_R(L) \subseteq [soc(M):R]$. ($\Leftarrow$) Suppose that $al \subseteq (0)$, for $a \in R$ and $L$ is a non-zero submodule of $M$, with $L \not\subseteq soc(M)$. It follows that $a \in ann_R(L)$, by hypothesis $a \in \sqrt{[soc(M):R]}$, that is $a \in \sqrt{([0]:soc(M):R]}$. Hence a zero submodule of an $R$-module $M$ is an app-primary submodule of $M$. 

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**Proposition 2.6** : Let $K$ be a proper submodule of an $R$-module $M$. Then $K$ is an app-primary submodule of $M$ if and only if for every $y \in M$ \( [K:y] \subseteq \sqrt{[K + soc(M); R]} \) with $y \notin K + soc(M)$.

**Proof**: 
\((\Rightarrow)\) Suppose that $K$ is an app-primary submodule of $M$, and $a \in [K:y]$, implies that $ay \in K$. Since $K$ is an app-primary submodule of $M$, and $y \notin K + soc(M)$, then $a \in \sqrt{[K + soc(M); R]}$. Thus $[K:y] \subseteq \sqrt{[K + soc(M); R]}$.

\((\Leftarrow)\) Let $ay \in K$, for all $a \in R$, $y \in M$, and suppose that $y \notin K + soc(M)$. It follows that $a \in [K:y]$ by hypothesis $a \in \sqrt{[K + soc(M); R]}$. Thus $K$ is an app-primary submodule of $M$.

**Proposition 2.7** : Let $K$ be a proper submodule of an $R$-module $M$ with $soc(M) \subseteq K$. Then $K$ is an app-primary submodule of $M$ if and only if $[K:r]I$ is an app-primary submodule of $M$ for each ideal $I$ of $R$.

**Proof**: 
\((\Rightarrow)\) Suppose that $K$ is an app-primary submodule of $M$, and $al \in [K:r]I$, for $a \in R$, $I$ is a submodule of $M$, it follows that $al \subseteq K + soc(M)$, but $K$ is an app-primary submodule of $M$, then by corollary (2.4) either $IL \subseteq K + soc(M)$ or $a \in \sqrt{[K + soc(M); R]}$. Since $soc(M) \subseteq K$, then $K + soc(M) = K$, it follows that $IL \subseteq K$ or $a \in \sqrt{[K;r]I}$, hence $L \subseteq [K;r]I$ or $a^nM \subseteq K$ for some $n \in Z^+$. Thus either $L \subseteq [K;r]I + soc(M)$ or $a^nM \subseteq K \subseteq [K;r]I + soc(M)$ for some $n \in Z^+$. Hence either $L \subseteq [K;r]I + soc(M)$ and $a \in \sqrt{[K;r]I + soc(M)]}$.

\((\Leftarrow)\) Follows by taking $I = R$, and using the fact $[K;r]R = K$.

**Proposition 2.8** : Let $K$ be a proper submodule of an $R$-module $M$. Then $K$ is an app-primary submodule of $M$ if and only if $[K:r]a \subseteq [K + soc(M); r]a^2$ for $a \in R$, $n \in Z^+$.

**Proof**: 
\((\Rightarrow)\) Suppose that $K$ is an app-primary submodule of $M$, and let $y \in [K:r]a$, such that $y \notin K + soc(M)$. Since $y \in [K:r]a$, it follows that $ay \in K$. But $K$ is an app-primary submodule of $M$, and $y \notin K + soc(M)$, then $a^n \in [K + soc(M); r]a^n$ for some $n \in Z^+$. That is $a^nM \subseteq K + soc(M)$, hence $a^n \notin K + soc(M)$ for all $y \in M$, it follows that $y \in [K + soc(M); r]a^n$. Thus $[K;r]a \subseteq [K + soc(M); r]a^n$.

\((\Leftarrow)\) Let $ay \in K$, for $a \in R$, $y \in M$, and suppose that $y \notin K + soc(M)$. Since $ay \in K$ it follows that $y \in [K;r]a \subseteq [K + soc(M); r]a^n$, implies that $y \in [K + soc(M); r]a^n$. That is $a^n \in K + soc(M)$ for all $y \in M$, hence $a^nM \subseteq K + soc(M)$. That is $a^n \in [K + soc(M); r]a^n$. Therefor $K$ is an app-primary submodule of $M$.

**Remark 2.9** : If $K$ is an app-primary submodule of an $R$-module $M$, then $[K:r]M$ need not to be an app-primary ideal of $R$. The following example explain that:

Let $M = Z_{12}$, $R = Z$, the submodule $K = (0)$ is an app-primary submodule of $Z_{12}$, since $soc(Z_{12}) = (2)$, hence for all $a \in Z$ and $y \in Z_{12}$ such that $ay \in K = (0)$, implies that either $y \in K + soc(M) = (0) + (2) = (2)$ or $a \in \sqrt{[K + soc(M); r]M} = \sqrt{(0) + (2)}Z_{12}$.

Thus that is if $2, 6 \in (0)$, for $2 \in Z$, $6 \in Z_{12}$, implies that either $6 \in (0) + soc(Z_{12}) = (2)$ or $2 \in \sqrt{(0) + soc(Z_{12})}Z_{12}$.

Then $[0]:Z_{12} = 12Z$ is not app-primary ideal of $Z$ because $4.3 \subseteq 12Z$, for $4, 3 \in Z$, but $3 \notin 12Z + soc(Z) = 12Z + (0) = 12Z$ and

and $4 \notin \sqrt{[12Z + soc(Z); z]Z} = \sqrt{12Z} = 6Z$.

The following proposition shows that under certain condition the residuel of an app-primary submodule is an app-primary ideal.

**Proposition 2.10** : Let $K$ be an app-primary submodule of an $R$-module $M$ with $soc(M) \subseteq K$. Then $[K:r]M$ is an app-primary ideal of $R$.

**Proof**: Let $al \in [K:r]M$, for $a \in R$, $I$ is an ideal of $R$, implies that $alM \subseteq K$, but $K$ is an app-primary submodule of $M$, then by corollary (2.4) either $IM \subseteq K + soc(M)$ or $a^n \in [K + soc(M); r]M$ for some $n \in Z^+$, that is $a^nM \subseteq K + soc(M)$. But $soc(M) \subseteq K$, then $K + soc(M) = K$, it follows that either $IM \subseteq K$ or $a^nM \subseteq K$, so either $I \subseteq [K:r]M$ or $a^n \in [K:r]M \subseteq [K:r]M + soc(R)$ or $a^n \in [K:r]M \subseteq [K:r]M + soc(R) = ([K:r]M + soc(R) : R)$. Thus $[K:r]M$ is an app-primary ideal of $R$.

**Remark 2.11** : Let $K$ be a proper submodule of an $R$-module $M$. If $[K:r]M$ is an app-primary ideal of $R$, then $K$ need not to be an app-primary submodule of $M$. The following example shows that:

Let $M = Z \oplus Z$, $R = Z$, and $K = (0) \oplus Z_2$, then $[K:r]M = (0)$ which is a prime ideal of $Z$ hence it is an app-primary submodule of $Z$ by remarks and examples (2.2)(4). But $K$ is not app-primary submodule of $M$, since $Z(0,3) = (0,6) \in K$, for $2 \in Z$, $(0,3) \in Z \oplus Z$, but $(0,3) \notin K + soc(Z \oplus Z) = (0) \oplus Z_2 + (0) = (0) \oplus Z_2$

and $2 \notin \sqrt{[(0) \oplus Z_2 + soc(Z \oplus Z); Z_2 \oplus Z]} = \sqrt{(0)} = (0)$.

**Proposition 2.12** : Let $K$ be a proper submodule of faithful multiplication $R$-module $M$. Then $K$ is an app-primary submodule of $M$ if and only if $[K:r]M$ is an app-primary ideal of $R$.

**Proof**: 
\((\Rightarrow)\) Suppose that $K$ is an app-primary submodule of $M$, and let $al \in [K:r]M$, for $a \in R$, $I$ is an ideal of $R$, it follows that $alM \subseteq K$. But $K$ is an app-primary submodule of $M$, then by corollary (2.4) we have either $IM \subseteq K + soc(M)$ or $a^nM \subseteq K + soc(M)$ for some $n \in Z^+$. Since $M$ is a faithful
multiplication, then \(soc(M) = soc(R)M\) [14, Cor. 2.14]. Hence either \(IM \subseteq [Kg_M]M + soc(R)M\) or \(a^nM \subseteq [Kg_M]M + soc(R)M\); it follows that either \(I \subseteq [Kg_M] + soc(R)\) or \(a^n \in [Kg_M] + soc(R) = \langle [Kg_M] + soc(R);g\rangle\). Hence \([Kg_M]\) is an app-primary ideal of \(R\).

\(\implies\) Suppose that \([Kg_M]\) is an app-primary ideal of \(R\), and let \(aL \subseteq K\), for \(a \in R\), \(L\) is a submodule of \(M\). Since \(M\) is a multiplication then \(L = JM\) for some ideal \(J\) of \(R\), that is \(aJ \subseteq K\), implies that \(aJ \subseteq [Kg_M]\). But \([Kg_M]\) is an app-primary ideal of \(R\), then by corollary \((2.4)\) either \(J \subseteq [Kg_M] + soc(R)\) or \(a^n \in [Kg_M] + soc(R);g\rangle = [Kg_M] + soc(R),\) hence either \(JM \subseteq [Kg_M]M + soc(R)M\) or \(a^nM \subseteq [Kg_M]M + soc(R)M\). Since \(M\) is a multiplication, then by \([14,\text{Cor. 2.14}]\) \(soc(M) = soc(R)M\). Thus either \(JM \subseteq K + soc(M)\) or \(a^nM \subseteq K + soc(M)\). That is either \(L \subseteq K + soc(M)\) or \(a^n \in [K + soc(M);g]\). Hence \(K\) is an app-primary submodule of \(M\).

**Proposition 2.13**: Let \(K\) be a proper submodule of a non-singular multiplication \(R\)-module \(M\). Then \(K\) is an app-primary submodule of \(M\) if and only if \([Kg_M]\) is an app-primary ideal of \(R\).

**Proof**:

\(\implies\) Suppose that \(K\) is an app-primary submodule of \(M\), and let \(as \in [Kg_M]\), for \(a,s \in R\), it follows that \(asM \subseteq K\). But \(K\) is an app-primary submodule of \(M\), then by corollary \((2.4)\) either \(sM \subseteq K + soc(M)\) or \(a^nM \subseteq K + soc(M)\). But \(M\) is non-singular then by \([11,\text{Cor. 1.26}]\) \(soc(M) = soc(R)M\), and since \(M\) is a multiplication then \(K = [Kg_M]M\). Hence either \(sM \subseteq [Kg_M]M + soc(R)M\) or \(a^nM \subseteq [Kg_M]M + soc(R)M\), it follows that either \(s \in [Kg_M] + soc(R)\) or \(a^n \in [Kg_M] + soc(R)\) is \(\langle [Kg_M] + soc(R);g\rangle\). Hence \([Kg_M]\) is an app-primary ideal of \(R\).

\(\implies\) Suppose that \([Kg_M]\) is an app-primary ideal of \(R\), and let \(JL \subseteq K\). If \(J\) is an ideal of \(R\) and \(L\) is a submodule of \(M\). Since \(M\) is a multiplication then \(L = JM\) for some ideal \(I\) of \(R\), that is \(JIM \subseteq K\), implies that \(JI \subseteq [Kg_M]\). But \([Kg_M]\) is an app-primary ideal of \(R\), then by proposition \((2.3)\) either \(I \subseteq [Kg_M] + soc(R)\) or \(J^nI \subseteq [Kg_M] + soc(R);g\rangle = [Kg_M] + soc(R),\) for some \(n \in Z^+\). Thus either \(I \subseteq [Kg_M]M + soc(R)M\) or \(J^nI \subseteq [Kg_M]M + soc(R)M\). Since \(M\) is non-singular, then by \([11,\text{Cor. 1.26}]\) \(soc(M) = soc(R)M\), and since \(M\) is a multiplication then \(K = [Kg_M]M\). Hence either \(IM \subseteq K + soc(M)\) or \(J^nI \subseteq K + soc(M)\). That is either \(L \subseteq K + soc(M)\) or \(J^nI \subseteq K + soc(M)\). Hence \(K\) is an app-primary submodule of \(M\).

**Proposition 2.14**: Let \(M\) be a faithfully finitely generated multiplication \(R\)-module. If \(A\) be an app-primary ideal of \(R\). Then \(AM\) is an app-primary submodule of \(M\).

**Proof**:

Let \(aL \subseteq AM\). for \(a \in R\), \(L\) be a submodule of \(M\). Since \(M\) is a multiplication, then \(L = IM\) for some ideal \(I\) of \(R\). That is \(aLM \subseteq AM\). But \(M\) is a finitely generated multiplication \(R\)-module, then by \([15,\text{Corollary Of Theo. 9}]\), we have \(aL \subseteq A + ann_R(M)\), and \(M\) is faithful, then \(ann_R(M) = (0)\), it follows that \(aL \subseteq A\). Now, by hypothesis \(A\) is an app-primary ideal of \(R\), then by corollary \((2.4)\) either \(L \subseteq A + soc(R)\) or \(a^n \in [A + soc(R);g]\) = \(A + soc(R)\). That is either \(IM \subseteq AM + soc(R)M\) or \(a^nM \subseteq AM + soc(R)M\), for some \(n \in Z^+\). But \(M\) is faithful multiplication \(R\)-module then by \([14,\text{Cor. 2.14}]\) \(soc(M) = soc(R)M\). Hence either \(L \subseteq AM + soc(M)\) or \(a^nM \subseteq AM + soc(M)\). Thus \(AM\) is an app-primary submodule of \(M\).

**Proposition 2.15**: Let \(M\) be a finitely generated multiplication non-singular \(R\)-module and \(B\) is an app-primary ideal of \(R\) with \(ann_R(M) \subseteq B\). Then \(BM\) is an app-primary submodule of \(M\).

**Proof**:

Let \(JK \subseteq BM\), for \(J\) is an ideal of \(R\) and \(K\) be a submodule of \(M\). Since \(M\) is a multiplication, then \(K = IM\) for some ideal \(I\) of \(R\). That is \(JIM \subseteq BM\). But \(M\) is a finitely generated multiplication, then by \([15,\text{Corollary Of Theorem. 9}]\) \(J \subseteq B + ann_R(M)\). But \(ann_R(M) \subseteq B\), then \(B + ann_R(M) = B\), it follows that \(JI \subseteq B\). Since \(B\) is an app-primary ideal of \(R\), then by proposition \((2.3)\) either \(I \subseteq B + soc(R)\) or \(J^nI \subseteq [B + soc(R);g]\) = \(B + soc(R)\) for some \(n \in Z^+\). Thus either \(IM \subseteq BM + soc(R)M\) or \(J^nI \subseteq BM + soc(R)M\). But \(M\) is non-singular then by \([11,\text{Corollary (1.26)}]\) \(soc(R)M = soc(M)\). Hence either \(K \subseteq BM + soc(M)\) or \(a^nM \subseteq BM + soc(M)\). Thus \(BM\) is an app-primary submodule of \(M\).

**Proposition 2.16**: Let \(K\) be a proper submodule of faithfully finitely generated multiplication \(R\)-module \(M\). Then the following statements are equivalent.

1) \(K\) is an app-primary submodule of \(M\).
2) \([Kg_M]\) is an app-primary ideal of \(R\).
3) \(K = AM\) for some app-primary ideal \(A\) of \(R\).

**Proof**:

(1) \(\iff\) (2) It follows by proposition \((2.12)\).

(2) \(\implies\) (3) Suppose that \([Kg_M]\) is an app-primary ideal of \(R\), and since \(M\) is multiplication \(K = [Kg_M]M = AM\) implies that \(A = [Kg_M]\) is an app-primary ideal of \(R\).

(3) \(\implies\) (2) Suppose that \(K = AM\) for some app-primary ideal \(A\) of \(R\). Since \(M\) is a multiplication, then \(K = [Kg_M]M = AM\). But \(M\) is faithfully finitely generated multiplication, implies that \(A = [Kg_M]\), hence \([Kg_M]\) is an app-primary ideal of \(R\).

**Proposition 2.17**: Let \(H\) be a proper submodule of non-singular faithfully generated multiplication \(R\)-module \(M\). Then the following statements are equivalent:

1) \(H\) is an app-primary submodule of \(M\).
2) \([Hg_M]\) is an app-primary ideal of \(R\).
3) \(H = BM\) for some app-primary ideal \(B\) of \(R\) with \(ann_R(M) \subseteq B\).
Proof :
(1) $\iff$ (2) It follows by proposition (2.13).
(2) $\implies$ (3) Suppose that $[H:M]$ is an app-primary ideal of $R$, and $H = [H:M]M$ for $M$ is a multiplication, then $H = BM$ and $B = [H:M]$ is an app-primary ideal of $R$ such that $ann_B(M) = [(0):_M] \subseteq [H:M]$. 
(3) $\implies$ (2) Suppose that $H = BM$ for some app-primary ideal $B$ of $R$ such that $ann_B(M) \subseteq B$. But $M$ is a multiplication, $H = [H:M]M$, since $M$ is finitely generated multiplication with $ann_B(M) \subseteq B$ and $[H:M]M = BM$, implies that $[H:M] = B + ann_B(M) = B$ because $ann_B(M) \subseteq B$, implies that $B + ann_B(M) = B$. Hence $[H:M]$ is an app-primary ideal of $R$.

Remark 2.18 : The intersection of two app-primary submodules of an $R$-module $M$ need not to be app-primary submodule of $M$. The following example shows that:

Let $M = Z$, $R = Z$, and $K = 2Z$, $L = 3Z$ are app-primary submodules of $M$, but $K \cap L = 2Z \cap 3Z = 6Z$ is not an app-primary submodule of $M$, since $2 \in 6Z$, but $3 \notin 6Z + soc(Z) = 6Z + (0) = 6Z$ and $2 \notin \sqrt{6Z + soc(Z)Z} = \sqrt{6Z} = 6Z$.

Proposition 2.19 : Let $K$ and $L$ be two app-primary submodule of an $R$-module $M$ such that $soc(M) \subseteq L$ or $soc(M) \subseteq K$. Then $K \cap L$ is an app-primary submodule of $M$.

Proof : Since $K \cap L \subseteq L$ and $L$ is a proper submodule of $M$, then $K \cap L$ is a proper submodule of $M$. Now, let $ay \in K \cap L$, for $a \in R$, $y \in M$, and suppose that $a^n \notin [K \cap L + soc(M):M]$ for some $n \in Z^+$, that is $a^n \notin K \cap L + soc(M)$, it follows that $a^n \notin K + soc(M)$ and $a^n \notin L + soc(M)$. Since $ay \in K \cap L$ implies that $ay \in K$ and $ay \in L$. But $K$ and $L$ be two app-primary submodule of an $R$-module $M$ and $a^n \notin K + soc(M)$ and $a^n \notin L + soc(M)$, it follows that $y \in K + soc(M)$ and $y \in L + soc(M)$, implies that $y \in (K + soc(M)) \cap (L + soc(M))$. If $soc(M) \subseteq L$ then $L + soc(M) = L$, that is $y \in (K + soc(M)) \cap L$, again since $soc(M) \subseteq L$, then by modular law we have $y \in (K \cap L) + soc(M)$. Similarly if $soc(M) \subseteq K$ we get $y \in (K \cap L) + soc(M)$. Hence $K \cap L$ is app-primary submodule of $M$.

The following propositions gives the behavior of app-primary submodules under $R$-homomorphism.

Proposition 2.20 : Let $f:M \longrightarrow M'$ be an $R$-epimorphism and $K$ be an app-primary submodule of $M$ with $Ker f \subseteq K$. Then $f(K)$ is an app-primary submodule of $M'$.

Proof : $f(K)$ is a proper submodule of $M'$. If not, we have $f(K) = M'$, that is $f(m) = f(K)$ for some $m \in M$, it follows that there exists $x \in K$ such that $f(x) = f(m)$, that is $f(x - m) = 0$, so $x - m \in Ker f \subseteq K$. Hence $m \in K$, since $m \in K$ and $M = K$ (since $K$ is a proper submodule of $M$), contradiction. Now let $ay' \in f(K)$, for $a \in R$, $y' \in M'$. Since $f$ is an epimorphism, then there exists $y \in M$ such that $f(y) = y'$. That is $f(ay) = af(y) \in f(K)$, implies that $f(ay) = f(x)$ for some $x \in K$, so $f(ay - x) = 0$, that follows that $ay - x \in Ker f \subseteq K$, hence $ay \in K$. But $K$ is an app-primary submodule of $M$, then either $y \in K + soc(M)$ or $a^nM \subseteq K + soc(M)$ for some $n \in Z^+$. It follows that $y' = f(y) \in f(K) + f(soc(M)) = f(K) + soc(f(M)) = f(K) + soc(f(M')) = f(K) + soc(f(M'))$, since $f$ is an epimorphism, or $a^n f(M) \subseteq f(K) + soc(M')$. Hence $f(K)$ is an app-primary submodule of $M'$.

Proposition 2.21 : Let $f : M \longrightarrow M'$ be an $R$-epimorphism and $K$ is an app-primary submodule of $M'$. Then $f^{-1}(K)$ is an app-primary submodule of $M$.

Proof : It is clear that $f^{-1}(K)$ is a proper submodule of $M$. Now, let $ay \in f^{-1}(K)$, for $a \in R$, $y \in M$, it follows that $f(ay) = af(y) \in K$. But $K$ is an app-primary submodule of $M'$, then either $f(ay) \subseteq K + soc(M')$ or $a^nM \subseteq K + soc(M')$ for some $n \in Z^+$. It follows that either $y \in f^{-1}(K) + f^{-1}(soc(M')) \subseteq f^{-1}(K) + soc(M)$ or $a^nM \subseteq f^{-1}(K) + f^{-1}(soc(M')) \subseteq f^{-1}(K) + soc(M)$. Hence $f^{-1}(K)$ is an app-primary submodule of $M$.

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