\( \alpha \)-almost similar operators

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**Abstract:**

The study focuses on \( \alpha \)-almost similar operator which is a new concept of the operator theory and also some basic concepts related to the concept \( \alpha \)-almost similar.

The study also defines a new concept called \( \beta \)-operator which is an expansion of the concept \( \theta \)-operator and the relationship of this concept with the \( \alpha \)-almost similar.

At the end of this research, we study some important relationships among similar, unitarily equivalent, and almost similar on the one hand and \( \alpha \)-almost similar on the other.

**Introduction**

We denote \( B(H_1,H_2) \) to the set of all bounded linear operators from a Hilbert space \( H_1 \) into a Hilbert space \( H_2 \), if \( H = H_1 = H_2 \) then we denote \( B(H) \) instead of \( B(H_1,H_2) \). The operator \( T \in B(H) \) is called self-adjoint if \( T = T^* \) where \( T^* \) is the adjoint of \( T[1] \).

An operator \( A \in B(H) \) is said to be isometric if \( A^*A = I[2] \). If \( A^*A = AA^* \), then \( A \) is called normal operator. And if \( A^*A = AA^* = I \) then \( A \) is said to be unitary [3]. If \( A^* = A \) and \( A^2 = A \) then \( A \) is said to be projection. If \( AA^*A = A \) then \( A \) is said to be partially isometric, equivalently \( A^*A \) is projection (i.e. \( (A^*A)^2 = A^*A \) [4]). Clearly every unitary operator is isometric and normal.

Two operators \( A \in B(H) \) and \( B \in B(H) \) are said to be similar and denoted by \( A \sim B \), if there exists an invertible operator \( X \) such that \( AXB \) is a subset of the same: space, point spectrum and approximate point spectrum [5].

Similarly, two operators \( A, B \in B(H) \) are said to be unitarily equivalent and denoted by \( A \cong B \), if there exists a unitary operator \( U \) such that \( UA = U^*B U \) [4]. If \( A, B \) are similar normal then they are unitarily equivalent by Foged- Putnam theorem [6].

Let \( A, B \) be two bounded linear operators on \( B(H) \). Then \( A, B \) are said to be almost similar and denoted by \( A \sim_a B \) if there exists an invertible operator \( X \) such that:

\[
A^*A = X^{-1}B^*B X \text{ and, } A^* + A = X^{-1}(B^* + B) X.
\]

The class of almost similar was first introduced by Jibril [7], we have extended this concept to \( \alpha \)-almost similar and demonstrated some different results.

An operator \( A \in B(H) \) is said to be \( \theta \)-operator if \( A^*A \) commutes with \( A^* + A \). The class of all \( \theta \)-operator in \( B(H) \) is denoted by \( \theta \). The class of \( \theta \)-operator be which has been widely studied by Campbell [8]. We have extended the concept of \( \theta \)-operator to another concept we called it \( \beta \)-operator, the class of \( \beta \)-operator in \( B(H) \) is denoted by \( \beta \).

Let \( T \in B(H) \) then the set of all complex number \( \lambda \) for which \( T - \lambda I \) is not invertible is called the spectrum of \( T \) and denoted by \( \sigma(T) \) that is, \( \sigma(T) = \{ \lambda \in \mathbb{C} ; (T - \lambda I) \text{ is not invertible} \} \). The complement of the spectrum of \( T \) is called resolvent set of \( T \). The spectrum of \( T \) can be split into many disjoint sets [9].

The point spectrum of the operator \( T \) is denoted by \( \sigma_p(T) \) is the set of all those \( \lambda \) for which \( T - \lambda I \) is not injective, that is \( \sigma_p(T) = \{ \lambda \in \mathbb{C} ; ker(T - \lambda I) \neq \{0\} \} \).

A scalar \( \lambda \) is said to be the approximate point spectrum for the operator \( T \) and denoted by \( \sigma_{ap}(T) \), if there exists a sequence of unit vector \( \{x_n\} \) such that \( \|\{(T - \lambda I)x_n\}\| \to 0 \) [9]. Let \( T \) be a linear transformation from a normed space \( X \) into a normed space \( Y \) \( (i.e. T: X \to Y) \). Then \( T \) is said to be compact if \( \overline{T(B)} \) is compact for every bounded
subset $\mathcal{B}$ of $X$, that is, $\overline{T(\mathcal{B})}$ is relatively compact for every bounded subset $\mathcal{B}$ of $X$ [9].

1. Basic concept on $\alpha$–almost similarity

**Definition 1.1.** Let $\alpha$ be a real number, two bounded linear operators $A, B \in \mathcal{B}(\mathcal{H})$ are said to be $\alpha$–almost similar and, denoted by $A \sim^{\alpha} B$. If there exist an invertible operator $X$ such that:

$$A' A = X^{-1} B^* B X \quad \text{(1)}, \quad A' + \alpha A = X^{-1} (B^* + \alpha B) X \quad \text{(2)}.$$

**Example 1.2.** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$, and define the invertible operator on $\mathbb{C}^2$ as follows:

$$X = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{take } \alpha = 2, \text{ then } A \sim^2 B.$$

To show that:

$$A' A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = X^{-1} B^* B X \quad \text{(1)}$$

and

$$A' + 2 \alpha A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = X^{-1} (B^* + 2B) X \quad \text{(2)}.$$

**Remark 1.3.** Every $1$– almost similar operators are almost similar and the converse are true.

The following example show almost similar operators are $\alpha$–almost similar independent when $\alpha \neq 1$.

**Example 1.4.** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$, and define the invertible operator on $\mathbb{C}^2$ as follows:

$$X = X^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \text{take } \alpha = -1. \text{ Then } A \sim^{-1} B. \text{ But } A \text{ is not almost similar to } B \text{ since } A' + A \neq X^{-1} (B^* + B) X,$$

indeed $A'^* A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 I, B^* + B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Since $X \neq X^{-1} (B^* + B) X$, we have $A \sim^1 B$. Then $A$ is invertible for every invertible operator $X$.

**Theorem 1.5.** Let $a \in \mathbb{R}$, the relation $\sim^a$ on $B(\mathcal{H})$ is equivalence relation.

Proof: (i) Reflexivity, let $A \in B(\mathcal{H})$ take $X = I$, $A'A = X^{-1} A'A X$ and, $A' + a A = X^{-1} (A' + a A) X$.

Then $A \sim^a A$.

(ii) Symmetry, suppose that $A, B \in B(\mathcal{H})$ and, $A \sim^a B$. Then there exists an invertible operator $X$ such that:

$$A' A = X^{-1} B^* B X \quad \text{(1)}, \quad A' + a A = X^{-1} (B^* + a B) X \quad \text{(2)}.$$

Now, pre-multiplying $X^{-1}$ and post-multiplying $X^{-1}$ yields $X^{-1} A'A X^{-1} = B^* B X^{-1} = B' B X^{-1}$.

Therefore,

$$A \sim^a B \quad \text{(1)}.$$

(iii) Transitivity, suppose that $A, B$, and $C \in B(\mathcal{H})$. And $A \sim^a B$, $B \sim^a C$, to show that $A \sim^a C$.

Since $A \sim^a B$, then there exists an invertible operator $X$ such that:

$$A'A = X^{-1} B^* B X \quad \text{(1)}, \quad A' + a A = X^{-1} (B^* + a B) X \quad \text{(2)}.$$

Also, since $B \sim^a C$, then there exists an invertible operator $Y \in B(\mathcal{H})$ such that:

$$B' B = Y^{-1} C^* C Y \quad \text{(3)} \quad \text{and, } B' + a B = Y^{-1} (C' + a C) Y \quad \text{(4)}.$$

Substituting (3) and (4) in (1) and (2) as follows:

$$A'A = X^{-1} [Y^{-1} C^* C' Y] X = X^{-1} Y^{-1} C^* C Y X Y = (Y^{-1} C^* C Y) X Y \quad \text{(5)}$$

Also, $A' + a A = X^{-1} [Y^{-1} (C' + a C) Y] X$.

Which implies that $A' + a A = (Y^{-1} C^* C Y) X$.

Then from (5) and (6) we get $A \sim^a C$.

**Proposition 1.6.** Let $A \in B(\mathcal{H})$, such that $A \sim^a 0$, then $A = 0$.

Proof: Since $A \sim^a 0$ then there exists an invertible operator $X$ such that:

$$A' A = X^{-1} 0^* 0 X = 0 \quad \text{(1)}, \quad A' + a A = X^{-1} (0^* + a 0) X = 0 \quad \text{(2)}.$$

Then $A' A = 0$ and $A' + a A = 0$. Now, $\|Ax\|^2 = (AX)\overline{(AX)} = |Ax|^2 \leq 0$.

Therefore $Ax = 0$ for all $x \in \mathcal{H}$. Thus $A = 0$.

**Remark 1.7.** Suppose that $A, B \in B(\mathcal{H})$ such that $A \sim^a B$, then clearly by using mathematical induction we can prove:

(i) $(A' A)^n = X^{-1} (B' B)^n X$.

(ii) $(A' + a A)^n = X^{-1} (B' + a B)^n X$.

For all natural number $n$.

**Proposition 1.8.** Let $A, B \in B(\mathcal{H})$ such that $A \sim^a B$, then $A$ is isometric if and only if $B$ is isometric.

Proof: Suppose that $A$ is isometric. Since $A \sim^a B$ this means that there exists an invertible operator $X$ such that:

$$A' A = X^{-1} (B' B) X \quad \text{(1)}, \quad A' + a A = X^{-1} (B' + a B) X \quad \text{(2)}.$$

Since $A$ is isometric then $A' A = I$ substituting in the equality (1) we have $I = A' A \Rightarrow X^{-1} (B' B) X$ which implies that $B' B = I$.

Thus, $B$ is isometric.

Conversely; by the same way we can prove that $A$ is isometric whenever $B$ is isometric.

**Proposition 1.9.** Let $a \in \mathbb{R}$, $A, B$ are two operators in $B(\mathcal{H})$ with $A \sim^a B$. Then:

(i) $A' A$ is onto if and only if $B' B$ is onto.

(ii) $A' + a A$ is onto if and only if $B' + a B$ is onto.

(iii) $A' A$ is one -to-one if and only if $B' B$ is one-to-

one.

(iv) $A' + a A$ is one-to-one if and only if $B' + a B$ is one-to-

one.

(v) $A' A$ is projection if and only if $B' B$ is projection.

Proof: Clearly.

**Remark 1.10.** Let $a \in \mathbb{R}$, $A, B$ are two operators in $B(\mathcal{H})$ with $A \sim^a B$. Then:

(iii) $A' A$ is one-to-one and onto if and only if $B' B$ is one-to-

and, onto.

(vii) $A' + a A$ is one-to-one and, onto if and only if $B' + a B$ is one-to-one and, onto.

Proof: immediately from proposition 1.9 above.

**Proposition 1.11.** Let $A \in B(\mathcal{H})$ and $A \sim I$, then $A$ is isometry.
Proof: Suppose that \( A \subseteq I \), then there exists an invertible operator \( X \) such that \( A'X = X^{-1}(I)X = X^{-1}X = I \ldots \) (1). Then \( A' = I \) (i.e. \( A \) is isometry).

**Proposition 1.12:** Let \( A, B \in B(\mathcal{H}) \) and \( A \subseteq B \) such that \( A \) is partially isometric then \( B \) is partially isometric.

Proof: \( A \subseteq B \) means that there exists an invertible operator \( X \) such that \( A'X = X^{-1}(B'B)X \ldots \) (1). Since \( A \) is paritally isometric then \( A'X \) is projection (i.e. \( (A'X)^2 = A'X \)). By squaring both sides in (1) we have \((X^{-1}(B'B)X)(X^{-1}(B'B)X) = (A'X)^2 = A'X \). Then \( X^{-1}(B'B)(B'B)X = X^{-1}(B'B)X \ldots \) (2).

Pre-multiplying and post-multiplying (2) by \( X \) and \( X^{-1} \) respectively we have, \((B'B)^2 = B'B\) (i.e. \( B'B \) is projection). Which implies that \( B \) is partially isometric.

**Proposition 1.13:** Let \( \alpha \in \mathbb{R} \). Then the transformation \( \varphi : B(\mathcal{H}) \rightarrow B(\mathcal{H}) \) that satisfies \( \varphi(A') = X^{-1}(B'B)X \) for every \( A \in B(\mathcal{H}) \) is an automorphism.

Proof: Suppose that \( \alpha \neq 0 \) and \( B \in B(\mathcal{H}) \) such that \( \varphi(A') = X^{-1}(B'B)X \). Then \( \varphi(A') = X^{-1}(B'B)B \) and \( \varphi(C') = X^{-1}(D'D)C \) for every \( A, C \in B(\mathcal{H}) \) and \( D, C \in B(\mathcal{H}) \).

**Proposition 1.14:** Let \( A, B \in B(\mathcal{H}) \) such that \( A, B \) are unitarily equivalent then \( A \subseteq B \).

Proof: Since \( A \) and \( B \) are unitarily equivalent then there exists a unitary operator \( U \) such that \( A = U'B'U \). Then \( A' = U'B'U \) which implies that \( A' = (U'B')U = U'B'U + U'B'U = U'B' \). And, \( A' + \alpha A = U'B'U + U'B'U = U'B' = U'B'B'U \). Thus, \( A \subseteq B \).

**Proposition 1.15:** Let \( A, B \in B(\mathcal{H}) \) such that \( A \subseteq B \).

Proof: \( A \subseteq B \) means that there is an invertible operator \( X \) such that \( A'X = X^{-1}(B'B)X \ldots \) (1). Then, \( A' + \alpha A = X^{-1}(B'B + \alpha AB)X \ldots \) (2).

From the equality (2) we have \( A' + \alpha A = X^{-1}(B'B + \alpha AB)X \), by post-adding to both sides \( \alpha A + \alpha A \) which implies that \( A' + \alpha A = X^{-1}(B'B + \alpha AB)X \). Then we have \( A' + \alpha A = X^{-1}(B'B + \alpha AB)X \ldots \) (3) since \( \lambda \) is real number. Now, we want to prove that \((A + \lambda I)'(A + \lambda I)X = X^{-1}(B + \lambda I)'(A + \lambda I)X \ldots \) (3).

Consider the following example: Let \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( X = X^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), be the operators on two-dimensional Hilbert space \( \mathbb{C}^2 \), take \( \alpha = 2 \),
$A \precsim B$. Also $A$ is similar to $B$ (i.e., $XA = BX$) but $A'A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = AA'$ and, $B'B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = BB'$. Then $A$ and $B$ are not normal operators.

2. The properties of self-adjoint operator on $\alpha$-almost similarity.

**Proposition 2.1:** Suppose that $A$, $B$ are self-adjoint operators in $B(\mathcal{H})$ with $A \precsim B$ (i.e. $A$ is similar to $B$), then $A \succeq B$, for every $a \in \mathbb{R}$.

Proof: Since $A$ and $B$ are similar operators, then there exists an invertible operator $X$ such that $XA = BX$ (i.e. $A = X^{-1}B X$).

Also, $A$ and $B$ are self-adjoint operators in $B(\mathcal{H})$, then

$A' = X^{-1}B' X$ ......... (1).

Also, $A' + aA = A + aA = X^{-1}B + a X^{-1}B X = X^{-1}(B + aB) X$ = $X^{-1}(B + aB) X$ ......... (2). From (1) and (2) we have $A \succeq B$.

**Remark 2.2:** The converse of the Proposition 2.1. above is not true in general.

For example: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and

$x = X^{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$ take $a = 2$. We know that $A \precsim B$ as in example 1.2. Moreover $A \precsim B$. But $A \neq A'$, also $B \neq B'$. Thus, $A$ and $B$ are not self-adjoint operators.

**Proposition 2.3:** Let $a = -1 \in \mathbb{R}$, $A, B \in B(\mathcal{H})$ and $A \precsim B$. If $A$ is self-adjoint then $B$ is self-adjoint.

Proof: Since $A \precsim B$, then there exist an invertible operator $X$ such that $A' = A = X^{-1}(B' - B) X$. Which implies that $0 = X^{-1}(B' - B) X$ ......... (1). Pre-multiplying and post multiplying (1) by $X$ and $X^{-1}$ respectively we have $0 = B' - B$. Then $B = B'$.

**Remark 2.4:** The converse of the Proposition 2.3 above is not true in general for example $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ = $A'$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B'$ and, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$, take $a \in \mathbb{R}$.

Then $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$X = X^{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $X = X^{-1} X = I$. Thus, $A$ is not $\alpha$-almost similar to $B$. Then $A$ is not (-1)-almost similar to $B$.

**Theorem 2.5** [4]: (Cartesian form) let $T$ be any operator, then there exist self-adjoint operators $A$ and $B$ such that $T = A + iB$. When $A = \frac{1}{2}(T + T^*)$ and,

$B = \frac{1}{2i}(T - T^*)$.

**Theorem 2.6:** Let $T \in B(\mathcal{H})$ then $T \precsim T^*$ if and only if $T$ is normal and $(T + T^*)^2 = 4T^*T$.

Proof: If $T \precsim T^*$ then clearly $(T + T^*)^2 = 4T^*T$ and $T$ is normal.

Conversely: If $4T^*T = (T + T^*)^2$ then $(T + T^*)^2 = T^* + 2T^*T + T^*T$. Hence, $T^* + 2T^*T + T^*T = 0$ which implies that $(T^* - T)^2 = 0 \Rightarrow (T^* - T)(T - T^*) = 0$. Let $S = T^* - T$, then $SS^* = 0 \Rightarrow S = 0 = SS^*x(x) = S^*(x)Sx = \|Sx\|^2$ for every $x$. Then $S^*x = 0$ for every $x \Rightarrow S = 0 \Rightarrow T^* - T = 0 \Rightarrow T = T^*$. 

**Remark 2.6:** If $T = T^*$ then $(T + \alpha T)^2 = (1 + \alpha)^2T^*T$ for every $\alpha \in \mathbb{R}$.

**Proposition 2.7:** Suppose that $(T + \alpha T)^2 = (1 + \alpha)^2T^*T$ then:

(i) If $\alpha = 1$ then $T$ is normal then $T = T^*$.

(ii) If $\alpha = -1$ then $T = T^*$.

(iii) If $\alpha \neq 1$, -1 then $T^2 = T^2$.

Proof: (i) directly as in theorem 2.6. And (ii) clearly. Now to prove (iii) let $\alpha \neq 1$, -1. $(T + \alpha T)^2 = (1 + \alpha)^2T^*T$ by taking adjoint to both sides we have $(T + \alpha T)^2 = (1 + \alpha)^2T^*T$. Then $T^2 + \alpha TT^* + \alpha T^*T + \alpha TT^* = T^2 + \alpha TT^* + \alpha^2T^2 \Rightarrow T^2 = T^2$.

**Theorem 2.8** [4]: If $T^*T$ is normal operator, then there exists a unitary operator $U$ such that $T^* = UT$.

3. The properties of $\beta$-operator on $\alpha$-almost similarity.

**Definition 3.1:** Let $A \in B(\mathcal{H})$, then $A$ is called a $\beta$-operator if $A'A$ commutes with $A' + aA$. The class of all $\beta$-operator in a Banach algebra on a Hilbert space $\mathcal{H}$ is denoted by $\beta$ i.e. $\beta = \{A : A \in B(\mathcal{H}) | [A'A', A' + aA] = 0\}$.

**Example 3.2:** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and take $\alpha = 3$ then $[A'A', A' + 3A] = 0$ i.e. $[A'A', A' + 3A] = (A' + 3A)(A' + aA)$ which implies that $A$ is $\beta$-operator.

**Proposition 3.3:** If $A \in B(\mathcal{H})$ is $\beta$-operator then $kA$ is $\beta$-operator for every real number $k$.

Proof: Clearly.

**Proposition 3.4:** If $A \in B(\mathcal{H})$ and $A \precsim B$ such that $B$ is $\beta$-operator then $A$ is $\beta$-operator.

Proof: $A \precsim B$ means that there exists an invertible operator $X$ such that $A'A = X^{-1}(B' + aB) X$. And, $A' + aA = X^{-1}(B' + aB) X$. Then, $[X^{-1}(B' + aB) X] [X^{-1}(B' + aB) X] = [A' + aA]$ $A'A$ .... (1)

And $[X^{-1}(B' + aB) X] [X^{-1}(B' + aB) X] = [A' + aA]$ $A'A$ .... (2).

From the equality (1) we have: $[X^{-1}(B' + aB) X] = [A' + aA] A'A$ .... (3).

Also, from the equality (2) we have: $[X^{-1}(B' + aB) X] = A'A [A' + aA]$ .... (4).

Since $B$ is $\beta$-operator then the left-hand side of the equality (3) and the equality (4) are equal. which imply that the right-hand side of the equality (3) and the equality (4) are equal. Hence $A$ is $\beta$-operator.

4. The relation among similarity, unitarily equivalent, quasi similarity and almost similarity.

**Proposition 4.1:** Let $A, B \in B(\mathcal{H})$ are orthogonal projection then $A$ and $B$ are $\alpha$-almost similar if and only if $A$ and $B$ are similar.

Proof: Suppose that $A \precsim B$ and $A, B$ are projection then by proposition 1.16 we get $A \succeq B$. 

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Conversely, suppose that $A$ and $B$ are similar operators then there exists invertible operator $X$ such that $A = X^{-1}BX$, since $A$ and $B$ are orthogonal projection then $A = A^* = A^2$, $B = B^* = B^2$. Which implies that $A^2 = X^{-1}B^2X$ then we have $A^*A = X^{-1}B^*BX$.

On the other hand, the second inequality follows from the fact that $A^* + \alpha A = (1 + \alpha)A = (1 + \alpha)X^{-1}BX = X^{-1}(B^* + \alpha B)X$, Thus, $A \preceq B$.

**Proposition 4.2.** Let $\alpha \in \mathbb{R}$, $A, B \in B(\mathcal{H})$ and $A, B$ are self-adjoint then $A$ and $B$ are unitarily equivalent if and only if $A \preceq B$.

**Proof:** Suppose that $A$ and $B$ are unitarily equivalence then by proposition 1.14 we have $A \preceq B$.

Conversely: Suppose that $A, B \in B(\mathcal{H})$ are self-adjoint with $A \preceq B$.

Now, $A \preceq B$ means that there exists an invertible operator $X$ such that $A^*A = X^{-1}(B^*B)X$ (1) and $A^* + \alpha A = X^{-1}(B^* + \alpha B)X$ (2).

Since $A, B$ are self-adjoint and $A \preceq B$ then they are similar operators (i.e, $A = X^{-1}BX$). Then $A$ and $B$ are both similar and self-adjoint operators then $A$ and $B$ are normal. Thus $A$ and $B$ are unitarily equivalent.

**Corollary 4.3.** Let $\alpha \in \mathbb{R}$, $A, B \in B(\mathcal{H})$ are self-adjoint and $A \preceq B$. Then $A$ and $B$ are unitarily equivalent.

**Proof:** directly from proposition 4.2 above.

**References**


المؤثرات الخطية المتشابهة تقريبا من النمط-α

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قسم الرياضيات، كلية علوم الحاسوب والرياضيات، جامعة تكريت، تكريت، العراق

الملخص
درسنا في هذه البحث المؤثرات الخطية المقيدة المتشابهة تقريبا من النمط-α وهو مفهوم جديد للنظرية المؤثرات الخطية. كذلك بعض المفاهيم الأساسية المتعلقة بمفهوم المؤثرات الخطية المقيدة المتشابهة تقريبا من النمط-α. كذلك عرفنا مفهوما جديدا والذي اطلقنا عليه اسم المؤثر من النمط-β والذي يعتبر توسعا للمؤثر من النمط-θ وعلاقة هذا المؤثر بالمؤثرات الخطية المتشابهة تقريبا من النمط-α. في نهاية هذا البحث درسنا بعض العلاقات المهمة بين التشابه، المؤثرات الاحادية المتكافئة، التشابه التقريبي من جهة وبين المؤثرات الخطية المتشابهة تقريبا من النمط-α من الجهة الأخرى.