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Dynamic Prey Predator Model and multiple forms of Harvest of Infected Prey

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ABSTRACT

In this paper, the dynamic of prey predator model was discussed when the relationship between them is functional response type III. In addition, when prey exposure to the disease as nonlinear function. Also the infected prey exposed to harvest as a nonlinear and as linear function. The bounded and positive solutions, periodic, conditions of equilibrium points and the stability were we discussed Some results were illustrated in numerical simulations, and show we can use the linear function of harvesting to control on the dices.

1. Introduction

One of the important issues in environmental systems is the effect of infectious diseases in addition to the environmental point of view. Therefore, researchers and environmental scientists attach great importance to the development of an important tool along with the experimental environment and described how the disease spread in population and transformation from susceptible to infected population. When diseases spread in populations, communities with other species in the same space compete for food, survival and predation. For example, when a bounty was placed on natural predators such as cougars, wolves, and coyotes in the Kaibab Plateau in Arizona, the deer population increased beyond the food supply, and then over half of the deer died of starvation in 1923-1925 [1]. Because no type of population can survive alone, researchers have provided many studied in describing interaction among populations. The first to describe in modern mathematical ecology was done by two researchers Lotka and Volterra, they describe the competition between prey and predator. But the most of models that involved the injury of one population species were originated from classical action of Kermack and McKendrick [2]. After these two wonderful works, the door become open for researchers to offer many studies in epidemiology and environmental science theory. Even in the last few decades, these models have become important tools for analyzing and understanding the spread of infectious diseases and controlling them. One of the first studies of prey predator model with disease was the study presented by researchers [3]. Many researchers [4,5 ,6] studied those models with diseases. [5,7,8, 9] they studied the role of disease in destabilizing the system. Harvesting is one of the means used to control disease and prevent its spread, but this method should be used with great care because misuse may expose species to extinction. There are many studied on this, [10] studied harvesting in prey predator model to form a controlled environment while ensuring the survival of species and continuation of harvest with controlled disease. Continuous harvesting in prey predator model in [9, 11]. The effect of harvesting as a nonlinear function was studied in [13].

This paper is organized as follows: In section two, we outline the mathematical model with some lemmas about natural solutions. Also we study in this suction the equilibriums points with sufficient conditions and its stability. In section three we add the function of harvesting as linear and other as nonlinear. Numerical solution to illustrate the behavior of interacting societies, that's in section four.

2.1 Mathematical Model

$$\frac{dx}{dt}=rx\left(1-\frac{x}{k}\right)-\beta_0\frac{x^2z}{b+x^2}-c\frac{xy}{x+y}$$
$$\frac{dy}{dt}=c\frac{xy}{x+y}-d_1y \qquad (1.1)$$
$$\frac{dz}{dt}=\beta_1\frac{x^2z}{b+x^2}-d_2z$$

where x', y' and $z' \geq 0$. Here x denoted to Susceptible prey, y Infected prey while z predator. All parameters greater than zero and denoted as follows:

r	Growth rate
k	Carrying capacity
c	Infection rate
β_0	predation rate of susceptible predator
β_1	The growth rate coefficient of predator due to its interaction with the susceptible prey
b	The half saturation constant.
d_1	Natural death rate of infected prey
d_2	Natural death rate of predator

Lemma 1. All solutions of (1.1) in R^3 are uniformly and bounded

Proof

Let $M(t)=x(t)+y(t)+z(t)$,

$\frac{dM}{dt}+\mu x=\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}+\mu x$ then

$$\frac{dM}{dt}\leq(r+\mu)x-r\frac{x^2}{k}$$

$$\frac{dM}{dt}\leq\frac{k}{4r}(r+\mu)^2,\ say\ \ v.$$

Then,

$$0<M(x(t),y(t),z(t))\leq\frac{v}{\mu}(1-e^{-\mu t})+(x(t),y(t),z(t))e^{-\mu t}$$

Lemma 2. If $\beta_1\leq d_2$ then $\lim_{x\to\infty}z(t)=0$

Proof.

then $\frac{\beta_1x^2}{b+x^2}\leq d_2$ Assume $\beta_1\leq d_2$ =

Then $\frac{dz}{dt}<0$ therefore $\lim_{z\to\infty}z(t)$ exist and nonnegative. We suppose

$\lim z(t)=0$ as $t\to\infty$, then $\exists p>0$ such that $\lim z(t)=p$ as $t\to\infty, p>\in$,

Let x^2 be the max positive constant such that for $t>t_0$. Now $x^2(t)\leq\max x^2$

$$z(t)=z(t_0)\exp\int_0\left(\frac{\beta_1x^2(s)}{b+x^2(s)}-d_2\right)ds$$

$$z(t)=z(t_0)\exp\int_0\left(\frac{\beta_1x^2(s)-bd_2-d_2x^2(s)}{b+x^2(s)}\right)ds$$

$$z(t)\leq z(t_0)\ \exp-\left(\frac{(d_2-\beta_1)x^2\max+bd_2}{b+x^2\max}\right)(t-t_0)\to o\quad as\quad t\to\infty$$

Its contradiction, then $\lim_{x\to\infty}z(t)=0$

Lemma 3. In the absent of predator the subsystem has no periodic orbit in R^2.

Proof:

Let $H=\frac{1}{xy}$, $h_1=rx\left(1-\frac{x}{k}\right)-c\frac{xy}{x+y}$ and $h_2=c\frac{xy}{x+y}-d_1y$,

then $\Delta(x,y)=\frac{\partial(h_1,H)}{\partial x}+\frac{\partial(h_2,H)}{\partial y}=-\frac{r}{ky}<0.$

Therefore, no periodic solutions.

Lemma 4: In the absent of infected prey the subsystem has no periodic orbit in R^2.

Proof: As in lemma 3.

2.2 Equilibrium Points and Stability.

There are five equilibrium points in system (1.1) as follows:

1. The trivial equilibrium point $E_0(0,0,0)$.

2. Second equilibrium point when no infected prey and no predator $E_1(k,0,0)$, in this case prey growth exponentially to the carrying capacity k according to Logistic equation.

3. Third equilibrium point when the system content only prey, $E_2(\bar{x},\bar{y},0)$ where

$\bar{x}=k\left(1-\frac{(c-d_1)}{r}\right), \bar{y}=\frac{(c-d_1)}{d_1}\bar{x}$, with condition $r>(c-d_1)$.

4. In absent of infected prey (The disease free equilibrium point) the equilibrium is $E_3(\hat{x},0,\hat{z})$, where

$$\hat{x}=\sqrt{\frac{d_2b}{(\beta_1-d_2)}}, \hat{z}=\frac{r}{\beta_0}\left(\frac{1}{\hat{x}}-\frac{1}{k}\right)\left(\frac{b\beta_1}{(\beta_1-d_2)}\right).$$

5. In the case of all population coexist, the fifth equilibrium is $E_4(x^*,y^*,z^*)$, where

$$x^*=\sqrt{\frac{d_2\beta_1}{(\beta_1-d_2)}}, y^*=\frac{c-d_1}{d_1}x^*, z^*=\left[r\left(1-\frac{x^*}{k}\right)-\frac{c-d_1}{x^*}\right]\frac{b+x^{*2}}{\beta_0x^*}$$

The Jacobean matrix of system (1.1) is

$$J=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z}\end{bmatrix}, \text{ where}$$

$$\frac{\partial f_1}{\partial x}=r-\frac{2r}{k}x-\beta_0\frac{2bx}{(b+x^2)^2}-c\frac{y^2}{(x+y)^2},\ \frac{\partial f_1}{\partial y}=-\frac{cy^2}{(x+y)^2},\ \frac{\partial f_1}{\partial z}=-\beta_0\frac{x^2}{b+x^2}$$

$$\frac{\partial f_2}{\partial x}=c\frac{y^2}{(x+y)^2},\ \frac{\partial f_2}{\partial y}=c\frac{x^2}{(x+y)^2}-d_1,\ \frac{\partial f_2}{\partial z}=0$$

$$\frac{\partial f_3}{\partial x}=\beta_1\frac{2bzx}{(b+x^2)^2},\ \frac{\partial f_3}{\partial y}=0,\ \frac{\partial f_3}{\partial z}=\beta_1\frac{x^2}{b+x^2}-d_2$$

Now by using this matrix we get the following

1. The trivial equilibrium point give $\lambda_1=r>0, \lambda_2=-d_1<0, \lambda_3=-d_2<0$, thus this point is saddle.
2. Equilibrium $E_1(k,0,0)$ also saddle because $\lambda_1=-r<0, \lambda_2=c-d_1>0,\quad \lambda_3=\frac{\beta_1 k^2}{b+k^2}-d_2>0$.
3. Characteristic equation of E_2 is $\lambda^3+A\lambda^2+B\lambda+C=0$, where

$A=-(j_{11}+j_{23}+j_{33})=\text{Arc } J$

$B=j_{11}j_{34}+j_{23}j_{33}+j_{11}j_{33}$.

$C=-j_{11}j_{23}j_{33}$

And

$$j_{11}=-r+2(c-d_1)-\frac{(c-d_1)^2}{d_1}\left(k-k\frac{(c-d_1)}{r}\right),\ j_{12}=-\frac{d_1^2}{c},\ j_{13}=-\beta_0\frac{\left(k-k\frac{(c-d_1)}{r}\right)^2}{b+\left(k-k\frac{(c-d_1)}{r}\right)^2}$$

$$j_{21}=\frac{(c-d_1)^2}{c},\ j_{22}=\frac{d_1^2}{c}-d_1,\ j_{33}=\beta_1\frac{\left(k-k\frac{(c-d_1)}{r}\right)^2}{b+\left(k-k\frac{(c-d_1)}{r}\right)^2}-d_2$$

Then from Routh Hurwitz criteria, this point is stable if $A>0, C>0$ and $AB-C>0$

4. The elements of Jacobean matrix near $E_3(\bar{x},0,\hat{z})$ are equal to zero except the following:

$$j_{11}=r-\frac{2r}{k}\sqrt{\frac{d_1b}{\beta_1-d_1}}-\left(\frac{r}{\sqrt{\frac{d_1b}{\beta_1-d_1}}}-\frac{r}{k}\right)\left(b+\frac{d_1b}{\beta_1-d_1}\right),\quad j_{13}=-\beta_0\frac{d_1}{\beta_1},\quad j_{22}=c-d_1$$

$$j_{31}=2\beta_1\frac{r}{\beta_0}\left(\frac{\left(1-\sqrt{\frac{d_1b}{\beta_1-d_1}}\right)}{b\left(\frac{b\beta_1}{\beta_1-d_1}\right)^2}\right),\ j_{33}=d_1-d_2$$

It's stabile if satisfied Routh Hurwitz criteria.

5. Equilibrium point $E_4=(x^*,y^*,z^*)$

The elements of Jacobean matrix in this case are

Where

$$j_{11}=\left(1-\beta_0+(-2+\beta_0)\frac{\sqrt{\frac{d_2b}{\beta_1-d_2}}}{k}\right)r-c\frac{(c-d_2)}{d_1+c-d_2}$$

$$j_{12}=-c\frac{\left(c^2+d_2^2+c-d_2-2cd_2\right)^2}{d_1^2},\ j_{13}=\frac{-\beta_0 d_2}{\beta_1},\ j_{21}=c\frac{\left(c^2+d_2^2+c-d_2-2cd_2\right)^2}{d_1^2}$$

$$j_{22}=\frac{c}{(1+c-d_2)^2}-d_1,\ j_{31}=\frac{2b\beta_1}{\beta_0\left(b+\frac{d_2b}{\beta_1-d_2}\right)}\left(r-r\frac{\sqrt{\frac{d_2b}{\beta_1-d_2}}}{k}-\frac{(c^2-cd_2)}{(1+c-d_2)}\right)$$

And other elements equal to zero. The characteristic equation near the interior equilibrium point is

$\lambda^3+A\lambda^2+B\lambda+C=0$

$A=-(j_{11}+j_{22})$

$B=-(-j_{11}j_{22}+j_{12}-j_{13})$

$C=-(-j_{12}j_{21}-j_{21}j_{31}j_{22})$

Then from Routh Hurwitz criteria, this point is stable if $A>0, C>0$ $AB-C>0$.

Lemma 5. The equilibrium E_4 is global stability in the first positive cone.

Proof. Since E_2 is locally asymptotically stable then we choose a Lyapunov function as follows

$$W(x,,y,z)=C_1\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+C_2y+C_3z$$

$$\frac{dW}{dt}=C_1\left(\frac{x-x^*}{x}\right)dx+C_2dy+C_3dz$$

$$\frac{dW}{dt}=rx-rx^*-\frac{rx^2}{k}x+\frac{rx}{k}x^*+\frac{\beta_0x^*xz}{b+x^2}+c\frac{x^*y}{x+y}-d_1y-(\beta_0-\beta_1)\frac{x^2z}{b+x^2}-d_2z<0$$

This point is stable if this condition satisfied,

$$rx^*+\frac{rx^2}{k}x>rx+\frac{rx}{k}x^*+\frac{\beta_0x^*xz}{b+x^2}+c\frac{x^*y}{x+y}$$

3. The Model with harvesting

Now the model with harvesting as follows

$$\frac{dx}{dt}=rx(1-\frac{x}{k})-\beta_0\frac{x^2z}{b+x^2}-c\frac{xy}{x+y}$$

$$\frac{dy}{dt}=c\frac{xy}{x+y}-Hy-d_1y \qquad (2.1)$$

$$\frac{dz}{dt}=\beta_1\frac{x^2z}{b+x^2}-d_2z$$

Where $H(y)$ is the function of harvesting, such that if $H(y)$ linear function then $H(y)=hy$ or $H(y)=\frac{hy}{\rho+y}$ when its nonlinear function [16] . In case of linear the equilibrium point is ,

Where $x_1^*=\sqrt{\frac{d_2b}{\beta_1-d_2}}$ *condition* $\beta_1>d_2$

$$z_1^*=\left(r-\frac{r x_1^*}{k}-\frac{cy_1^*}{x_1^*+y_1^*}\right)\frac{b+x_1^{*2}}{\beta_0 x_1^*}\quad y_1^*=\frac{c-d_1-h}{d_1+h}x_1^*$$

and

The stability in this case

$$j_{11}=r-\frac{2x_1^*}{k}-\beta_0\frac{2b\,x_1^*z_1^*}{\left(b+x_1^{*2}\right)^2}-c\frac{y_1^{*2}}{\left(x_1^*+y_1^*\right)^2},\ j_{12}=c\frac{x_1^{*2}}{\left(x_1^*+y_1^*\right)^2},\ j_{13}=-\beta_0\frac{x_1^{*2}}{b+x_1^{*2}}$$

$$j_{21}=c\frac{y_1^{*2}}{\left(x_1^*+y_1^*\right)^2}\ j_{22}=c\frac{x_1^{*2}}{\left(x_1^*+y_1^*\right)^2}-d_1-h,\ j_{31}=\beta_1\frac{2b\,x_1^*z_1^*}{\left(b+x_1^{*2}\right)^2}.\ j_{33}=\beta_1\frac{x_1^{*2}}{\left(b+x_1^{*2}\right)^2}-d_2$$

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0$$

$$A == -(j_{11} + j_{22} + j_{33})$$

$$B = (j_{11}j_{22} + j_{11}j_{33} + j_{22}j_{33} - (j_{12}j_{21} + j_{13}j_{31})$$

$$C = (-j_{11}j_{22}j_{33} + j_{12}j_{21}j_{33} + j_{13}j_{31}j_{22})$$

Then from Routh Hurwitz criteria, this point is stable if $A > 0, C > 0$ and $AB - C > 0$

When H(y) nonlinear function then in this case then the equilibrium pint

$E_6 = (x_2^{\bullet}, y_2^{\bullet}, z_2^{\bullet})$

Where $x_2^{\bullet} = x_1^{\bullet} = x^*$

$y_2^{\bullet} = Q + \sqrt{\frac{Q^2 - 4d_1B}{2d_1}}$ *where* $Q = d_1\rho + d_1x_2^{\bullet} - cx_2^{\bullet} + h$, $U = d_1x_2^{\bullet}\rho - cx_2^{\bullet}\rho + hx_2^{\bullet}$

$Q > 4d_1U$ **with condition**

And $z_2^{\bullet} = z_1^* = z^*$

The Jacobean matrix is

$$j_{11} = r - \frac{2x_2^{\bullet}}{k} - \beta_0 \frac{2bx_2^{\bullet}z_2^{\bullet}}{\left(b + x_2^{\bullet 2}\right)^2} - c\frac{y_2^{\bullet 2}}{\left(x_2^{\bullet} + y_2^{\bullet}\right)^2},\ j_{12} = c\frac{x_2^{\bullet 2}}{\left(x_2^{\bullet} + y_2^{\bullet}\right)^2},\ j_{13} = -\beta_0\frac{x_2^{\bullet 2}}{b + x_2^{\bullet 2}}$$

$$j_{21} = c\frac{y_2^{\bullet 2}}{\left(x_2^{\bullet} + y_2^{\bullet}\right)^2},\ j_{22} = c\frac{x_2^{\bullet 2}}{\left(x_2^{\bullet} + y_2^{\bullet}\right)^2} - d_1 - h,\ j_{31} = \beta_1\frac{2bx_2^{\bullet}z_2^{\bullet}}{\left(b + x_2^{\bullet 2}\right)^2},\ j_{33} = \beta_1\frac{x_2^{\bullet 2}}{\left(b + x_2^{\bullet 2}\right)^2} - d_2$$

The characteristic equations of this point is $\lambda^3 + A\lambda^2 + B\lambda + C = 0$, where $A = -(j_{11} + j_{22} + j_{33})$

$C = -j_{11}j_{22}j_{33} + j_{12}j_{21}j_{33} + j_{13}j_{31}j_{22}$, $B = j_{11}j_{22} + j_{11}j_{33} + j_{22}j_{33} -$

Then from Routh Hurwitz criteria, the boundary point is stable if $A > 0, C > 0$

and $AB - C > 0$

4. Numerical Solution

In this section, we employ Mathematic Programing to show the behavior of each populations of the system without harvesting also with harvesting. After many attempts and after commitment with conditions of existence and stability of equilibrium, we fixed the parameters as $r = k = \rho = 1, \beta_0 = 0.5 > \beta_1 = 0.4, d_1 = 0.14, d_2 = 0.16, c = 0.3$ and $b = 0.5$.

The initial values are $x = 0.9, y = 0.4$ and $z = 0.6$.

First we show the behavior of system (1.1) ended to coexist prey and growth the infected prey, while predator disappearance, figure 1.

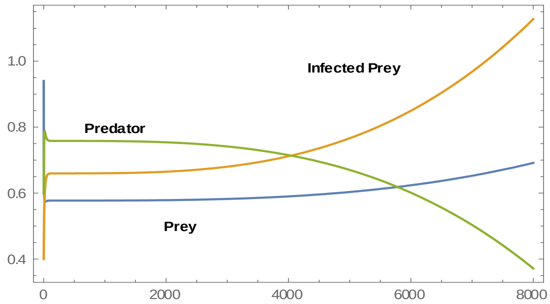

Fig 1

Now we employ the harvesting to show the effect this on the behavior, this in two case. First, when the function of harvesting is nonlinear. Put the parameter $h = 0.08$, we show the harvesting help us to eliminate the disease and it's useful to predator, predator population growth up with prey, figure 2.

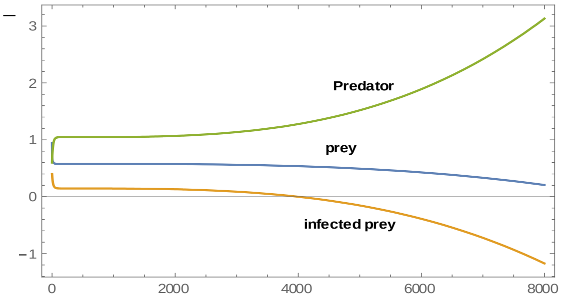

Fig 2

Finally, the linear function of harvesting with same parameter, we obtained same result but the different is the predator disappears slower than the nonlinear function of harvesting, figure 3.

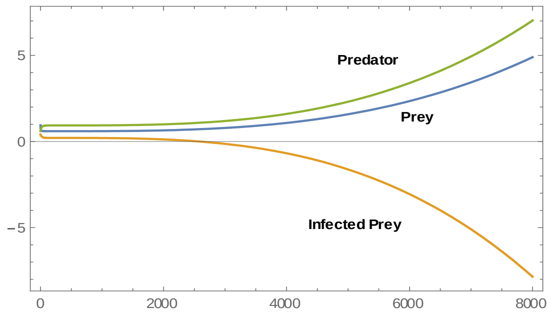

Fig 3

Now we can say, use the harvesting (linear function) to control to the disease and not to transform to epidemic. The optimal harvesting may be use the nonlinear function at least in such model.

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ديناميكية نموذج الفريسة والمفترس واشكال حصاد متعدد للفريسة المصابة

سفيان عباس وهيب ، مصطفى حاتم منصور

قسم الرياضيات ، كلية علوم الحاسوب والرياضيات ، جامعة تكريت ، تكريت ، العراق

الملخص

في هذا البحث، تم مناقشة ديناميكية نموذج الفريسة والمفترس عندما تكون دالة الاستجابة من النوع الثالث بينهما. بالإضافة إلى ذلك، عندما تتعرض الفريسة للمرض على شكل دالة غير خطية . كما تتعرض الفريسة المصابة للحصاد كدالة غير خطية تارة وتارة اخرى الى حصاد كدالة خطية. ايضا تم مناقشة الحلول المقيدة والموجبة، والسلوك الدوري وايجاد نقاط الاتزان وشروطها وشروط استقراريتها. بعض النتائج كانت قد تم توضيحها في المحاكاة العددية, حيث وجدنا ان الحصاد عندما يكون دالة خطية يمكن السيطرة على المرض.