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# On $\mathcal{S}$-normed spaces

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## ABSTRACT

The study focused on expanding the concept of 2-normed spaces by developing a new definition ($\mathcal{S}$-normed space), and the study concentrated on the convergent of sequences and Cauchy sequences in our definition, as well as some other branches such as linear transformation and contraction.

## 1. Introduction

The concept of linear 2-normed spaces was introduced by Siegfried Gähler [1, 2, 3] in 1960, a German mathematician who worked in German academy of science, Berlin. Published this concept in the series of papers in German language, this subject has been studied by many mathematicians: A. White, Y J Cho, R W Freese and others who contributed to expanionality of this branch of mathematics since many researchers and scientists have obtained various results in this space, later the theory of 2-normed space was generalized and developed by S.Gähler then they trying to expand this generalization as well as many other subjects.

The study also give special attention to other results on this subject [4, 5, 6, 7, 8].

## 2. Preliminaries

In this part, the study dealt with two basic definitions as well as some properties are being focused on, each supported by an example for clarification .

**Definition 2.1. [9]** Let $\mathcal{X}$ be a vector space with $\dim\mathcal{X} > 1$, over the field F, where F is the field of real or complex numbers. The real valued function $\|.,.\|: \mathcal{X}\times\mathcal{X} \to \mathcal{R}$ which satisfying these conditions for all $x, y, z \in \mathcal{X}$ and $\alpha \in$ F:

1. $\| x,y \| = 0$ if and only if $x$ and $y$ is linearly dependent.
2. $\| x,y \| = \| y,x \|$.
3. $\| x,\alpha y \| = |\alpha| \, \| x,y \|$.
4. $\| x ,y + z \| \le \| x,y \| + \| x, z \|$.

Is called a 2-norm on $\mathcal{X}$ and the pair $(\mathcal{X}, \|.,.\|)$ is said to be a 2-normed space over the field F .

Note in any 2-normed space $(\mathcal{X}, \|.,.\|)$ the 2-norm $\|.,.\|$ is non-negative .

classical example of the 2-normed space $(\mathcal{R}^2, \|.,.\|)$ that 2-norm $\|.,.\|$ on $\mathcal{R}^2$ is defined by $\| x,y \| = |x_1 y_2 - y_1 x_2|$ where $x = (x_1, x_2)$ , $y = (y_1, y_2) \in \mathcal{R}^2$. This is the area of parallelogram determined by the vectors $x$ and $y$ .

**Proposition 2.2. [9]** Let $(\mathcal{X}, \|.,.\|)$ be a 2-normed space over the field F. Then

1. $\| x, y \| = \| x, y + \alpha x \|$ , $\forall x, y \in \mathcal{X}$ and $\alpha \in$ F;
2. If $y$ and $z$ are linearly independent in $\mathcal{X}$ and $\| x, y \| = \| x, z \| = 0$ $\forall x \in \mathcal{X}$, then $x = 0$ .

**Definition 2.3. [10]** Let $n \in \mathcal{N}$ and $\mathcal{X}$ be real vector space such that $\dim \ge n$ . A real valued function $\| ., \dots, . \|$ on $\mathcal{X}^n$ which satisfying the following four properties

1. $\| x_1 ,\dots, x_n \| = 0$ if and only if $x_1 ,\dots, x_n$ are linearly dependent;
2. $\| x_1 ,\dots, x_n \|$ is invariant under permutation;
3. $\| x_1,\dots, \alpha x_n \| = |\alpha| \, \| x_1 ,\dots, x_n \|$ for any $\alpha \in \mathcal{R}$;
4. $\| x_1 ,\dots, x_{n-1}, y + z \| \le \| x_1 ,\dots, x_{n-1}, y \| + \| x_1 ,\dots, x_{n-1}, z \|$,

Is called an n-norm on $\mathcal{X}$ and the pair $(\mathcal{X}, \|., \dots, .\|)$ is called an $n$-normed space .

Note that in an n-normed space $(\mathcal{X}, \|., \dots, .\|)$, we have, for instance, $\| x_1 ,\dots, x_n \| \ge 0$ and $\| x_1$

$, \dots, x_{n-1}, x_n \parallel = \parallel x_1, \dots, x_{n-1}, x_n + \alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1} \parallel$. For all $x_1, \dots, x_n \in \mathcal{X}$ and $\alpha_1, \dots, \alpha_{n-1} \in \mathcal{R}$

A trivial example of $n$-normed space is $\mathcal{X} = \mathcal{R}^n$ equipped with the following $n$-norm:

$$\parallel x_1, \dots, x_n \parallel_E = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right).$$

Where $x_i = (x_{i1}, \dots, x_{in}) \in \mathcal{R}^n$, for each $i = 1, \dots, n$. (The subscript $E$ is for Euclidean) .

## 3. Definitions and characterizations

The study also presents a new concept,($\mathcal{S}$-normed space), where we studied the convergent of sequence and Cauchy sequence in $\mathcal{S}$-normed space applied followed by some possible cases .

**Definition 3.1.** Let $\mathcal{X}$ be a vector space with dim$\mathcal{X}$>1 over the field F, where F is the field of real or complex numbers, ℮ $= \{\mathcal{S} \subseteq \mathcal{X} : \mathcal{S}$ is finite and independent set$\}$.

The real valued function $\parallel .,. \parallel : \mathcal{X} \times$ ℮ $\to \mathcal{R}$ which satisfying these conditions $\forall x, y \in \mathcal{X}$, $\mathcal{S} \in$ ℮ and $\alpha \in$ F:

1. $\parallel x, \mathcal{S} \parallel = 0$ if and only if there exist $y \in \mathcal{S}$ such that $x, y$ are linearly dependent.
2. $\parallel \alpha x, \mathcal{S} \parallel = |\alpha| \parallel x, \mathcal{S} \parallel$.
3. $\parallel x + y, \mathcal{S} \parallel \leq \parallel x, \mathcal{S} \parallel + \parallel y, \mathcal{S} \parallel$,

is called $\mathcal{S}$-norm on $\mathcal{X}$, and the pair $(\mathcal{X}, \parallel .,. \parallel)$ is said to be $\mathcal{S}$-normed space, over the field F .

The researcher is supporting his definition by this example define, when $\mathcal{X}$ be 2-normed space and ℮ $= \{\mathcal{S} \subseteq \mathcal{X} : \mathcal{S}$ is finite and independent set$\}$, the $\mathcal{S}$-norm defined as : $\parallel x, \mathcal{S} \parallel = \min\{\parallel x, y \parallel : y \in \mathcal{S}\}$ Then $\mathcal{X}$ is $\mathcal{S}$-normed space.

**Proposition 3.2.** The $\mathcal{S}$-normed space has the following properties:

$| \parallel x, \mathcal{S} \parallel - \parallel y, \mathcal{S} \parallel | \leq \parallel x - y, \mathcal{S} \parallel, \forall x, y \in \mathcal{X}$ and $\mathcal{S} \in$ ℮

Proof:

$\parallel x, \mathcal{S} \parallel = \parallel (x - y) + y, \mathcal{S} \parallel \leq \parallel x - y, \mathcal{S} \parallel + \parallel y, \mathcal{S} \parallel$

$\Rightarrow \parallel x, \mathcal{S} \parallel - \parallel y, \mathcal{S} \parallel \leq \parallel x - y, \mathcal{S} \parallel$, ... (1)

$\parallel y, \mathcal{S} \parallel = \parallel -y, \mathcal{S} \parallel = \parallel x - y - x, \mathcal{S} \parallel \leq \parallel x - y, \mathcal{S} \parallel + \parallel x, \mathcal{S} \parallel$

$\Rightarrow \parallel y, \mathcal{S} \parallel - \parallel x, \mathcal{S} \parallel \leq \parallel x - y, \mathcal{S} \parallel$

$\Rightarrow \parallel x, \mathcal{S} \parallel - \parallel y, \mathcal{S} \parallel \geq - \parallel x - y, \mathcal{S} \parallel$, ... (2)

From (1) and (2) we get :

$- \parallel x - y, \mathcal{S} \parallel \leq \parallel x, \mathcal{S} \parallel - \parallel y, \mathcal{S} \parallel \leq \parallel x - y, \mathcal{S} \parallel$

$\Rightarrow | \parallel x, \mathcal{S} \parallel - \parallel y, \mathcal{S} \parallel | \leq \parallel x - y, \mathcal{S} \parallel, \forall x, y \in \mathcal{X}$ and $\mathcal{S} \in$ ℮ .

**Definition 3.3**. Let $\mathcal{X}$ be $\mathcal{S}$-normed space then the sequence $\{x_n\}$ in $\mathcal{X}$ is convergent to $x \in \mathcal{X}$ if:

$\forall \epsilon > 0, \exists \mathcal{K} \in \mathcal{N}$, such that $\forall n > \mathcal{K} : \parallel x_n - x, \mathcal{S} \parallel < \epsilon$ , for every $\mathcal{S} \in$ ℮.

If $\{x_n\}$ is convergent to $x$ , we write : $\{x_n\} \longrightarrow x$ .

**Lemma 3.4.** Let $\mathcal{X}$ be $\mathcal{S}$-normed space and $\{x_n\}$,$\{y_n\}$ be sequences in $X$, then :

1. If, $\{x_n\} \longrightarrow x$ and $\{y_n\} \longrightarrow y$, then $\{x_n + y_n\} \longrightarrow x + y$.
2. If, $\{x_n\} \longrightarrow x$, then $\{\alpha x_n\} \longrightarrow \alpha x$ .

Proof:

1. $\because \{x_n\} \longrightarrow x$,

$\therefore \forall \epsilon > 0, \exists \mathcal{K}_1 \in \mathcal{N}$, such that $\forall n > \mathcal{K}_1 : \parallel x_n - x, \mathcal{S} \parallel < \frac{\epsilon}{2}$, for every $\mathcal{S} \in$ ℮

$\because \{y_n\} \longrightarrow y$,

$\therefore \forall \epsilon > 0, \exists \mathcal{K}_2 \in \mathcal{N}$, such that $\forall n > \mathcal{K}_2 : \parallel y_n - y, \mathcal{S} \parallel < \frac{\epsilon}{2}$, for every $\mathcal{S} \in$ ℮

Take $\mathcal{K} = \max\{\mathcal{K}_1, \mathcal{K}_2\}$

Then,$\forall n > \mathcal{K}$: $\parallel (x_n + y_n) - (x + y), \mathcal{S} \parallel = \parallel (x_n - x) + (y_n - y), \mathcal{S} \parallel$

$\leq \parallel x_n - x, \mathcal{S} \parallel + \parallel y_n - y, \mathcal{S} \parallel$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, for every $\mathcal{S} \in$ ℮

$\therefore \{x_n + y_n\} \longrightarrow x + y$.

2. $\because \{x_n\} \longrightarrow x$,

$\therefore \forall \epsilon > 0, \exists \mathcal{K} \in \mathcal{N}$, such that $\forall n > \mathcal{K}$: $\parallel x_n - x, \mathcal{S} \parallel < \frac{\epsilon}{|\alpha|}$ , for every $\mathcal{S} \in$ ℮

We have : $\forall n > \mathcal{K}$, $\parallel \alpha x_n - \alpha x, \mathcal{S} \parallel = |\alpha| \parallel x_n - x, \mathcal{S} \parallel < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon$, for every $\mathcal{S} \in$ ℮

$\therefore \{\alpha x_n\} \longrightarrow \alpha x$ .

The following theorem shows that if the sequence is converge then the convergence point is unique.

**Theorem 3.5.** If $\{x_n\}$ is convergent sequence in $\mathcal{S}$-normed space $\mathcal{X}$, then the convergent point, is unique .

Proof :

Suppose $\{x_n\}$ is convergent to, $x$ and $y$ in $\mathcal{S}$-normed space $\mathcal{X}$ such that $x \neq y$

Since $\{x_n\} \longrightarrow x$ ,

then $\forall \varepsilon > 0, \exists \mathcal{K}_1 \in \mathcal{N}$ such that $\forall n > \mathcal{K}_1 : \parallel x_n - x, \mathcal{S} \parallel < \frac{\varepsilon}{2}, \forall \mathcal{S} \in$ ℮

Since $\{x_n\} \longrightarrow y$

$\Rightarrow \forall \varepsilon > 0, \exists \mathcal{K}_2 \in \mathcal{N}$ such that $\forall n > \mathcal{K}_2 : \parallel x_n - y, \mathcal{S} \parallel < \frac{\varepsilon}{2}, \forall \mathcal{S} \in$ ℮

For every $z \neq 0$ take $\mathcal{S} = \{z\}$ such that $\{x - y\}, z$ are linearly independent ,

So, $\parallel x - y, \mathcal{S} \parallel \neq 0 \Rightarrow \parallel x - y, \mathcal{S} \parallel = \varepsilon > 0$

$\varepsilon = \parallel x - y, \mathcal{S} \parallel = \parallel x - x_n + x_n - y, \mathcal{S} \parallel \leq \parallel x_n - x, \mathcal{S} \parallel + \parallel x_n - y, \mathcal{S} \parallel < \varepsilon$

and that is a contradiction.

So that $x = y$ .

**Definition 3.6**. Let $\mathcal{X}$ be $\mathcal{S}$-normed space, then the sequence $\{x_n\}$ in $\mathcal{X}$ is called Cauchy sequence if :

$\forall \epsilon > 0, \exists \mathcal{K} \in \mathcal{N}$, such that: $\forall n, m > \mathcal{K}, \parallel x_n - x_m, \mathcal{S} \parallel < \epsilon$, for every $\mathcal{S} \in$ ℮ .

**Definition 3.7**. The $\mathcal{S}$-normed space $\mathcal{X}$, is called complete $\mathcal{S}$-normed space, if every Cauchy sequence in $\mathcal{X}$, converges to a point in $\mathcal{X}$.

**Definition 3.8**. The $\mathcal{S}$-normed space $\mathcal{X}$, is said to be $\mathcal{S}$-Banach space if it is complete $\mathcal{S}$-normed space .

**Definition 3.9**. Let $(\mathcal{X}, \parallel .,. \parallel)$ be $\mathcal{S}$-normed space,

Let $x_0 \in \mathcal{X}, r > 0$ *and* $\mathcal{S} \in$ ℮, the set $B_{\{\mathcal{S}\}}(r, x_0) = \{x \in \mathcal{X} : \parallel x - x_0, \mathcal{S} \parallel < r\}$.

We call it an open ball with respect to $\mathcal{S}$ with center $x_0$ and redius $r$.

**Definition 3.11**. Let $\mathcal{X}$ be $\mathcal{S}$-normed space, the sequence $\{x_n\}$ in $\mathcal{X}$ is called $\mathcal{S}$-bounded if:

$\forall \mathcal{S} \in \mathbb{e}, \exists \mathcal{M}_\mathcal{S} > 0$ such that $\| x_n, \mathcal{S} \| \leq \mathcal{M}_\mathcal{S}, \forall n$.

**Lemma3.12**. Every Cauchy sequence in $\mathcal{S}$-normed space is $\mathcal{S}$-bounded.

Proof : Suppose $\{x_n\}$ is Cauchy sequence and take $\epsilon = 1$

then $\exists\ \mathcal{K} \in \mathcal{N}$ such that $\forall n > \mathcal{K}\ \| x_n - x_m, \mathcal{S} \| < 1, \forall \mathcal{S} \in \mathbb{e}$

let $m = k+1 \Rightarrow \forall n > \mathcal{K}\!:\ \| x_n - x_{k+1}, \mathcal{S} \| < 1$

since $\forall n > \mathcal{K}, \| x_n, \mathcal{S} \| - \| x_{k+1}, \mathcal{S} \| \leq \| x_n - x_{k+1}, \mathcal{S} \| < 1$

$\Longrightarrow \forall n > \mathcal{K}\!:\ \| x_n, \mathcal{S} \| < 1 + \| x_{k+1}, \mathcal{S} \|$

Take $\mathcal{M}_\mathcal{S} = Max\{\| x_1, \mathcal{S} \|, \| x_2, \mathcal{S} \|, \dots, \| x_k, \mathcal{S} \|, 1 + \| x_{k+1}, \mathcal{S} \|\}$

$\Longrightarrow \| x_n, \mathcal{S} \| < \mathcal{M}_\mathcal{S}, \forall\, n \in \mathcal{N}$ .

$\Longrightarrow \{x_n\}$ is $\mathcal{S}$-bounded.

**Theorem 3.13**. Every convergent sequences in $\mathcal{S}$-normed space is Cauchy sequence

Proof: Let $\{x_n\}$ be a convergent sequence, in $\mathcal{S}$-normed space, such that $\{x_n\} \longrightarrow x$,

Let $> 0$ , since $\{x_n\} \longrightarrow x$ then $\exists\, \mathcal{K} \in \mathcal{N}$ such that $\forall n > \mathcal{K}\!:\ \| x_n - x, \mathcal{S} \| < \frac{\epsilon}{2}$ ,

$\forall \mathcal{S} \in \mathbb{e}$, for $m > \mathcal{K}\!:\ \| x_m - x, \mathcal{S} \| < \frac{\epsilon}{2}, \forall \mathcal{S} \in \mathbb{e}$

Since $\forall n, m > \mathcal{K}\!: \| x_n - x_m, \mathcal{S} \| < \| x_n - x, \mathcal{S} \| + \| x_m - x, \mathcal{S} \| < \mathcal{E}, \forall \mathcal{S} \in \mathbb{e}$

So $\{x_n\}$, is Cauchy sequence in $\mathcal{S}$-normed space .

The prove of the following corollary consequence from lemma (3.12) and lemma (3.13).

**Corollary3.14**. Every convergent sequence in $\mathcal{S}$-normed space is bounded .

## 4. Linear transformation on $\mathcal{S}$-normed spaces

In this part of the study, the linear transformation and contraction mapping is being defined in $\mathcal{S}$-normed space and with a study of several propositions.

**Definition 4.1**. Let, $\mathcal{X}$ be $\mathcal{S}$-normed space and $\mathcal{Y}$ is normed space, then

$\mathcal{F}\!: \mathcal{X} \times \mathbb{e} \longrightarrow \mathcal{Y}$ , is called $\mathcal{S}$-linear transformation if it satisfies the conditions:

$\forall\, x \in \mathcal{X}, \forall \mathcal{S} \in \mathbb{e}$ and $\alpha\ \in$ F;

1. $\mathcal{F}(x + y, \mathcal{S}) = \mathcal{F}(x, \mathcal{S}) + \mathcal{F}(y, \mathcal{S})$,
2. $\mathcal{F}(\alpha x, \mathcal{S}) = \alpha\, \mathcal{F}(x, \mathcal{S})$.

**Definition 4.2**. A linear $\mathcal{S}$-normed space $\mathcal{F}$ is said to be bounded, if $\exists$ M$> 0$ such that $\| \mathcal{F}_{(x,\mathcal{S})} \| \leq M \| x, \mathcal{S} \|, \forall\, x \in \mathcal{X}, \forall \mathcal{S} \in \mathbb{e}$.

**Propoition4.3**. Let $\mathcal{X}$ be $\mathcal{S}$-normed space and $\mathcal{F}\!: \mathcal{X} \times \mathbb{e} \longrightarrow \mathcal{Y}$ is bounded $\mathcal{S}$-linear transformation. Let $\mathcal{S}$ belong to $\mathbb{e}, x \in \mathcal{X}$. If there exist $y \in \mathcal{S}$ such that $x, y$ are linraly dependent then $(x, \mathcal{S}) \in ker\, \mathcal{F}$ , where $ker\, \mathcal{F} = \{(x, \mathcal{S}) : \mathcal{F}(x, \mathcal{S}) = 0\}$.

Proof:

Since $\mathcal{F}$ is bounded

Then there exist $\mathcal{M} > 0$ such that $\| \mathcal{F}(x, \mathcal{S}) \| \leq \mathcal{M} \| x, \mathcal{S} \|$

since there exit $y \in \mathcal{S}$, such that $x$ and $y$ are linearly dependent

$\Longrightarrow \| x, \mathcal{S} \| = 0$

Hence, $0 \leq \| \mathcal{F}(x, \mathcal{S}) \| \leq \mathcal{M} \| x, \mathcal{S} \| = \mathcal{M}.0 = 0$

Therefore, $\mathcal{F}(x, \mathcal{S}) = 0$

Thus, $(x, \mathcal{S}) \in ker\, \mathcal{F}$.

Let $\mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$ be the set of all bounded $\mathcal{S}$-linear transformation on the $\mathcal{S}$-normed space $\mathcal{X} \times \mathbb{e}$.

We can define over $\mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$ operations (+) and ( . ) in this way

For $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$

$(\mathcal{F}_1 + \mathcal{F}_2)_{(x,\mathcal{S})} = \mathcal{F}_1(x, \mathcal{S}) + \mathcal{F}_2(x, \mathcal{S}), \forall\, x \in \mathcal{X}$ and $\forall \mathcal{S} \in \mathbb{e}$.

$(\alpha.\mathcal{F}_1)_{(x,\mathcal{S})} = \alpha.\mathcal{F}_1(x, \mathcal{S}), \forall x \in \mathcal{X}, \forall \mathcal{S} \in \mathbb{e}$, and $\alpha \in F$

**Propoition 4.4.** The set $\mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$ with two operation defined above is vector space

Proof: Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y}) \Rightarrow \exists\, \mathcal{M}_1, \mathcal{M}_2 > 0$ such that $\| \mathcal{F}_1(x, \mathcal{S}) \| \leq \mathcal{M}_1 \| x, \mathcal{S} \|$

And $\| \mathcal{F}_2(x, \mathcal{S}) \| \leq \mathcal{M}_2 \| x, \mathcal{S} \|$

Then,

$\| (\mathcal{F}_1 + \mathcal{F}_2)_{(x,\mathcal{S})} \| = \| \mathcal{F}_1(x, \mathcal{S}) + \mathcal{F}_2(x, \mathcal{S}) \| \leq \| \mathcal{F}_1(x, \mathcal{S}) \| + \| \mathcal{F}_2(x, \mathcal{S}) \|$

$\leq (\mathcal{M}_1 + \mathcal{M}_2) \| x, \mathcal{S} \|$

$\| (\alpha.\mathcal{F}_1)_{(x,\mathcal{S})} \| = \| \alpha.\mathcal{F}_1(x, \mathcal{S}) \| = |\alpha| \| \mathcal{F}_1(x, \mathcal{S}) \|$

$\leq |\alpha|\, \mathcal{M}_1 \| x, \mathcal{S} \|$

So $\mathcal{F}_1 + \mathcal{F}_2 \in \mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$ and $\alpha, \mathcal{F}_1 \in \mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$

By the same way in proposition4.4 we can prove other conditions of vector space.

**Propoition4.5.** Let $\mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$ be the set of all bounded $\mathcal{S}$-linear transformation, for every $F \in \mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$ , then $\mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$ is normed space where,

$\| \mathcal{F} \| = sup_{y \in \mathcal{S}} \{ \frac{\| \mathcal{F}(x,\mathcal{S}) \|}{\| x,\mathcal{S} \|} :$ $x, y$ are linearly independent }

we call $\| \mathcal{F} \|$ the norm of bounded $\mathcal{S}$-linear transformation $\mathcal{F}$.

Proof: The vector space $\mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$ with the function $\| . \|\!: \mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y}) \rightarrow \mathcal{R}$, is normed space because it satisfies :

1. $\| \mathcal{F} \| \geq 0$

$\because \| x, \mathcal{S} \| > 0, \forall y \in \mathcal{S}$ , $x, y$ are linearly independent and $\| F(x, \mathcal{S}) \| \geq 0, \forall y \in \mathcal{S}$, $x \in \mathcal{X}$.

$\therefore \frac{\| \mathcal{F}(x,\mathcal{S}) \|}{\| x,\mathcal{S} \|} \geq 0$ , $\forall\ y \in \mathcal{S}$ , $x, y$ are linearly independent

$\therefore sup_{y \in \mathcal{S}} \{ \frac{\| \mathcal{F}(x,\mathcal{S}) \|}{\| x,\mathcal{S} \|} : x, y$ are linearly independent$\} \geq 0$

$\therefore \| \mathcal{F} \| \geq 0, \forall \mathcal{F} \in \mathcal{B}(\mathcal{X} \times \mathbb{e}, \mathcal{Y})$

2. $\| \mathcal{F} \| = 0$ if and only if $\mathcal{F}=0$

Let $\| \mathcal{F} \| = 0 \Leftrightarrow sup_{y \in \mathcal{S}} \{ \frac{\| \mathcal{F}(x,\mathcal{S}) \|}{\| x,\mathcal{S} \|} : x, y$ are linearly independent$\} = 0$

$\Leftrightarrow \frac{\| \mathcal{F}(x,\mathcal{S}) \|}{\| x,\mathcal{S} \|} = 0$ $x, y$ are linearly independent, $\forall y \in \mathcal{S}$,

$\because \| x, \mathcal{S} \| > 0$ $x, y$ are linearly independent, $\forall y \in \mathcal{S}$,

$\Leftrightarrow \| \mathcal{F}(x, \mathcal{S}) \| = 0$ , $x, y$ are linearly independent, $\forall y \in \mathcal{S}$,

$\Leftrightarrow \mathcal{F}(x,\mathcal{S}) = 0$ , $x,y$ are linearly independent, $\forall y \in \mathcal{S}$,
$\Leftrightarrow \mathcal{F}=0$ .

3. $\forall \alpha \in$ F, $\| \alpha\mathcal{F} \| = |\alpha| \, \| \mathcal{F} \|$

$\| \alpha F \| = sup_{y\in\mathcal{S}} \{ \frac{\|(\alpha\mathcal{F})(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$ : $x,y$ are linearly independent }

$= sup_{y\in\mathcal{S}} \{ \frac{\|\alpha\mathcal{F}(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$ : $x,y$ are linearly independent }

$= sup_{y\in\mathcal{S}} \{ \frac{|\alpha| \, \|\mathcal{F}(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$ : $x,y$ are linearly independent }

$= |\alpha| \; sup_{y\in\mathcal{S}} \{ \frac{\|\mathcal{F}(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$ : $x,y$ are linearly independent }

$= |\alpha| \, \| \mathcal{F} \|$ .

4. $\| \mathcal{F}_1 + \mathcal{F}_2 \| \leq \| \mathcal{F}_1 \| + \| \mathcal{F}_2 \|$ , $\forall \ \mathcal{F}_1$and $\mathcal{F}_2 \in \mathcal{B}(\mathcal{X}\times$ e$,\mathcal{Y})$.

$\| \mathcal{F}_1 + \mathcal{F}_2 \| = sup_{y\in\mathcal{S}}\{\frac{\|(\mathcal{F}_1+\mathcal{F}_2)(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$ : $x,y$ are linearly independent}

$= sup_{y\in\mathcal{S}}\{\frac{\|\mathcal{F}_1(x,\mathcal{S}) + \mathcal{F}_2(x,\mathcal{S})\|}{\| x,\mathcal{S} \|}$ : $x,y$ are linearly independent}

$\leq sup_{y\in\mathcal{S}}\{\frac{\|\mathcal{F}_1(x,\mathcal{S})\|+\|\mathcal{F}_2(x,\mathcal{S})\|}{\| x,\mathcal{S} \|}$ : $x,y$ are linearly independent}

$= sup_{y\in\mathcal{S}}\{\frac{\|\mathcal{F}_1(x,\mathcal{S})\|}{\|x,\mathcal{S}\|} + \frac{\|\mathcal{F}_1(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$ : $x,y$ are linearly independent}

$\leq sup_{y\in\mathcal{S}}\{\frac{\|\mathcal{F}_1(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$: $x,y$ are linearly independent}+ $sup_{y\in\mathcal{S}}\{\frac{\|\mathcal{F}_2(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$: $x,y$ are linearly independent}

$\therefore \| \mathcal{F}_1 + \mathcal{F}_2 \| \leq \| \mathcal{F}_1 \| + \| \mathcal{F}_2 \|$.

**Proposition 4.6.** Let $\mathcal{X}$ be $\mathcal{S}$-normed space and $\mathcal{Y}$ be Banach space, then $\mathcal{B}(\mathcal{X}\times \mathrm{e},\mathcal{Y})$ is a Banach space .

Proof : let, $\{\mathcal{F}_n\}$ be Cauchy sequence in $\mathcal{B}(\mathcal{X} \times \mathrm{e},\mathcal{Y})$, and $m = n + p$, where $p \in \mathcal{N}$

$\Rightarrow \ \forall \mathcal{E} > 0$ , $\exists \ \mathcal{K} \in \mathcal{N}$, such that $\forall n,m > \mathcal{K}$: $\| \mathcal{F}_n - \mathcal{F}_m, \mathcal{S} \| < \mathcal{E}$ , $\forall \mathcal{S} \in \mathrm{e}$.

Then, $\forall x \in \mathcal{X}$, and $\mathcal{S} \in \mathrm{e}$ such that $x,y$ are linearly independent $\forall y \in \mathrm{e}$

We have $\| \mathcal{F}_n(x,\mathcal{S}) + \mathcal{F}_{n+p}(x,\mathcal{S}) \| \leq \| \mathcal{F}_n - \mathcal{F}_m, \mathcal{S} \| \, \| x,\mathcal{S} \|$

$\therefore$ we get $\{\mathcal{F}_n(x,\mathcal{S})\}$ is Cauchy sequence in $\mathcal{Y}$

Since $\mathcal{Y}$ is complete then the sequence $\{\mathcal{F}_n(x,\mathcal{S})\}$ is convergent to a pont in $\mathcal{Y}$ say $\mathcal{F}(x,\mathcal{S})$

$\therefore \{\mathcal{F}_n(x,\mathcal{S})\} \longrightarrow \mathcal{F}(x,\mathcal{S})$

Now, we will show that $\{\mathcal{F}_n\} \longrightarrow \{\mathcal{F}\}$

$\| (\mathcal{F}_n - \mathcal{F})_{(x,\mathcal{S})} \| = \| \mathcal{F}_n(x,\mathcal{S}) - \mathcal{F}(x, \mathcal{S})\| = \lim_{p\to\infty} \| \mathcal{F}_n(x,\mathcal{S}) + \mathcal{F}_{n+p}(x,\mathcal{S}) \|$

$\lim_{p\to\infty} \| \mathcal{F}_n - \mathcal{F}_{n+p}, \mathcal{S} \| \, \| x,\mathcal{S} \| < \mathcal{E} \, \| (x,\mathcal{S}) \|$

$\therefore \ x \in \mathcal{X}$ and $\mathcal{S} \in \mathrm{e}$ such that $x,y$ are linearly independent $\forall y \in \mathrm{e}$

$\Longrightarrow \frac{\|(\mathcal{F}_n+\mathcal{F})(x,\mathcal{S})\|}{\|x,\mathcal{S}\|} \leq \mathcal{E}$

$\therefore \ \| \mathcal{F}_n - \mathcal{F} \| = sup_{y\in\mathcal{S}}\{\frac{\|(\mathcal{F}_n+\mathcal{F})(x,\mathcal{S})\|}{\|x,\mathcal{S}\|}$ : $x,y$ are linearly independent} $\leq \mathcal{E}$

$\therefore \{\mathcal{F}_n\} \longrightarrow \{\mathcal{F}\}$.

We will show that $\mathcal{F}$ is bounded and $\mathcal{S}$-linear transformation

$\because \{\mathcal{F}_n\}$ is Cauchy sequence

$\therefore \exists \mathcal{M} > 0$, such that , $\| \mathcal{F}_n \| < \mathcal{M}, \forall n.$

$\| \mathcal{F}(x,\mathcal{S}) \| = \| \left((\mathcal{F} - \mathcal{F}_n) + \mathcal{F}_n\right)(x,\mathcal{S}) \| \leq \| (\mathcal{F} - \mathcal{F}_n)_{(x,\mathcal{S})} \| + \| \mathcal{F}_n(x,\mathcal{S}) \|$

$\leq \| \mathcal{F}_n - \mathcal{F}_m \| \; \|x,\mathcal{S}\| + \| \mathcal{F}_n \| \, \|x,\mathcal{S}\| \leq (\mathcal{E} + \mathcal{M})\|x,\mathcal{S}\|$.

$\therefore \mathcal{F}(x,\mathcal{S})$ is bounded.

$\because \{\mathcal{F}_n(x,\mathcal{S})\} \longrightarrow \mathcal{F}(x,\mathcal{S})$, $\forall x \in \mathcal{X}$ and $\forall \mathcal{S} \in \mathrm{e}$

Then $\mathcal{F}(x + y,\mathcal{S}) = \lim_{n\to\infty} \mathcal{F}_n(x + y,\mathcal{S})$

$= \lim_{n\to\infty}(\mathcal{F}_n(x,\mathcal{S}) + \mathcal{F}_n(y,\mathcal{S})) = \mathcal{F}(x,\mathcal{S}) + \mathcal{F}(y,\mathcal{S})$

And $\mathcal{F}(\alpha x,\mathcal{S}) = \lim_{n\to\infty} \mathcal{F}_n(\alpha x,\mathcal{S}) = \lim_{n\to\infty} \alpha\mathcal{F}_n(x,\mathcal{S}) = \alpha\mathcal{F}(x,\mathcal{S})$.

$\therefore \mathcal{F}$ is $\mathcal{S}$-linear transformation.

$\therefore \mathcal{B}(\mathcal{X}\times \mathrm{e},\mathcal{Y})$, is Banach space.

**Definition 4.7.** Let $\mathcal{X}$ be $\mathcal{S}$-normed space, the mapping $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ is called $\mathcal{S}$-contraction, if there exist: $\mathcal{C} \in [0,1)$ such that, $\| \mathcal{T}x - \mathcal{T}y, \mathcal{S} \| \leq \mathcal{C} \, \| x - y, \mathcal{S} \|$ , for all $x \in \mathcal{X}$ and $\mathcal{S} \in \mathrm{e}$ .

**Definition 4.8.** Let $\mathcal{X}$ be $\mathcal{S}$-normed space, the mapping $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ is called sequentially continuous if :

$x_n \longrightarrow x$ then $\mathcal{T}_{x_n} \longrightarrow x$ .

**Lemma 4.9.** Let,$\mathcal{X}$ be $\mathcal{S}$-normed space, then every contraction $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$, is sequentially continuous .

Proof : since, $\mathcal{T}$ is contraction

Then, $\exists \ \mathcal{C} \in [0,1)$ such that: $\| \mathcal{T}x - \mathcal{T}y, \mathcal{S} \| \leq \mathcal{C} \, \| x - y, \mathcal{S} \|$ , for all $x,y \in \mathcal{X}$ and $\mathcal{S} \in \mathrm{e}$

Let, $\{x_n\}$ be a sequence in $\mathcal{X}$, such that: $\{x_n\} \longrightarrow x$, then

$\| \mathcal{T}x_n - \mathcal{T}x, \mathcal{S} \| \leq \mathcal{C} \, \| x_n - x, \mathcal{S} \| \longrightarrow 0$ , as $n \longrightarrow \infty$

$\Longrightarrow \mathcal{T}x_n \longrightarrow \mathcal{T}x$.

$\Longrightarrow \mathcal{T}$ is sequentially continuous .

**Lemma 4.10** Let $(\mathcal{X}, \|.,.\|)$ be $\mathcal{S}$-Banach, the $\mathcal{S}$-contraction map $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ has a unique fixed point in $\mathcal{X}$. .

Proof : $\because \mathcal{T}$ is contraction,

$\therefore \exists \ \mathcal{C} \in [0,1)$ such that: $\| \mathcal{T}_x - \mathcal{T}_y, \mathcal{S} \| \leq \mathcal{C} \, \| x - y, \mathcal{S} \|$ , for all $x,y \in \mathcal{X}$ and $\forall \mathcal{S} \in \mathrm{e}$

Similarly, $\| \mathcal{T}_x^n - \mathcal{T}_y^n, \mathcal{S} \| \leq \mathcal{C}^n \, \| x - y, \mathcal{S} \|$, for all $x \in \mathcal{X}$ and $\mathcal{S} \in \mathrm{e}$, $n = 1,2,3, \ldots$

Let $x_0 \in \mathcal{X}$ and $x_n = \mathcal{T}_{x_{n-1}} \Longrightarrow x_n = \mathcal{T}^n{}_{x_0}$

To show that $\{x_n\}$ is Cauchy in $\mathcal{X}$, we take $= n + p$ , $\forall \mathcal{S} \in \mathrm{e}$, where $p \in \mathcal{N}$

$\| x_n - x_m, \mathcal{S} \| = \| x_n - x_{n+p}, \mathcal{S} \|$

$\leq \| x_n - x_{n+1}, \mathcal{S} \| + \| x_{n+1} - x_{n+2}, \mathcal{S} \| + \ldots + \| x_{m+p-1} - x_{m+p}, \mathcal{S} \|$

$= \| \mathcal{T}_{x_0}^n - \mathcal{T}_{x_1}^n, \mathcal{S} \| + \| \mathcal{T}_{x_0}^{n+1} - \mathcal{T}_{x_1}^{n+1}, \mathcal{S} \| + \cdots + \| \mathcal{T}_{x_0}^{n+p-1} - \mathcal{T}_{x_1}^{n+p-1}, \mathcal{S} \|$

$\leq \mathcal{C}^n \, \| x_0 - x_1, \mathcal{S} \| + \mathcal{C}^{n+1} \, \| x_0 - x_1, \mathcal{S} \| + \cdots + \mathcal{C}^{n+p-1} \, \| x_0 - x_1, \mathcal{S} \|$

$\leq \mathcal{C}^n \, \| x_0 - x_1, \mathcal{S} \| \, (1 + \mathcal{C} + \mathcal{C}^2 + \cdots)$

$= \frac{c^n}{1-c} \, \| x_0 - x_1, \mathcal{S} \|$ , $\forall \mathcal{S} \in \mathrm{e}$

So, $\| x_n - x_m, \mathcal{S} \| \leq \frac{c^n}{1-c} \, \| x_0 - x_1, \mathcal{S} \| \longrightarrow 0$ , as $n \longrightarrow \infty$ , $\forall \mathcal{S} \in \mathrm{e}$

$\Longrightarrow \| x_n - x_m, \mathcal{S} \| \longrightarrow 0$ , as $n \longrightarrow \infty$, $\forall \mathcal{S} \in \mathrm{e}$

∴ $\{x_n\}$ is Cauchy sequence in $\mathcal{X}$
∵$\mathcal{X}$ is $\mathcal{S}$-Banach
∴ $\{x_n\}$ is converges to a point in $\mathcal{X}$ say $x$.
∵ $\mathcal{T}$ is contraction map then $\mathcal{T}$ is sequentially continuous
∴ $\mathcal{T}_x = lim\ \mathcal{T}_{x_n} = \lim \mathcal{T}_{x_{n+1}} = x$ , as $n \longrightarrow \infty$
∴$\mathcal{T}$ has a fixed point in $\mathcal{X}$.
Now, we will show that a fixed point is unique
Let $y \in \mathcal{X}$ and $y$ be another fixed point of $\mathcal{T}$ such that $y \neq x$
then $\| x - y, \mathcal{S} \| = \| \mathcal{T}_x - \mathcal{T}_y, \mathcal{S} \| \leq \mathcal{C} \| x - y, \mathcal{S} \|$ , when $\mathcal{C} \geq 1$
Since $\mathcal{C} \in [0,1) \Longrightarrow$ contraction
$y = x$
∴ $\mathcal{T}$ has a unique fixed point .

## 5. Conclusion.

In this research, the researcher gives a new definition. They study some topics and characteristics over the new definition, of these cases is convergence of sequences, linear transformations and contraction. One of the results obtained by the researcher is that the set of all bounded $\mathcal{S}$-linear transformation is normed space and the $\mathcal{S}$-contraction map over $\mathcal{S}$-Banach space has a unique fixed point.

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# حول فضاءات $\mathcal{S}$− المعيارية

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**الملخص**
في هذا البحث, قمنا بتوسيع مفهوم الفضاءات 2−المعيارية من خلال وضع تعريف جديد (فضاءات $\mathcal{S}$−المعيارية)، ودراسة تقارب المتتابعات و المتتابعات الكوشيه على تعريف فضاءات $\mathcal{S}$−المعيارية, وكذلك بعض الفروع الأخرى مثل التحويلات الخطية والتحويلات الخطية الانكماشية.