IASJ
المجلات الاكاديمية العراقية

**Tikrit Journal of Pure Science**
*ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)*

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

# On a unique solution of fractional differential system

**Muayyad Mahmood Khalil**
*Department of Mathematics, College of Education for pure Sciences, Tikrit University , Tikrit , Iraq*

https://doi.org/10.25130/tjps.v24i3.381

**ARTICLE INFO.**

**Article history:**
**-Received: 12 / 11 / 2018**
**-Accepted: 5 / 3 / 2019**
**-Available online: / / 2019**

**Keywords:** unique solution, differential system, fractional, Banach, fixed point.

**Corresponding Author:**
**Name:**
**Muayyad Mahmood Khalil**

**E-mail:** medomath80@gmail.com

**Tel:**

## ABSTRACT

The aim of the study is to investigate the existence and uniqueness of solutions for a semi linear fractional differential system via Banach fixed point theorem. The study proved the existence and uniqueness of solution for a fractional differential system with initial conditions by using contraction mapping theorem, existence and uniqueness results are obtained. Some examples are chosen to illustrate the validity of our results.

## 1. Introduction

Suppose we have a fractional differential system of the form

$$D^{\alpha}_{t_0,t}x(t) = Ax(t) + f\left(t, x(t), D^{\beta}_{t_0,t}x(t)\right), t > t_0$$

With initial conditions (ICs) (1.1)

$x(t_0) = x_0, x'(t_0) = x_1$

Where $x(t) \in R^n, x_k \in R^n (k = 0,1), A \in R^{n\times n}, 1 < \alpha < 2, 0 < \beta < \alpha - 1$ and $f: [t_0, \infty) \times R^n \times R^n \to R^n$ is a given function. We will show the existence and uniqueness of the above system after we present some definitions and lemmas in the next section, which will help us in our result.

The theory of fixed point in many references is the standard gist of gaining the requisite and adequate conditions for existence and uniqueness of fractional differential equations. Particularly, Banach fixed point theorem is fundamental for a unique solution, but there is exigency to a vigorous supposition to be stratified, like, contraction rule [1][2][3]. The problem of how to prove that the solution exists and unique for fractional differential equations submitted by many of authors [4][5][6][2][7][8]. It is a requisite theoretical part of some applications, which are inspected also by many authors [2][8][9]. It emerges in many fields like fluid dynamics, electronic, chemical kinetics, and biological models. A well-known example is the equations of analysis of the electric circuit, to a look to a little of these results see [10]. Finally, our goal of this study is to investigate the existence and uniqueness of solutions for a semi linear fractional differential system via Banach fixed point theorem.

## 2. Preliminaries

The study presents some definitions and mathematical properties, which will be useful in our research

**Definition (2.1) [11]:** we called the function

$$E_{\alpha}(v) = \sum_{k=0}^{\infty} \frac{v^k}{\Gamma(\alpha k+1)}, v \in \mathbb{C}, Re(\alpha) > 0 \ldots\ldots\ldots (2.1)$$

Mittag-Leffler function with one parameter.

And

$$E_{\alpha,\beta}(v) = \sum_{k=0}^{\infty} \frac{v^k}{\Gamma(\alpha k+\beta)}, v \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0 \ldots\ldots\ldots (2.2)$$

Mittag-Leffler function with two parameter. And both of $(2.1), (2.2)$ satisfying the conditions $E_{\alpha,1}(v) = E_{\alpha}(v), and\ E_1(v) = e^v$

**Definition (2.2) [11]:** if we differentiate Mittag-Leffler function then the derivative will be as follows

$$\frac{d}{dv}E_{\alpha,1}(av^{\alpha}) = \sum_{k=1}^{\infty} \frac{a^k v^{\alpha k-1}}{\Gamma(\alpha k)} = av^{\alpha-1}\sum_{k=0}^{\infty} \frac{(av^{\alpha})^k}{\Gamma(\alpha k+\alpha)} = av^{\alpha-1}E_{\alpha,\alpha}(av^{\alpha}) \ldots\ldots\ldots (2.3)$$

And

$$\frac{d}{dv}\left(v^{\beta-1}E_{\alpha,\beta}(av^{\alpha})\right) = v^{\beta-2}E_{\alpha,\beta-1}(av^{\alpha}) \ldots\ldots\ldots (2.4)"$$

**Definition (2.3) [12]:** let $\alpha > 0$, for a function $f: [t_0, T] \to R$, the fractional integral of order $\alpha$ of $f$ is defined as

$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)ds$ … … … (2.5)

**Definition (2.4) [12]:** the caputo derivative of a continuous function $f:[t_0,T]\to R, n-1<\alpha\le n$ is given by

$$D_{t_0,t}^{\alpha}f(t)=\begin{cases}I^{n-\alpha}f^{(n)}(t), n-1\alpha<n\\ f^{(n)}(t), \alpha=n\end{cases} \quad \dots\dots\dots (2.6)$$

We can write it in another form by

$$D_{t_0,t}^{\alpha}f(t)=I^{n-\alpha}\left(D_{t_0,t}^{n}f(t)\right)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s)ds \quad \dots\dots\dots (2.7)$$

For more properties of (2.7) see [13] and for theory of FDEs see [12], and $\Gamma$ is the symbol of gamma function [11].

**Note (2.5):** the study will use Caputo operator in all mathematical procedures of this paper.

**Definition (2.6) [3]:** the Laplace transform ($LT$) of the caputo derivative is given by

$$L\{D_{t_0,t}^{\alpha}f(t)\}(s)=s^{\alpha}L\{f(t)\}(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(t_0), n-1<\alpha<n \quad \dots\dots\dots (2.8)$$

**Lemma (2.7) [1]:** the $LT$ of the Mittag-Leffler function is given by

$$L\{t^{\beta-1}E_{\alpha,\beta}(\pm\theta t^{\alpha})\}(s)=s^{\alpha-\beta}(s^{\alpha}\pm\theta)^{-1}, Re(s)>\theta^{\frac{1}{\alpha}} \quad \dots\dots\dots (2.9)$$

So

$$L\{E_{\alpha}(\pm\theta t^{\alpha})\}(s)=s^{\alpha-1}(s^{\alpha}\pm\theta)^{-1}, Re(s)>\theta^{\frac{1}{\alpha}} \quad \dots\dots\dots (2.10)$$

**Definition (2.8) [1]:** the convolution of the function $f$ and $g$ is given by

$$f*g(t)=\int_0^t g(t-s)f(s)ds \quad \dots\dots\dots (2.11)$$

**Lemma (2.9) [1]:** the inverse of Laplace transforms of the convolution defined as follows

$$L^{-1}\{F(s)G(s)\}=L^{-1}\{F(s)\}*L^{-1}\{G(s)\} \quad \dots\dots\dots (2.12)$$

**Definition (2.10) [13]:** For $P\in C[t_0,T]$, $P$ is compact iff $P$ is closed, bounded, and eqicontinuous. We notice that compact operator on a Banach space are always completely continuous. This called Arzela-Ascoli theorem.

**Definition (2.11) [13]:** let $K$ be Banach space, and let $G:K\to K$ be a contraction mapping with Lipschitz condition $W$. Then $G$ has a unique fixed point, which means there exist a unique $Q\in K$ such that $G(Q)=Q$.

**Lemma (2.12)**: For $1<\alpha<2$, the solution of (1.1) is

$$x(t)=E_{\alpha,1}(A(t-t_0)^{\alpha})x_0+(t-t_0)E_{\alpha,2}(A(t-t_0)^{\alpha})x_1+\int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)d\tau$$

**Proof:** By operate with Laplace transform on (1.1)

$$L\left\{D_{t_0,t}^{\beta}x(t)\right\}(s)=L\{Ax(t)\}(s)+L\left\{f\left(t,x(t),D_{t_0,t}^{\beta}x(t)\right)\right\}(s)$$

$$\Rightarrow s^{\alpha}L\{x(t)\}(s)-s^{\alpha-1}x_0-s^{\alpha-2}x_1=L\{Ax(t)\}(s)+L\left\{f\left(t,x(t),D_{t_0,t}^{\beta}x(t)\right)\right\}(s)$$

Let

$$L\left\{f\left(t,x(t),D_{t_0,t}^{\beta}x(t)\right)\right\}(s)=F\left(s,x(s),D_{t_0,t}^{\beta}x(s)\right)$$

$$\Rightarrow L\{x(t)\}(s)[s^{\alpha}-A]=s^{\alpha-1}x_0+s^{\alpha-2}x_1+F\left(s,x(s),D_{t_0,t}^{\beta}x(s)\right)$$

Which is satisfying

$$L\{x(t)\}(s)=[s^{\alpha}-A]^{-1}s^{\alpha-1}x_0+[s^{\alpha}-A]^{-1}s^{\alpha-2}x_1+[s^{\alpha}-A]^{-1}F\left(s,x(s),D_{t_0,t}^{\beta}x(s)\right)$$

Now, applying inverse of Laplace transform and refer to lemma (2.9), we obtain

$$x(t)=E_{\alpha,1}(A(t-t_0)^{\alpha})x_0+(t-t_0)E_{\alpha,2}(A(t-t_0)^{\alpha})x_1+L^{-1}\left\{[s^{\alpha}-A]^{-1}F\left(s,x(s),D_{t_0,t}^{\beta}x(s)\right)\right\}(t) \quad \dots\dots\dots (2.12)$$

By convolution theorem of LT on $L^{-1}\left\{[s^{\alpha}-A]^{-1}F\left(s,x(s),D_{t_0,t}^{\beta}x(s)\right)\right\}(t)$ we have

$$L^{-1}\left\{[s^{\alpha}-A]^{-1}F\left(s,x(s),D_{t_0,t}^{\beta}x(s)\right)\right\}(t)=L^{-1}\{[s^{\alpha}-A]^{-1}\}(t)*L^{-1}\left\{F\left(s,x(s),D_{t_0,t}^{\beta}x(s)\right)\right\}(t)$$

$$\Rightarrow L^{-1}\left\{[s^{\alpha}-A]^{-1}F\left(s,x(s),D_{t_0,t}^{\beta}x(s)\right)\right\}(t)=\int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)d\tau$$

Here, proof is completed.

Suppose $C$ be a Banach space of all continuous vector functions such that $C=C([t_0,T],R^n=C(J,R^n)$, we define a norm by $\|x\|=\underbrace{sup}_{t\in J}\|x(t)\|$, the norm of n-vector function $x(t)=(x_1(t),x_2(t),\dots,x_n(t))$ in $R^n$ is defined by

$$\|x(t)\|=(\sum_{k=1}^{n}|x_k(t)|^2)^{\frac{1}{2}}$$

Let $X=\left\{x:x\in C, D_{t_0,t}^{\beta}x\in C\right\}$. Then $X$ is a Banach space endowed with the norm

$$\|x\|_X=\|x\|+\left\|D_{t_0,t}^{\beta}x\right\|$$

From lemma (2.12), we convert IVP (1.1) to $Hx=x, x\in X$, where $H:X\to X$ is defined by

$$Hx(t)=E_{\alpha,1}(A(t-t_0)^{\alpha})x_0+(t-t_0)E_{\alpha,2}(A(t-t_0)^{\alpha})x_1+\int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)d\tau \quad \dots\dots\dots (2.14)$$

Note that problem (1.1) has a solution if the operator (2.14) has a fixed point.

## 3. Main result

The study will use Banach fixed point theorem to obtain an asset and unique solution for (1.1).

**Theorem (3.1):** let $f$ be a be continuous function where $f$ is a mapping from$[t_0,\infty)\times R^n\times R^n toR^n$ and satisfying Lipschitz condition

$\|f(t,x_1(t),y_1(t)-f(t,x_2(t),y_2(t)\| \le W\{\|x_1(t)-x_2(t)\|+\|y_1(t)-y_2(t)\|\}, t\in J, w>0$

Then, (1.1) has a unique solution whenever $\frac{4}{\alpha}WM_\alpha(T-t_0)^\alpha$, and $\frac{4\Gamma(\alpha-1)\frac{4}{\alpha}WM_\alpha(T-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\max\left\{1,\frac{T-t_0}{\alpha-\beta}\right\}<1$.

**Proof:** the proof is divided into two parts:

**I)** We will prove $H(B_r)\subset B_r$, that is the study show that $H(x)\in B_r$ is closed ball of radius $r>0$ in $X$, that is, $B_r=\{x\in X,\|x\|_X\le r\}$. Let $M_0\ge \underbrace{sup}_{t\in J}\|f(t,0,0)\|$, and $\|E_{\alpha,i}(A(t-t_0)^\alpha)\|\le M_i, i=1,2,\alpha-1,\alpha$ and put

$$r>\max\left\{\left(M_1\|x_0\|+M_2\|x_1\|(T-t_0)+\frac{1}{\alpha}M_0M_\alpha(T-t_0)^\alpha\right)\left(\frac{1}{2}-\frac{2}{\alpha}WM_\alpha(T-t_0)^\alpha\right)^{-1},\left(\frac{M_1\|x_1\|}{\Gamma(2-\beta)}(T-t_0)^{1-\beta}+\frac{\Gamma(\alpha)\left(M_\alpha\|A\|\|x_0\|+\frac{M_{\alpha-1}M_0}{\alpha-1}\right)}{\Gamma(\alpha-\beta+1)}(T-t_0)^{\alpha-\beta}\right)\left(\frac{1}{2}-\frac{2\Gamma(\alpha-1)WM_{\alpha-1}}{\Gamma(\alpha-\beta+1)}(T-t_0)^{\alpha-\beta}\right)^{-1}\right\}$$

For $x\in B_r$, we obtain

$$\|Hx(t)\|\le\|E_\alpha(A(t-t_0)^\alpha)\|\|x_0\|+(t-t_0)\|E_{\alpha,2}(A(t-t_0)^\alpha)\|\|x_1\|+\int_{t_0}^t(t-s)^{\alpha-1}\|E_{\alpha,\alpha}(A(t-s)^\alpha)\|\left\|f\left(s,x(s),D_{t_0,t}^\beta x(s)\right)\right\|ds$$

$$\le M_1\|x_0\|+M_2\|x_1\|(t-t_0)+M_\alpha\int_{t_0}^t(t-s)^{\alpha-1}\left\|f\left(s,x(s),D_{t_0,t}^\beta x(s)\right)-f(s,0,0)+f(s,0,0)\right\|ds$$

$$\le M_1\|x_0\|+M_2\|x_1\|(t-t_0)+M_\alpha\int_{t_0}^t(t-s)^{\alpha-1}\left(\left\|f\left(s,x(s),D_{t_0,t}^\beta x(s)\right)-f(s,0,0)\right\|+\|f(s,0,0\|\right)ds$$

By using Lipchitz condition,

$$\|Hx(t)\|\le M_1\|x_0\|+M_2\|x_1\|(t-t_0)+M_\alpha\int_{t_0}^t(t-s)^{\alpha-1}\left(W\left(\|x(s)+\|\left\|D_{t_0,t}^\beta x(s)\right\|\right)+\|f(s,0,0\|\right)ds$$

$$\le M_1\|x_0\|+M_2\|x_1\|(t-t_0)+2WM_\alpha\|x\|_X\int_{t_0}^t(t-s)^{\alpha-1}ds+M_0M_\alpha\int_{t_0}^t(t-s)^{\alpha-1}ds$$

$$\le M_1\|x_0\|+M_2\|x_1\|(t-t_0)+\frac{1}{\alpha}M_0M_\alpha(t-t_0)^\alpha+\frac{2}{\alpha}WM_\alpha(T-t_0)^\alpha\|x\|_X$$

Differentiate $Hx(t)$, and equation (2.4),

$$H'x(t)=A(t-t_0)^{\alpha-1}E_{\alpha,\alpha}(A(t-t_0)^\alpha)x_0+E_{\alpha,1}(A(t-t_0)^\alpha)x_1+\int_{t_0}^t(t-\tau)^{\alpha-2}E_{\alpha,\alpha-1}(A(t-\tau)^\alpha)f\left(\tau,x(\tau),D_{t_0,t}^\beta x(\tau)\right)d\tau$$

Therefore,

$$\|H'x(t)\|\le\|A(t-t_0)^{\alpha-1}\|\|E_{\alpha,\alpha}(A(t-t_0)^\alpha)\|\|x_0\|+\|E_{\alpha,1}(A(t-t_0)^\alpha)\|\|x_1\|+\int_{t_0}^t(t-\tau)^{\alpha-2}\|E_{\alpha,\alpha-1}(A(t-\tau)^\alpha)\|\left\|f\left(\tau,x(\tau),D_{t_0,t}^\beta x(\tau)\right)\right\|d\tau$$

$$\le M_\alpha\|A\|\|x_0\|(t-t_0)^{\alpha-1}+M_1\|x_1\|+M_{\alpha-1}\int_{t_0}^t(t-\tau)^{\alpha-2}(2W\|x\|_X+M_0)d\tau$$

$$\le M_\alpha\|A\|\|x_0\|(t-t_0)^{\alpha-1}+M_1\|x_1\|+\frac{1}{\alpha-1}M_{\alpha-1}M_0(t-t_0)^{\alpha-1}+\frac{2}{\alpha-1}WM_{\alpha-1}(t-t_0)^{\alpha-1}\|x\|_X$$

Now

$$\left\|D_{t_0,t}^\beta Hx(t)\right\|\le\frac{1}{\Gamma(1-\beta)}\int_{t_0}^t(t-\tau)^{-\beta}\|H'x(s)\|ds$$

$$\le\frac{M_1\|x_1\|}{\Gamma(2-\beta)}(t-t_0)^{1-\beta}+\frac{1}{\Gamma(1-\beta)}\left(M_\alpha\|A\|\|x_0\|+\frac{M_{\alpha-1}M_0}{\alpha-1}+\frac{2WM_{\alpha-1}\|x\|_X}{\alpha-1}\right)\int_{t_0}^t(t-\tau)^{-\beta}(s-t_0)^{\alpha-1}ds$$

$$\le\frac{M_1\|x_1\|}{\Gamma(2-\beta)}(t-t_0)^{1-\beta}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)}\left(M_\alpha\|A\|\|x_0\|+\frac{M_{\alpha-1}M_0}{\alpha-1}+\frac{2WM_{\alpha-1}\|x\|_X}{\alpha-1}\right)(t-t_0)^{\alpha-\beta}$$

If $x\in B_r$, then $\|x\|_X\le r$. Therefore

$$\|Hx\|_X=\|Hx\|+\left\|D_{t_0,t}^\beta Hx\right\|$$

$$\le M_1\|x_0\|+M_2\|x_1\|(T-t_0)+\frac{1}{\alpha}M_0M_\alpha(T-t_0)^\alpha+\frac{2}{\alpha}WM_\alpha(T-t_0)^\alpha r+\frac{M_1\|x_1\|}{\Gamma(2-\beta)}(T-t_0)^{1-\beta}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)}\left(M_\alpha\|A\|\|x_0\|+\frac{M_{\alpha-1}M_0}{\alpha-1}\right)(T-t_0)^{\alpha-\beta}+\frac{2r\Gamma(\alpha-1)WM_{\alpha-1}}{\Gamma(\alpha-\beta+1)}(T-t_0)^{\alpha-\beta}$$

Then

$$\|Hx\|_X\le\left(\frac{1}{2}-\frac{2}{\alpha}WM_\alpha(T-t_0)^\alpha\right)r+\frac{2}{\alpha}WM_\alpha(T-t_0)^\alpha r+\left(\frac{1}{2}-\frac{2\Gamma(\alpha-1)WM_{\alpha-1}}{\Gamma(\alpha-\beta+1)}(T-t_0)^{\alpha-\beta}\right)r+\frac{2r\Gamma(\alpha-1)WM_{\alpha-1}}{\Gamma(\alpha-\beta+1)}(T-t_0)^{\alpha-\beta}$$

$$=\left(\frac{1}{2}-\frac{2}{\alpha}WM_\alpha(T-t_0)^\alpha+\frac{2}{\alpha}WM_\alpha(T-t_0)^\alpha\right)r+\left(\frac{1}{2}-\frac{2\Gamma(\alpha-1)WM_{\alpha-1}}{\Gamma(\alpha-\beta+1)}(T-t_0)^{\alpha-\beta}+\frac{2\Gamma(\alpha-1)WM_{\alpha-1}}{\Gamma(\alpha-\beta+1)}(T-t_0)^{\alpha-\beta}\right)r=r$$

Thus, $\|Hx\|_X$, then $Hx\in B_r$, whenever $x\in B_r$.

**II)** We next show that $H$ is contracting mapping on $X$. For $x,y\in X$ and for each $t\in J$ we obtain

$$\|(Hx)(t)-(Hy)(t)\|\le M_\alpha\int_{t_0}^t(t-s)^{\alpha-1}\left\|f\left(s,x(s),D_{t_0,t}^\beta x(s)\right)-f\left(s,y(s),D_{t_0,t}^\beta y(s)\right)\right\|ds$$

$$\le M_\alpha W\int_{t_0}^t(t-s)^{\alpha-1}\left\{\|x(s)-y(s)\|+\left\|D_{t_0,t}^\beta x(s)-D_{t_0,t}^\beta y(s)\right\|\right\}ds$$

$$\le\frac{2}{\alpha}M_\alpha W(t-t_0)^\alpha\|x-y\|_X$$

On the other hand

$$\left\|D_{t_0,t}^\beta Hx(t)-D_{t_0,t}^\beta Hy(t)\right\|\le\frac{1}{\Gamma(1-\beta)}\int_{t_0}^t(t-s)^{-\beta}\|H'x(s)-H'y(s)\|ds$$

$$\le\frac{2M_{\alpha-1}W}{\Gamma(1-\beta)}\|x-y\|_X\int_{t_0}^t(t-s)^{-\beta}.(s-t_0)^{\alpha-2}ds$$

$\leq \frac{2M_{\alpha-1}W\Gamma(\alpha-1)}{\Gamma(\alpha-\beta)}(t-t_0)^{\alpha-\beta-1}\|x-y\|_X$

Then

$\|Hx\|_X = \|Hx\| + \left\|D^{\beta}_{t_0,t}Hx\right\| \leq \left(\frac{2}{\alpha}M_\alpha + \frac{2M_{\alpha-1}\Gamma(\alpha-1)}{\Gamma(\alpha-\beta)}(T-t_0)^{-\beta-1}\right)W(T-t_0)^{\alpha}\|x-y\|_X$

The study conclude that $H$ is a contraction since $\left(\frac{2}{\alpha}M_\alpha + \frac{2M_{\alpha-1}\Gamma(\alpha-1)}{\Gamma(\alpha-\beta)}(T-t_0)^{-\beta-1}\right)W(T-t_0)^{\alpha} < 1$, and the statement of the theorem follows by the classical Banach fixed point theorem. This finishes the proof.

## 4. Examples

In this section we will discuss some examples on fractional differential systems that has a solution for $\alpha \in (1,2)$

**Example 4.1** consider the initial value problem

$D^{1.5}_{0,t}x(t) = \frac{e^{-t}}{e^t+9}\left(|x(t)| - \frac{1}{1+D^{1.5}_{0,t}|x(t)|}\right), t \in [0,1]$ ... ... ... (4.1)

With initial conditions $x(0) = 0, x'(0) = 1$.

The study is going to prove that (4.1) has a unique solution. Referring to the system(1.1), we have $A = 0$ and

$f\left(t, x(t), D^{1.5}_{0,t}x(t)\right) = \frac{e^{-t}}{e^t+9}\left(|x(t)| - \frac{1}{1+D^{1.5}_{0,t}|x(t)|}\right)$

Whenever $f: [0,1] \times R \times R \to R$.

In order to show that the initial value problem has a unique solution, we will check the conditions of theorem (3.1)

$\left|f\left(t, x(t), D^{1.5}_{0,t}x(t)\right) - f\left(t, y(t), D^{1.5}_{0,t}y(t)\right)\right|$

$= \frac{e^{-t}}{e^t+9}\left||x(t)| - |y(t)| - \frac{1}{1+D^{1.5}_{0,t}|x(t)|} + \frac{1}{1+D^{1.5}_{0,t}|y(t)|}\right|$

$= \frac{e^{-t}}{e^t+9}\left||x(t)| - |y(t)| + \frac{(D^{1.5}_{0,t}|x(t)| - D^{1.5}_{0,t}|y(t)|)}{(1+D^{1.5}_{0,t}|x(t)|)(1+D^{1.5}_{0,t}|y(t)|)}\right|$

$\leq \frac{1}{10}\left(|x(t)| - |y(t)| + \left|D^{1.5}_{0,t}(|x(t)| - |y(t)|)\right|\right)$

$\leq \frac{1}{10}\left(|x(t) - y(t)| - \left|D^{1.5}_{0,t}(|x(t)| - |y(t)|)\right|\right)$

Which is satisfying the Lipchitz condition with $W = \frac{1}{10}$.

Because of $|E_{\alpha,\alpha}(A(t-s)^{\alpha})| \leq M_\alpha$, and $|E_{\alpha,\alpha-1}(A(t-s)^{\alpha})| \leq M_{\alpha-1}$,

For $A = 0$, we have

$E_{1.5,1.5}(0) = \frac{1}{\Gamma(1.5)} = \frac{2}{\sqrt{\pi}}$

And

$E_{1.5,0.5}(0) = \frac{1}{\Gamma(0.5)} = \frac{1}{\sqrt{\pi}}$

Now

$\frac{4}{\alpha}WM_\alpha(T-t_0)^{\alpha} = \frac{4}{1.5}.\frac{2}{10\sqrt{\pi}}(1-0)$

And

$\frac{4\Gamma(\alpha-1)\frac{4}{\alpha}WM_{\alpha-1}(T-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\max\left\{1, \frac{T-t_0}{\alpha-\beta}\right\} = \frac{4\Gamma(0.5)(0.1)\frac{1}{\Gamma(0.5)}}{\Gamma(2)}\max\{1,1\} = \frac{4}{10} < 1$

Hence, the initial value problem (4.1) has a unique solution.

**Example 4.2**

Assume we have the following initial value problem

$D^{1.8}_{0,t}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \begin{pmatrix}1 & 2\\0 & 3\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} + \frac{1}{e^t+39}\begin{pmatrix}\frac{|x_1(t)|}{1+|x_1(t)|} + \left|D^{0.7}_{0,t}x_1(t)\right|\\ \frac{|x_2(t)|}{1+|x_2(t)|} + \left|D^{0.7}_{0,t}x_2(t)\right|\end{pmatrix}$ ... (4.2)

Whenever $t \in [0,0.5]$, and $x(0) = \begin{pmatrix}0\\0\end{pmatrix}, x'(0) = \begin{pmatrix}1\\1\end{pmatrix}$

For the system (4.2) we have $A = \begin{pmatrix}1 & 2\\0 & 3\end{pmatrix}$, and

$f\left(t, x(t), D^{0.7}_{0,t}x(t)\right) = \frac{1}{e^t+39}\begin{pmatrix}\frac{|x_1(t)|}{1+|x_1(t)|} + \left|D^{0.7}_{0,t}x_1(t)\right|\\ \frac{|x_2(t)|}{1+|x_2(t)|} + \left|D^{0.7}_{0,t}x_2(t)\right|\end{pmatrix}$

Where $f: [0,0.5] \times R^2 \times R^2 \to R^2$

We will check the conditions of theorem (3.1) to show that (4.2) has a unique solution

$\left\|f\left(t, x(t), D^{0.7}_{0,t}x(t)\right) - f\left(t, y(t), D^{0.7}_{0,t}y(t)\right)\right\| = \frac{1}{e^t+39}\begin{pmatrix}\frac{|x_1(t)|}{1+|x_1(t)|} - \frac{|y_1(t)|}{1+|y_1(t)|} + \left|D^{0.7}_{0,t}x_1(t)\right| - \left|D^{0.7}_{0,t}y_1(t)\right|\\ \frac{|x_2(t)|}{1+|x_2(t)|} - \frac{|y_2(t)|}{1+|y_2(t)|} + \left|D^{0.7}_{0,t}x_2(t)\right| - \left|D^{0.7}_{0,t}y_2(t)\right|\end{pmatrix}$

Then

$\left\|f\left(t, x(t), D^{0.7}_{0,t}x(t)\right) - f\left(t, y(t), D^{0.7}_{0,t}y(t)\right)\right\|^2 = \frac{1}{(e^t+39)^2}\left(\frac{|x_1(t)|}{1+|x_1(t)|} - \frac{|y_1(t)|}{1+|y_1(t)|} + \left|D^{0.7}_{0,t}x_1(t)\right| - \left|D^{0.7}_{0,t}y_1(t)\right|\right)^2 + \frac{1}{(e^t+39)^2}\left(\frac{|x_2(t)|}{1+|x_2(t)|} - \frac{|y_2(t)|}{1+|y_2(t)|} + \left|D^{0.7}_{0,t}x_2(t)\right| - \left|D^{0.7}_{0,t}y_2(t)\right|\right)^2$

$= \frac{1}{(e^t+39)^2}\left(\frac{|x_1(t)| - |y_1(t)|}{(1+|x_1(t)|)(1+|y_1(t)|)} + \left|D^{0.7}_{0,t}x_1(t)\right| - \left|D^{0.7}_{0,t}y_1(t)\right|\right)^2 + \frac{1}{(e^t+39)^2}\left(\frac{|x_2(t)| - |y_2(t)|}{(1+|x_2(t)|)(1+|y_2(t)|)} + \left|D^{0.7}_{0,t}x_2(t)\right| - \left|D^{0.7}_{0,t}y_2(t)\right|\right)^2$

But we know that

$||x(t)| - |y(t)|| \leq |x(t) - y(t)|$

So, we obtain

$\left\|f\left(t, x(t), D^{0.7}_{0,t}x(t)\right) - f\left(t, y(t), D^{0.7}_{0,t}y(t)\right)\right\|^2 \leq \frac{1}{1600}\left(|x_1(t) - y_1(t)| + \left|D^{0.7}_{0,t}x_1(t) - D^{0.7}_{0,t}y_1(t)\right|\right)^2 + \frac{1}{1600}\left(|x_2(t) - y_2(t)| + \left|D^{0.7}_{0,t}x_2(t) - D^{0.7}_{0,t}y_2(t)\right|\right)^2$

Therefore

$\left\|f\left(t, x(t), D^{0.7}_{0,t}x(t)\right) - f\left(t, y(t), D^{0.7}_{0,t}y(t)\right)\right\| \leq \frac{1}{40}(|x_1(t) - y_1(t)| + |x_2(t) - y_2(t)|) + \left(\left|D^{0.7}_{0,t}x_1(t) - D^{0.7}_{0,t}y_1(t)\right| + \left|D^{0.7}_{0,t}x_2(t) - D^{0.7}_{0,t}y_2(t)\right|\right)$

$\leq \frac{\sqrt{2}}{40}\left(\|x(t) - y(t)\| + \left\|D^{0.7}_{0,t}x(t) - D^{0.7}_{0,t}y(t)\right\|\right)$

Now, the Lipchitz condition is hold with $W = \frac{\sqrt{2}}{40}$".

Next, we calculate the bound of Mittag-Leffler functions, by using the maximum norm of $A$.

We have

$\|E_{1.8,0.7}(A(t-s)^{\alpha})\| \leq 1.6$

And

$\|E_{0.8,0.7}(A(t-s)^{\alpha})\| \leq 11.5$

Also, we have

$$\frac{4}{\alpha} W M_\alpha (T - t_0)^\alpha = \frac{4}{1.8} . \frac{\sqrt{2}}{40} . 1.6(0.5 - 0) < 1$$

And

$$\frac{4\Gamma(\alpha-1)\frac{4}{\alpha} W M_{\alpha-1} (T-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \max\left\{1, \frac{T-t_0}{\alpha-\beta}\right\} =$$

$$\frac{4\Gamma(0.8)\frac{\sqrt{2}}{40}(11.5)(0.5)^{1.1}}{\Gamma(2.1)} \max\left\{1, \frac{0.5}{1.1}\right\} < 1$$

Therefore, the initial value problem (4.2) has a unique solution.

## 5. Conclusion

The study discussed the existence and uniqueness results for a fractional differential system with initial conditions. The study proved the existence and uniqueness of the solution throughout applying well-known Banach fixed point theorem and the study concluded that the solution of this system is existed and unique for value of $\alpha$ belongs to the interval $1 < \alpha < 2, 0 < \beta < \alpha - 1$. Therefore, this study and the others, which similar to it are very important for fractional order differential equations and their applications, which can be extended with another condition in the future studies.

## References


[1] Kilbas, A.A.; Trujillo J.J. and Srivastava, H.M. (2006). Theory and Application of Fractional Differential Equations. First edition, Amsterdam: Elsevier: 539.

[2] Miller, K. and Ross, B.(1993). Introduction to the Fractional Calculus and Fractional Differential Equations. First edition. New York: John Wiley and Sons Inc.:376.

[3] Podlubny,I. and Thimann, K.V.(1998). Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. First edition, San Diego: Academic Press:366.

[4] Yu,C. and Gao,G.(2005). Existence of Fractional Differential Equations. *Journal of Mathematical Analysis and Applications.*,**310**:26-29

[5] Zhai, C. (2009). Positive Solutions for Third Order Sturm-Liouville Boundary Value Problem with p - Laplacian. *Electronic Journal of Differential Equations*, **2009 (154)**:1-9

[6] Delbosco, D. and Rodino, L. (1996). Existence and Uniqueness for a Nonlinear Fractional Differential Equations, *Journal of Mathematical Analysis and Applications*, **204**:609-625.

[7] Benchohra, M. and Hamani, S. (2008). Boundary Value Problem for Differential Equations with Fractional Order, *Surveys in Mathematics and Its Application*, **3**:1-12.

[8] Plotnikov, V.A. and Vityuk. (1999). Differential Equations with a Multivalued Right-Hand Side, Asymptotic Methods. First edition, Odessa: Astroprint: 356.

[9] Girg, P.; Roca, F. and Villegas, S.(2005). Semi Linear Sturm - Liouville Problem with Periodic Nonlinearity. *Nonlinear Analysis*, **61**:1157-1178.

[10] Jaradat, O.K.; Al-Omari, A. and Momani, S.(2008). Existence of the Mild Solution for Fractional Semi Linear Initial Value Problem. *Nonlinear Analysis*.**69 (9):**3153-3159.

[11] Mourad, E.; Ismail, H. and Koelink, E.(2005). Theory and Applications of Special Functions. 1st edition, USA: Krishnasami Alladi: Springer:504.

[12] N'guerekata, G.M. (2009). A Cauchy Problem for Some Fractional Abstract Differential Equations with Nonlocal Condition, *Nonlinear Analysis*. **70 (5)**:1873-1876.

[13] Granas, A. and Dugundji, J. (2003). Fixed Point Theory. First edition, New York: Springer-Verlag: 707.


# حول الحل الوحيد لنظام تفاضلي كسري


مؤيد محمود خليل

*قسم الرياضيات ، كلية التربية للعلوم الصرفة ، جامعة تكريت ، تكريت ، العراق*



**الملخص**

ان هدف الدراسة هو التحقق من وجود ووحدانية الحلول للأنظمة التفاضلية الكسرية شبه الخطية بواسطة مبرهنة بناخ للنقطة الثابتة. وقد أثبتت الدراسة وجود ووحدانية الحل لنظام تفاضلي كسري ذي شروط ابتدائية باستخدام مبرهنة التطبيق الانكماشي، وحصلنا على نتائج الوجود والوحدانية كما أخترنا بعض الأمثلة التي توضح مدى صلاحية النتائج التي توصلنا اليها.