



On fixed point theorem in complete quasi-metric space under F-contraction mapping

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<https://doi.org/10.25130/tjps.v24i2.358>

ARTICLE INFO.

Article history:

-Received: 25 / 10 / 2018

-Accepted: 31 / 12 / 2018

-Available online: / / 2019

Keywords: fixed point; F-contractions; quasi-metric space

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ABSTRACT

In this paper, a fixed point theorem under F-contraction mapping was considered and proved in complete quasi-metric space. This theorem was considered by Piri and Kumam in [1].

Subject Classification : 30C45; 30C10; 47B38

1.Introduction

Since 1922 when Banach in [2] presents his contraction theorem of fixed point, hundreds of fixed point theorems have been presented and proved in many published papers, where some of them are generalized by Banach's work [2] and some of them are not [3]. Banach and many others [1,2,4,5,6] consider contraction mappings in complete metric space.

In [2], Ahmed et.al. presented new results concerned with F-contraction mapping in complete metric spaces, while Abodayeh et. al. in [7] study some fixed point theorems related to the quasi-metric spaces for some type of contraction mapping .

Many applications are presented and considered in the literature which concerned with contraction mapping, [5] for instance, while different types of contraction mappings are presented and considered [6]. The structure of this article begins with preliminaries which may needed to prove theorems in the main results section. The main results section state two theorems, the first one is the generalization of a theorem presented in [1] by Piri and Kumam – the statement of this theorem is recalled at the end of this section. The second theorem is concerned with the uniqueness of fixed point, these two theorems are proved in this article as mentioned above.

In the beginning, the definition of quasi-metric space is recalled:

preliminaries

Definition 1[7]

Let X be a non-empty set and $d: X \times X \rightarrow [0, \infty)$ be a given function which satisfies:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$. for all $x, y \in X$
2. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Then (X, d) is called quasi-metric space.

Clearly that every metric space is quasi-metric space but the reverse is not true in general .

The next definition states the most important deference between metric spaces and quasi ones, which presented by Jlili and Samet[8].

Definition 2 [8]

Let (X, d) be a quasi-metric space, (x_n) be a sequence in X and $x \in X$. Then the sequence (x_n) converges to x if and only if

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0$$

At this point we refer to the research paper [7] in which the authors define left and right Cauchy sequence as well as the left and right completeness of quasi-metric spaces.

The contraction mapping introduced by Wardowski [9] which called F-contraction mapping defined as following:

Definition 3[9]

Let (X, d) be a metric space. A mapping $T: X \rightarrow X$ is called F- contraction mapping if there exists $\tau > 0$ and $F: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$\forall x, y \in X, [d(T_x, T_y) > 0 \Rightarrow \tau + F(d(T_x, T_y)) \leq F(d(x, y))] \quad (1)$$

And F satisfies the following conditions:

F1: F is strictly increasing, i. e., $\forall x, y \in \mathbb{R}, F(x) < F(y)$ such that $x < y$.

F2: For any sequence $(\alpha_n)_{n=1}^\infty, \alpha_n > 0, \forall n \in \mathbb{N}, \lim_{n \rightarrow \infty} \alpha_n = 0$ iff $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$

F3: $\exists k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} F(\alpha) = 0$

The set of all functions that satisfies conditions F1 to F3 is denoted by \mathcal{F} . For examples of these functions the reader referred to [9].

Note that from F1 and equation (1), every F-contraction mapping is continuous function.

Since the set of all quasi-metric spaces contains all metric spaces, so, what we say about the quasi-metric space is applied on metric space, namely, in this article two fixed point theorems concerned with F-contraction mapping are proved for quasi-metric space to generalize these theorems.

Piri and Kumam in [1], consider some fixed point theorems concerned with F-contraction mapping in complete metric spaces, in this article, generalizing of the next theorem in [1] into quasi-metric space.

Theorem (1) [2]

Let T be a self-mapping of a complete metric space X into itself. Suppose $F \in \mathcal{F}$ and there exist $\tau > 0$ such that

$$\forall x, y \in X, [d(T_x, T_y) > 0 \Rightarrow \tau + F(d(T_x, T_y)) \leq F(d(x, y))]$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $(T^n x_0)_{n=1}^\infty$ converges to x^* .

2. Main Results

To generalize **Theorem (1)** into the space of quasi-metric space the following main theorem regard to this article is reformed and proved.

Theorem (2)

Let T be a continuous self-mapping of a quasi-complete metric space X into itself. Let F be an F-contraction mapping and there exist $\tau > 0$ s. t. (1) is hold $\forall x, y \in X$. Then there is $x^* \in X$ s. t. $x^* = T_{x^*}$ or T has a fixed point $x^* \in X$.

Proof

Define a sequence $(x_n)_{n=1}^\infty$ recurrently by choosing some element $x_0 \in X$, and the other elements defined as follows:

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \dots, x_{n+1} = Tx_n = T^{n+1}x_0, \quad \forall n \in \mathbb{N} \quad (2)$$

Note that for any $n \in \mathbb{N}$, if $d(x_n, Tx_n) = 0$, then the proof is complete. To avoid this case, assume that $\forall n \in \mathbb{N}$,

$$0 < d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n) \quad (3)$$

For any $n \in \mathbb{N}$ we have

$$\tau + F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n))$$

That is,

$$F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n)) - \tau$$

By repeating this process, we get:

$$\begin{aligned} F(d(Tx_{n-1}, Tx_n)) &\leq F(d(x_{n-1}, x_n)) - \tau \\ &= F(d(Tx_{n-2}, Tx_{n-1})) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &= F(d(Tx_{n-3}, Tx_{n-2})) - 2\tau \\ &\leq F(d(x_{n-3}, x_{n-2})) - 3\tau \\ &\vdots \\ &\leq F(d(x_0, x_1)) - n\tau \end{aligned} \quad (4)$$

From (4) we get $\lim_{n \rightarrow \infty} F(d(Tx_{n-1}, Tx_n)) = -\infty$

Which gives $\lim_{n \rightarrow \infty} d(Tx_{n-1}, Tx_n) = 0$ i. e.,

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0 \quad (5)$$

Claim that the sequence $(x_n)_{n=1}^\infty$ is Cauchy sequence and assume that there exists $\epsilon > 0$ and two sequences, namely, $(p(n))_{n=1}^\infty$ and $(q(n))_{n=1}^\infty$ of natural numbers such that $\forall n \in \mathbb{N}$:

$$p(n) > q(n) > n, \quad d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \quad d(x_{p(n)-1}, x_{p(n)}) < \epsilon \quad (6)$$

So, we have

$$\begin{aligned} \epsilon &\leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \\ &\leq d(x_{p(n)}, x_{p(n)-1}) + \epsilon \\ &= d(x_{p(n)}, Tx_{p(n)}) + \epsilon \end{aligned}$$

which leads together with (5) to

$$\lim_{n \rightarrow \infty} d(x_{p(n)}, x_{q(n)}) = \epsilon \quad (7)$$

On the other side, from (5) there exists $N \in \mathbb{N}$, such that

$$d(x_{p(n)}, Tx_{p(n)}) < \frac{\epsilon}{4} \quad \text{and} \quad d(x_{q(n)}, Tx_{q(n)}) < \frac{\epsilon}{4}, \quad \forall n \geq N \quad (8)$$

Next, without any effect of quasi-metric space, we claim the following

$$d(Tx_{p(n)}, Tx_{q(n)}) = d(x_{p(n)+1}, x_{q(n)+1}) > 0 \quad \forall n \geq N \quad (9)$$

Seeking for contradiction, there exists $m \geq N$ such that

$$d(x_{p(m)+1}, x_{q(m)+1}) = 0 \quad (10)$$

It follows from (6), (8) and (10) that

$$\begin{aligned} \epsilon &\leq d(x_{p(m)}, x_{q(m)}) \leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)}) \\ &\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, x_{q(m)}) \\ &= d(x_{p(m)}, Tx_{p(m)}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)}, Tx_{q(m)}) \\ &< \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} = \frac{\epsilon}{2} \end{aligned}$$

By the relation (9) and the assumption of the theorem, it follows that:

$$\tau + F(d(Tx_{p(n)}, Tx_{q(n)})) \leq F(d(x_{p(n)}, x_{q(n)})), \quad \forall n \geq N \quad (11)$$

So we have a contradiction, i. e., $(x_n)_{n=1}^\infty$ is Cauchy sequence.

Consequently, by completeness of (X, d) the sequence $(x_n)_{n=1}^\infty$ is converges in X .

The continuity of T yields that:

$$d(Tx, x) = \lim_{n \rightarrow \infty} d(Tx_n, x_n) = d(Tx_{n+1}, x_n) = d(x^*, x^*) = 0$$

Which complete the proof. \square

Theorem (3)

Let T be a continuous self-mapping of a quasi-complete metric space X into itself. Let F be an F -contraction mapping and there exist $\tau > 0$ s. t. (1) is hold $\forall x, y \in X$. Then if T has a fixed point $x^* \in X$ this fixed point is unique.

Proof

In theorem (2) above the existence of fixed for the mapping T has been proved. In this theorem, we will show that T has at most one fixed point.

Actually, if $x, y \in X$ are two different fixed points of T . That is

$$Tx = x \neq y \neq Ty \text{ then} \\ d(Tx, Ty) = d(x, y) > 0$$

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Which yields to:

$$F(d(x, y)) = F(d(Tx, Ty)) < \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \text{ which is a contradiction .}$$

Therefore, the fixed point is unique. \square

3. Conclusions

The concept of fixed point theorem considered as an important tool in the study of existence and uniqueness of solution for many types of operators, so, this paper provides the existence and uniqueness proof for F -contraction mappings. Constructing an arbitrary example in order to declare and emphasize the proofs stated above is important.

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حول نظرية النقطة الثابتة في الفضاءات المترية التامة الكاذبة بوجود التطبيقات الانكماشية

من نوع F

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الملخص

في هذا البحث نوقشت نظرية النقطة الثابتة بوجود التطبيقات الانكماشية من نوع F وتم برهانها في الفضاءات المترية التامة الكاذبة. وهذه النظرية قد نوقشت من قبل الباحثان بيروي وكمام في المصدر [2].