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Chaotic diffeomorphism on manifold (smooth manifold)

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ABSTRACT

In this paper we concern in studying chaotic homeomorphisms deals with study and investigate of chaotic homeomorphisms on smooth manifolds. For connections more precisely to chaotic diffeomorphisms maps on smooth manifolds.

New relation between diffeomorphism an chaotic, transitive, dense and nowhere dense in smooth manifolds have been found. Product of two diffeomorphisms dense maps on smooth manifold is dense. Product of two chaotic maps on smooth manifold is chaotic.

Introduction

One of the basic concepts of modern mathematics is the smooth manifolds. It is formalization of the object which has independently arisen in many mathematical disciplines (and also in applications of mathematics - mathematical physics, mechanics and other sciences) on which it is possible to use (generalize) coordinates, but there is no general system of coordinates applicable to all points at once. It is a natural and important object.

Statement of a design of smooth manifolds is attractive as it allows to concern some initial concepts of such mathematical discipline as topology, and it is motivated to make clear their need and naturalness.

Dynamical systems with chaotic behavior are now a subject of intensive studying. First of all it is connected with the fact that chaos is a rather general property of the most various nonlinear processes inherent in various fields of natural sciences: from biology to chemical kinetics. The reason of such behavior is not in complexity of dynamic systems and not in manifestation of external influences.

Definition .1[6]

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets. A map $f : U \to V$ is called differentiable of class C^r , if the functions $f^i = f^i(x^1,...,x^m), \quad i = 1,2,...,n$ which give f , they have partial derivatives up to order r .

We say that f is a smooth map (of class C^∞) of Euclidean spaces, if f has partial derivatives for each order.

Definition .2 [6]

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets. A map $f : U \to V$ is said to be C^r - diffeomorphism if f and f^{-1} are C^r - differentiable if it satisfies the following conditions:

1) If f is bijective.

2) Where f and f^{-1} are smooth.

3) **Definition .3 [2].** A homeomorphism is often called a C-diffeomorphism. If the map $g : u \to v$ is a diffeomorphism, then the sets u and v are said to be diffeomorphic.

Definition .4

A C^k $n-$ dimensional manifold M is a non-empty (second countable, Hausdorff) topological space such that :

a) M is the union of open subsets U_α , and each U_α is equipped with a homeomorphism φ_α , taking U_α to an open set in $\mathbb{R}^n$, i.e. $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$.

b) If $U_\alpha \cap U_{\beta \neq \phi}$, then the overlap map $\varphi_\beta \varphi^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map . (see figure 1).

Each pair $(U_\alpha, \varphi_\alpha)$ is called chart on M and the collection $A = \{(U_\alpha, \varphi_\alpha)\}$ of all charts is called a smooth atlas on M .

The space M taken together with atlas A will be called a smooth manifold of dimension n or smooth $n-manifold$ or C^∞ $n-manifold$.

Moreover , if the topological space M satisfies condition (a) only , the manifold will be called a topological manifold or simply a manifold.

Theorem. 5 [3]

1) Every composition of diffeomorphisms is diffeomorphism.

2) Every diffeomorphism is a homeomorphism and an open map.

3) Every finite product of diffeomorphisms between smooth manifold is diffeomorphism **.**

4) Diffeomorphic is an equivalence relation on the class of all smooth manifold with or without boundry

Definition. 6 [3]

Let (M, f) be a topological manifold system , the dynamic is obtained by iterating the map . then , f is said to be chaotic on M provided that for any non-empty open sets U *and* V in M , there is a periodic point $p \in M$ such that $U \cap O_f(p) \neq \phi$.

Definition. 7 [3]

Let (M, f) be a topological manifold system , then , the map $f : M \to M$ is chaotic if for every pair of not empty open subsets $U, V \subset M$ the are periodic points $x \in U$ *and* $n \in N$ such that $f^n(x) \in V$.

Definition . 8 [5]

The chaotic manifold is a manifold changed by the time into homeomorphic manifolds either with fixed point p

Definition .9 [4]

Recall that a subset A of a manifold M is called dense in M if $cl(A) = M$.

Definition .10[6]

A manifold changed by the time into homeomorphic manifold , either with fixed point $p_i, i = 1,2,\dots,n$ or with no fixed point called the chaotic manifold.

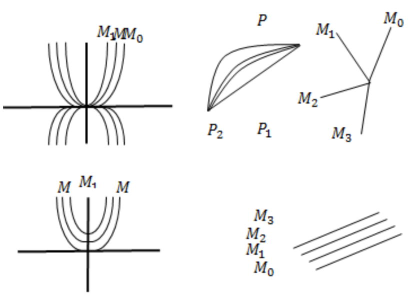

Fig. (1)

Remark 11.

- The topological property that precludes such decomposition is called topological transitivity .

Let M and N be 2 $n-dimensional$ smooth manifolds, the where $n \geq 1$, and $\dim M = \dim N$. then we can introduce the following some new definitions.

Definition 12.

Let M *and* N be a smooth manifolds, and $f : M \to N$ be a diffeomorphism map. We say that f is C^k – diffeo transitive if $\bigcup_{n>0} f^n(w)$ is dense in N for every non – empty open subset w of M .

Definition 13.

Let M *and* N be a smooth manifolds, and $f : M \to N$ be a diffeomorphism map. Then the set $A \subseteq M$ is called diffeo – transitive set if for every pair of non – empty open sets U & V in M and N respectively , with $A \cap U \neq \Phi$ and $A \cap V \neq \Phi$ there is positive integer n such that $f^n(U) \cap V \neq \Phi$.

Definition 14.

Let M *and* N be a smooth manifolds, $f : M \to N$ be a diffeomorphism map , then the closed f – invariant set $A \subseteq M$ is called diffeo – mixing set if , given any non – empty open subsets $U \subseteq M$ *and* $V \subseteq N$ with $A \cap U \neq \Phi$ and $A \cap V \neq \Phi$ then $\exists N > 0$ such that $f^n(U) \cap V \neq \Phi \quad \forall \quad n > N$.

Theorem 15.

Let N *and* M be smooth manifolds , then a map $f : N \to M$ is continuous.

Proof:

Let $A_N \subset N$ where A_N atlas and N smooth manifold, and let $A_M \subset M$, where A_M atlas and M smooth manifold

Since A_N is an atlas on N , it follows that $U_{\alpha \in \lambda} A_N \subset N$ where $\Psi = \{(U_\alpha, \psi)\}_{\alpha \in \lambda}$ of $n-$ dimension charts in N .

Since A_M is atlas on M , it follows that $V_{\xi \in \beta} A_M \subset M$ where $\Phi = \{(V_\xi, \phi)\}_{\xi \in \beta}$ of $n-$ dimension charts in M .

Since Ψ and Φ a families of charts on N and M respectively, then the maps ψ and ϕ are difeomorphisms and $\psi \circ \phi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is difffeomorphism (smooth connection).

Suppose that the maps $f = \psi \circ f \circ \phi^{-1}$ for charts A_n and A_m are smooth on their domains

Then f is smooth. △

Definition. 16

Let M be a smooth manifold and A subset of M .

1. A subset A of a smooth manifold M is called dense in M if $cl(A)=M$.
2. A subset A is said to be dense if for any x M either x in A or it is a limit point for A .
3. A subset A of a smooth manifold M is said to be nowhere dense , if it's closure has an empty interior , that $\operatorname{int}(cl(A))=1$

Definition 17 . [1]

Two manifolds N and M are said to be C^k- conjugated if there is C^k- diffeomorphism $h:N\to M$ such that $h\circ f=g\circ h$ we will call C^k- conjugacy.

Theorem .18 let M *and* N be smooth manifolds and the map $f:M\to N$ is diffeomorphism map , then the following are equivalent:

1. The map f is diffeomorphism transitive .
2. Any closed subset $A\subset M$ is nowhere dense , where $f(A)\subseteq \mathrm{N}$.
3. For all subset $A\subset M$ and , a subset A is either dense or nowhere dense.
4. Any subset $A\subseteq M$ and $f^{-1}(A)\subseteq A$ with nonempty interior is dense.

Proof: 1. is direct.

2. let A be closed subset of smooth manifold .

Since $f(A)\subset N$ and $f:M\to N$

Then $\operatorname{int}(cl(A))=\phi$, then A is nowhere dense by definition [3.8]

3. Since $A\subset M$ and $f(A)\subset N$

Then if A closed subset then A is nowhere dense and if A is open subset then $cl(A)=M$

Then A is dense by definition [9]

4. Let A be any subset of M .

Since $f^{-1}(A)\subset A$, by hypothesis

Then A is closed subset of M

Since $f^{-1}(A)\subset A=\phi$ and closed

Since $f-$ diffeomorphism map . then the closed subset $A=M$, then A is dense by definition .

Theorem . 19

Let a map $g:M\to M$ be diffeomorphism on a smooth manifold M , and a map $f:N\to N$ be diffeomorphism on a smooth manifold N, then the set of periodic points of $f\times g$ where $f\times g:N\times M\to N\times M$ is dense in $N\times M$ if and only if , for both of f and g , the sets of periodic points in N and M are dense in N with atlas $\{(V_i,\varphi_i)\}$, and in M with atlas $\{(V_j,\psi_j)\}$.

Proof:

$\Longrightarrow$ Suppose that the set of periodic points of f is dense in N and the set of periodic points of g is dense in M .

Since $\{U_i,\varphi_i\}$ atlas in N and $\{V_j,\varphi_j\}$ atlas in M it should be proved that $\{(V_i\times V_j,\varphi_i\times\psi_j)\}$ atlas in $N\times M$. the set of periodic points.

let $\lambda\subset N\times M$ be any non – empty open set. Then there exist non – empty open sets $U\subset N$ and $V\subset M$ with $U\times V\subset\lambda$. this means that , there exists a point $x\in U$ such that $f^n(x)=x$ with $n\geq 1$.

Similarly , there exists $y\in V$ such that $g^m(y)=y$ with $m\geq 1$. for $p=(x,y)\in\lambda$ and $k=m\times n$ we get $(f\times g)^k(p)=(f\times g)^k(x,y)$ $=(f^k(x),g^k(y))=(x,y)=p$.

We conclude , it contains a periodic point and thus the set of periodic points of $f\times g$ is dense in $N\times M$.

$\Longleftarrow$ Conversely, if $U\subset N$ and $V\subset M$ be non-empty open subsets. Then $U\times V$ is a non-empty open subset of $N\times M$.

Since the set of the periodic points of $f\times g$ is dense in $N\times M$, there exists a point $p=(x,\mathrm{y})$ and $\mathrm{p}\in(\mathrm{U}\times\mathrm{V})$ such that $(f\times g)^n(x,y)=((f^n(x),g^n(y))=(x,y)$ for some n .

We obtain $f^n(x)=x$ for $x\in U$ and similarly we conclude that $g^n(y)=y$ for $y\in V$ also.

By hypotheses denseness of periodic points carries over from factors to products . the converse conclude.

Definition .20

Let M *and* N be smooth manifolds, and $f:M\to N$ be a continuous map . If for every non empty open subsets $U\subset M$ *and* $V\subset N$ there exists a positive integer m_0 such that, for every $m\geq m_0$, $f^m(U)\cap V\neq\phi$ then f is called mixing.

Theorem. 21 Let $f:N\to N$ and $g:M\to M$ be chaotic and topological mixing diffeomorphism maps on smooth manifold systems N & M , with atlases $\{(U_i,\theta_i)\}$ *and* $\{(V_j,\psi_j)\}$ respectively .then the product $f\times g:N\times M\to N\times M$ is chaotic with atlas $\{(U_i\times V_j,\phi_i,\psi_j)\}$.

Proof: the map $f\times g$ has dense periodic points (by theorem 3.11)

And it is topological mixing by hypotheses and hence topologically transitive .

Thus, the two conditions of chaos are satisfied.

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التشاكلات التفاضلية الفوضوية في المنطويات (المنطويات الناعمة)

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الملخص

في هذه الرسالة تم دراسة التشاكلات التفاضلية الفوضوية في اتصالات المنطويات الناعمة.

ويمكن تقديم النتائج التي تم التوصل اليها كما يأتي:

1. الحصول على بين الدوال المتشاكلة التفاضلية الفوضوية، التعدي، الكثيفة، في المنطويات الناعمة.
2. الوصول الى حاصل ضرب تطبيقين كثيفة ومتشاكلة تفاضلية يكون كثيفا في المنطويات الناعمة.
3. حاصل ضرب تطبيقين متشاكلة تفاضلية فوضوية يكون فوضويا في المنطويات الناعمة.