ON (sub- super) asymptotic martingales

Hassan H- Ebrahim¹, Juwan Abbas-Ali²

1 Department of Mathematics, College of Computer Sciences & Mathematics, University of Tikrit, Tikrit, Iraq
2 Department of Mathematics, College of Education for Pure Science, University of Tikrit, Tikrit, Iraq

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1- Introduction

Theory of asymptotic martingales (amart) has been developed and extensively studied in recent years by Bellow[5], Edgar and Sucheston [7], Chacon and Sucheston[6]. These authors are the first to believe that the notion merits a name: asymptotic martingale. A systematic presentation of amart theory paralleling the martingale theory, including for the first time the optional sampling theorem, the Riesz decomposition, the descending and the parameter cases, was given by Edgar among other. It was shown that every real valued amart and every vector-valued uniform amart has a Riesz decomposition.

The amart combines several useful properties of the martingale, submartingale, supermartingale. Thus the class of martingales is closed under linear combinations, the class of supermartingales under infimum, the class of submartingales under supremum, but the class of amart is closed under all three operations[2].

In (1975) R. V. Chacon and L. Sucheston, reduced on convergence of vector asymptotic martingale[8].

We recall that the definition of asymptotic martingale, A sequence \( \{X_n, \mathcal{F}_n, n \geq 1\} \) if and only if for every \( \varepsilon > 0 \), there exist \( t_0 \in T \) such that for every \( t, \sigma \in T, t, \sigma \geq t_0 \) a.s.

We have

\[ ||\int X_t \, dp - \int X_\sigma \, dp|| < \varepsilon. \]

Remark 2.3

1-Martingale ⇒ asymptotic martingale ⇒ super asymptotic martingale.

In this paper we introduce a new class of definitions (sub - super) asymptotic martingale through the concept of asymptotic martingale. we investigate and prove some properties of asymptotic martingale and (sub - super) asymptotic martingale.

2 – (sub - super) asymptotic martingale.

In this section we introduce a new class of definitions (sub - super) asymptotic martingale and prove some properties of asymptotic martingale and (sub - super) asymptotic martingale.

Definition 2.1

A sequence \( \{X_n, \mathcal{F}_n, n \geq 1\} \) is called a sub asymptotic martingale (for short - subamart) if and only if for every \( \varepsilon > 0 \), there exist \( t_0 \in T \) such that for every \( t, \sigma \in T, t, \sigma \geq t_0 \) a.s. We have

\[ ||\int X_t \, dp - \int X_\sigma \, dp|| \leq \varepsilon. \]

Definition 2.2

A sequence \( \{X_n, \mathcal{F}_n, n \geq 1\} \) is called a super asymptotic martingale (for short - superamart) if and only if for every \( \varepsilon > 0 \), there exist \( t_0 \in T \) such that for every \( t, \sigma \in T, t, \sigma \geq t_0 \) a.s. We have

\[ ||\int X_t \, dp - \int X_\sigma \, dp|| \leq \varepsilon. \]
2. sub asymptotic martingale ⇔ super asymptotic martingale.

**Example 2.4**
Let $X_t$ be integrable random variable and $\mathcal{F}_t \subseteq \mathcal{F}_1 \subseteq \ldots$ be a filter on probability space $(\Omega, \mathcal{F}, P)$, $E|X| < \infty$ and define $X_0 = a/2$, then $\{X_n, \mathcal{F}_n\}$ is asymptotic martingale.

**Theorem 2.5**
If $\{X_n, \mathcal{F}_n, n \geq 1\}$ and $\{Y_n, \mathcal{F}_n, n \geq 1\}$are asymptotic martingale, then $\{X_n + Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

With corresponding statements for subasymptotic martingale and supersymptotic martingale.

**Proof**
Since $\{X_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$.
Then $\|X_t, \sigma \mathcal{F}_t - \int X_t \mathcal{F}_t \mathcal{F}_t \| < 1/2 \varepsilon$ ... (2.5.1)
since $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$.
Then $\|Y_t \mathcal{F}_t - \int Y_t \mathcal{F}_t \mathcal{F}_t \| < 1/2 \varepsilon$ ... (2.5.2)
We want to prove that
\[
\|X_t, \sigma \mathcal{F}_t + Y_t \mathcal{F}_t, \sigma \mathcal{F}_t - \int (X_t + Y_t) \mathcal{F}_t \mathcal{F}_t \| < \varepsilon
\]
Thus $\{X_n + Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

**Theorem 2.6**
If $\{X_n, \mathcal{F}_n, n \geq 1\}$ and $\{Y_n, \mathcal{F}_n, n \geq 1\}$ are asymptotic martingale, then $\{X_n - Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

With corresponding statements for subasymptotic martingale and supersymptotic martingale.

**Proof**
Since $\{X_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$.
Then $\|X_t, \sigma \mathcal{F}_t - \int X_t \mathcal{F}_t \mathcal{F}_t \| < 1/2 \varepsilon$ ... (2.6.1)
since $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$.
Then $\|Y_t \mathcal{F}_t - \int Y_t \mathcal{F}_t \mathcal{F}_t \| < 1/2 \varepsilon$ ... (2.6.2)
We want to prove that
\[
\|X_t \mathcal{F}_t - \int (X_t - Y_t) \mathcal{F}_t \mathcal{F}_t \| < \varepsilon
\]
Thus $\{X_n - Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

**Theorem 2.7**
If $\{aX_n, \mathcal{F}_n, n \geq 1\}$ and $\{bY_n, \mathcal{F}_n, n \geq 1\}$ are asymptotic martingale, then $\{aX_n + bY_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

With corresponding statements for subasymptotic martingale and supersymptotic martingale.

**Proof**
Since $\{aX_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$.
Then $|a|\|X_t \mathcal{F}_t - \int X_t \mathcal{F}_t \mathcal{F}_t \| < 1/2 \varepsilon$ ... (2.7.1)
since $\{bY_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$.
Then $|b|\|Y_t \mathcal{F}_t - \int Y_t \mathcal{F}_t \mathcal{F}_t \| < 1/2 \varepsilon$ ... (2.7.2)
We want to prove that
\[
\|aX_t \mathcal{F}_t + bY_t \mathcal{F}_t \mathcal{F}_t \| < \varepsilon
\]
We have from (2.7.1) and (2.7.2) we get
\[
\|aX_t \mathcal{F}_t + bY_t \mathcal{F}_t \mathcal{F}_t \| = \|aX_t \mathcal{F}_t - \int aX_t \mathcal{F}_t \mathcal{F}_t \| + \|bY_t \mathcal{F}_t - \int bY_t \mathcal{F}_t \mathcal{F}_t \| \leq |a|\|X_t \mathcal{F}_t - \int X_t \mathcal{F}_t \mathcal{F}_t \| + |b|\|Y_t \mathcal{F}_t - \int Y_t \mathcal{F}_t \mathcal{F}_t \| < 1/2 \varepsilon + 1/2 \varepsilon = \varepsilon
\]
Thus $\{aX_n + bY_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

**Theorem 2.8**
If $\{\alpha Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale, then $\{\alpha Y_n, \mathcal{F}_n\}$, for $\alpha < 0$, is a super asymptotic martingale $\{\alpha Y_n, \mathcal{F}_n\}$, for $\alpha > 0$, is a sub asymptotic martingale.

**Proof**
For $\alpha < 0$, since $\{Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$,
then $\|Y_t \mathcal{F}_t - \int Y_t \mathcal{F}_t \mathcal{F}_t \| \leq \varepsilon_1$.
That is implies
\[
|\alpha|\|Y_t \mathcal{F}_t - \int Y_t \mathcal{F}_t \mathcal{F}_t \| = \|\alpha Y_t \mathcal{F}_t - \int \alpha Y_t \mathcal{F}_t \mathcal{F}_t \| \leq |\alpha|\varepsilon_1 = \varepsilon
\]
That is $\{\alpha Y_n, \mathcal{F}_n\}$ is a super asymptotic martingale.

For $\alpha > 0$, since $\{Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$,
then $\|Y_t \mathcal{F}_t - \int Y_t \mathcal{F}_t \mathcal{F}_t \| \leq \varepsilon_1$.
That is implies
\[
|\alpha|\|Y_t \mathcal{F}_t - \int Y_t \mathcal{F}_t \mathcal{F}_t \| = \|\alpha Y_t \mathcal{F}_t - \int \alpha Y_t \mathcal{F}_t \mathcal{F}_t \| \leq |\alpha|\varepsilon_1 = \varepsilon
\]
That is $\{\alpha Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale.

**Theorem 2.9**
If $\{X_{1,n}, \mathcal{F}_n\}, \ldots, \{X_{m,n}, \mathcal{F}_n\}$ are a super asymptotic martingale, then $\{\min (X_{1,n}, \ldots, X_{m,n}), \mathcal{F}_n\}$ is a super asymptotic martingale.

**Proof**
We want to prove
\[
\|\min(X_{1,n}, \ldots, X_{m,n}) \mathcal{F}_n - \int \min(X_{1,n}, \ldots, X_{m,n}) \mathcal{F}_n \mathcal{F}_n \| \leq \varepsilon
\]
Since $\{X_{1,n}, \mathcal{F}_n\}, \ldots, \{X_{m,n}, \mathcal{F}_n\}$ are a super asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T$, $t, \sigma \geq t_0$.
That is
\[
\|X_{r,n} \mathcal{F}_n - \int X_{r,n} \mathcal{F}_n \mathcal{F}_n \| \leq \varepsilon, \ldots, \|X_{m,n} \mathcal{F}_n - \int X_{m,n} \mathcal{F}_n \mathcal{F}_n \| \leq \varepsilon
\]
Consider
\[
\|\min(X_{1,n}, \ldots, X_{m,n}) \mathcal{F}_n - \int \min(X_{1,n}, \ldots, X_{m,n}) \mathcal{F}_n \mathcal{F}_n \| \leq \|X_{1,n} \mathcal{F}_n - \int X_{1,n} \mathcal{F}_n \mathcal{F}_n \| \leq \varepsilon \ldots (2.9.1)
\]
\[ \int \min(X_{t_1,n}, \ldots X_{tm,n}) dp - \int \min(X_{\sigma,t_1}, \ldots X_{\sigma,tm}) dp \leq \varepsilon \ldots \ldots (2.9.2) \]

\[ \frac{t}{t_0} \geq \tau_0, \quad \text{Then} \quad \int Y_t dp - \int Y_0 dp = \int \sum_{k=1}^\infty (X_1 + X_2 + \ldots + X_k) dp = \int (X_1 + X_2 + \ldots + X_n) dp = \int X_1 dp + \int X_2 dp + \ldots + \int X_n dp - \int X_1 dp - \int X_2 dp - \int X_n dp \]

\[ \int X_1 dp - \int X_n dp \leq \varepsilon \ldots \ldots (2.10.2) \]

\[ \frac{t}{t_0} \geq \tau_0, \quad \text{Then} \quad \int Y_t dp - \int Y_0 dp = \int \sum_{k=1}^\infty (X_1 + X_2 + \ldots + X_k) dp = \int (X_1 + X_2 + \ldots + X_n) dp = \int X_1 dp + \int X_2 dp + \ldots + \int X_n dp - \int X_1 dp - \int X_2 dp - \int X_n dp \]

\[ \int X_1 dp - \int X_n dp \leq \varepsilon \ldots \ldots (2.10.2) \]

\[ \frac{t}{t_0} \geq \tau_0, \quad \text{Then} \quad \int Y_t dp - \int Y_0 dp = \int \sum_{k=1}^\infty (X_1 + X_2 + \ldots + X_k) dp = \int (X_1 + X_2 + \ldots + X_n) dp = \int X_1 dp + \int X_2 dp + \ldots + \int X_n dp - \int X_1 dp - \int X_2 dp - \int X_n dp \]

\[ \int X_1 dp - \int X_n dp \leq \varepsilon \ldots \ldots (2.10.2) \]

\[ \frac{t}{t_0} \geq \tau_0, \quad \text{Then} \quad \int Y_t dp - \int Y_0 dp = \int \sum_{k=1}^\infty (X_1 + X_2 + \ldots + X_k) dp = \int (X_1 + X_2 + \ldots + X_n) dp = \int X_1 dp + \int X_2 dp + \ldots + \int X_n dp - \int X_1 dp - \int X_2 dp - \int X_n dp \]

\[ \int X_1 dp - \int X_n dp \leq \varepsilon \ldots \ldots (2.10.2) \]
\[ \int_{\mathcal{F}_t} = \int A Y_\omega + \mathcal{A} Y_\omega \quad \ldots \quad (2.15.4) \]
Subtracting (2.15.4) from (2.15.3), then using (2.15.1), we have
\[ \int A Z_t = \int A Y_\omega + \int Y_\omega \int Y_t \leq \int A Z_\omega + \varepsilon \quad \ldots \quad (2.15.5) \]
and (2.15.5) and (2.15.5) we have
\[ \int Z_t \leq \int Z_\omega + \varepsilon \quad (2.15.6) \]
This, together with (2.15.2), yields
\[ \int |Z_t - Z_\omega| \leq 2\varepsilon \quad \ldots \quad (2.15.6) \]
This shows that the net \((Z_t)_t \in \mathbb{T} \) is cauchy, hence convergent.

**Definition 2.16**

Let \((\Omega, \mathcal{F}, P)\) be a probability space \(\{X_1, X_2, \ldots\}\) a sequence of integrable random variable on \((\Omega, \mathcal{F}, P)\)
and \(\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots\) an increasing sequence of sub \(\sigma\)-field of \(\mathcal{F}_n\), is measurable that is \(X_n : (\Omega, \mathcal{F}) \rightarrow [ \mathbb{R}, \mathcal{B}(\mathbb{R}) ] \).
The sequence \(\{X_n, \mathcal{F}_n\}\) is said to be a Martingale if and only if for all \(n = 1, 2, \ldots\), \(E[|X_n|]\mathcal{F}_n = X_n\) a.e. and asubmartingale if and only if \(E[|X_n|]\mathcal{F}_n \geq X_n\) a.s.; and supermartingale if and only if \(E[|X_n|]\mathcal{F}_n \leq X_n\) a.e.

**Example 2.17**

Let \(B_t^1\) be a Brownian motion we compute \(E[B_t^1, t] \mathcal{F}_s\) for \(t \geq s\).
\[ E[B_t^1, t] \mathcal{F}_s] = E[B_t^1, t] B_s^1 B_s^2 = E[B_t^1, t] B_s^1 B_s^2 \mathcal{F}_s] \quad \text{t} \geq s \]
\[ E[B_t^1, t] \mathcal{F}_s] = E[B_t^1, t] B_s^1 B_s^2 \mathcal{F}_s] + 2 E[B_t^1, t] B_s^1 B_s^2 \mathcal{F}_s] \quad s \geq t \]
Using independence, \(E[B_t^1, t] \mathcal{F}_s] = E[B_t^1, t] B_s^1 B_s^2 \mathcal{F}_s] = E[B_t^1, t] \mathcal{F}_s] \quad \text{t} \geq s \]
Of course \(E[B_t^1, t] \mathcal{F}_s] = B_t^1 \mathcal{F}_s] \quad \text{t} \geq s \]
In the last term we use propert (if \(Y \in \mathcal{F}\), \(E[XY] < \infty\), then \(E(XY \mathcal{F}) = XE(Y \mathcal{F})\))
\[ E[(B_t^1, t) \mathcal{F}_s] = B_s^1 B_s^2 \mathcal{F}_s] = B_s^1 B_s^2 \mathcal{F}_s] = 0 \]
Since \(B_t^1, t\) is independence of \(\mathcal{F}_s\),
Thus \(E[B_t^1, t] \mathcal{F}_s] = B_t^1 \mathcal{F}_s] \quad \text{t} \geq s \]
Thus \(B_t^1\) is a martingale.

**Theorem 2.18**

Every martingale is asymptotic martingale.

**Proof**

The adapted sequence \((X_n)_{n \in \mathbb{N}}\) is defined so that martingale if and only if
\[ \int |X_n| < \infty \quad \text{for all } n \quad \text{and } E[X_n \mathcal{F}_n] = X_n \quad \text{for all } n, m \in \mathbb{N} \quad \text{with } n \geq m \quad \text{in particular } X_1 = X_{\omega} \]
If \(t \in \tau\), choose \(n \in \mathbb{N}\) with \(n \geq t\).
Thus \((\int X_t)\) is constant, the sequence \((X_n)_{n \in \mathbb{N}}\) is asymptotic martingale.

**Theorem 2.19**

Let \((X_n, n \geq 1)\) be a martingale for increasing family \((F_n, n \geq 1)\) of \(\sigma\)-field. Let \((G_n, n \geq 1)\) be another increasing family of \(\sigma\)-field with \(G_n \subseteq F_n\) for all \(n \in \mathbb{N}\). Then \(Y_n = E(X_n | G_n)\) is an asymptotic martingale for \((G_n, n \geq 1)\) and not martingale.

**Proof**

we want to prove \((Y_n, G_n, n \geq 1)\) is asymptotic martingale.
for every \(\varepsilon > 0\), there exist \(t_0 \in T\) such that for every \(\ell, s \in T, \ell, s \geq t_0\),
\[ ||Y_{t_0} - Y_\ell dp - \int Y_t dp|| = ||\int E(X_t | G_t) dp - \int E(X_{t_0} | G_t) dp|| = ||\int E(X_{t_0} | G_t) dp - \int E(E(X_{t_0} | G_{t_0}) dp)|| = \]
Since \(G_n \subseteq F_n\)
\[ ||\int E(X_{t_0} | G_t) dp - \int E(X_{t_0} | G_{t_0}) dp|| = ||\int X_t dp - \int X_\ell dp|| \quad \text{by theorem(1.17)} \]
\[ ||\int X_t dp - \int X_\ell dp|| < \varepsilon \]
Thus \((Y_n, G_n, n \geq 1)\) is asymptotic martingale.

We want to prove \((Y_n, G_n, n \geq 1)\) is not martingale.
\[ E(Y_{t_0} | G_{t_0}) = E(E(X_{t_0} | G_{t_0}) G_{t_0}) = E(X_{t_0} | G_{t_0}) = X_{\omega} \neq Y_{t_0} \]

**Theorem 2.20**

If \((X_n, F_n, n \geq 1)\) and \((Y_n, F_n, n \geq 1)\) are martingale, then \((X_n, F_n, n \geq 1)\) is martingale.

**Proof**

By theorem (2.18) and theorem (2.20), then \((X_n, F_n, n \geq 1)\) is asymptotic martingale.

**References**


حول المارتنكل المقارب الجزئي والخاص
حسن حسين إبراهيم ¹، جوان عباس علي ²

¹ قسم الرياضيات، كلية علوم حاسوب والرياضيات، جامعة تكريت، تكريت، العراق
² قسم الرياضيات، كلية التربية لعلوم الصرفة، جامعة تكريت، تكريت، العراق

الملخص
في هذا البحث قمنا صنف جديد من مفاهيم وتعريف المارتنكل المقارب بنوعيه الجزئي والخاص ويردها بعض خصائص المارتنكل المقارب الجزئي والخاص.