Tikrit Journal of Pure Science Vol. 25 (4) 2020

TJPS

IASJ

المجلات الاكاديمية العراقية

Tikrit Journal of Pure Science

ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

Some Topological Concepts Via Graph Theory

Taha H. Jasim[1] , Aiad I. Awad[2]

[1]*Department of Mathematics, College of Computer Science and Mathematics, Tikrit University , Tikrit , Iraq*

[2] *Department of Mathematics, College of Computer Science and Mathematics , Mosul University, Mosul , Iraq*

https://doi.org/10.25130/tjps.v25i4.280

ARTICLE INFO.

Article history:

-Received: 29 / 1 / 2020

-Accepted: 7 / 5 / 2020

-Available online: / / 2020

Keywords: topological graph, graph closure, graph interior, graph exterior, graph boundary, graph limit point.

Corresponding Author:

Name: Aiad I. Awad

E-mail: aiad19ibr@gmail.com

Tel:

ABSTRACT

Due it is difficult to find applications in topological spaces, which are branches of pure mathematics, the importance of this paper is to find applications in graph theory. Via some concepts of topological space we generalizes to a graph like (graph interior, graph closure graph exterior, graph boundary, graph limit point). The relations among them were studied. At least many theorems were proofed as a characterization and some examples introduced to explain the subject.

1. Introduction and preliminaries

Topology is one of branches of mathematics, which is interested in studying the construct, components and peculiarities of all different spaces, so that these characteristics[1]. If Y is a non-empty set, a collection τ from partial sets from Y is called a topology at Y,if the following provision carry $Y, \emptyset \in \tau$, the finite "intersection" from any two sets at τ belong into τ, and the "union" from any numeral from sets at τ belong to τ [2]. Both element in the topology is said to be open set, her complement is a closed set [3]. The closure of a subset U briefly, $Cl(U)$ is the smallest closed set that include U [4]. The interior of a partial set U briefly, $Int(U)$ is the largest open set that is include in U [4]. The exterior of U is the interior from $C(U)$ [1].The boundary from U is $Cl(U) \cap Cl(C(U))$ the set of points that belong to interior don't the exterior of A, and limit point [1].

A graph G is defined as a non-empty set M of elements called "vertices" and we symbolize it sometimes by $M(G)$ with the N family of unordered pairs of vertices set and each element of E is called "edge" and we symbolize it sometimes by $N(G)$ [5]. Sometimes we express the graph G of his vertices set and his family edges N of the ordered pairs $(M(G), N(G))$ [5]. The numeral of vertices in a graph G is the order from G, and the numeral of edges is the volume from G [6]. Also, the statement may contain an edge that reaches to a vertex itself called a "loop"[2]. Two or more edges that link the same pair of different vertices are refer to "parallel edges"[7]. Let $G = (M(G), N(G))$ be a graph; we name U a "subgraph" of G if $M(U) \subseteq M(G)$ and $N(U) \subseteq N(G)$, in whose state we write $U \subseteq G$ [7]. The number of edges on the vertex m is defined by a degree and denoted by the symbol $\deg(m)$ [6]. A simple graph that does not contain "loops and double edge" [5].

In this research, we found a new definition of a relation to extract a topology of any graph and study some peculiarities. Conduct the research, and some terms peculiarities on (closure, interior, exterior, boundary and limit point) for "topological graph" will be studied.

2. Construct A Topology Via Graph

Definition 2.1 : Suppose that $G = (M(G), N(G))$ be a graph, $m \in M(G)$ then we define the post stage mR is the set of all vertices which is not neighborhood of m. S_G is the collection of (mR) is called subbasis of graph. $B_G = \bigcap_{i=1}^{n} S_{G_i}$ is called bases of graph. Then the union of B_G is form a topology on G and $(M(G), \tau_G)$ is called topological graph.

Remark 2.2 : Every topological construct τ_G onto a graph G is topological graph.

Proof : Let τ_G be a topological construct for a graph G. Now, we prove that τ_G is a topological graph.

(i) Since $Y = \cup_{j\in J} B_j$ where $B_j \in B_G$, then $B_j = \cap_{j=1}^{n} S_j$, where $S_j \in S_G$ and $S_j = v_j R$, $v \in Y$. Then $Y = \cup_{j\in I} (\cap_{j=1}^{n} v_j R)$, and so $Y \in \tau_G$. Also, as is obvious that $\emptyset \in \tau_G$ by complement from Y.

(ii) Let $A_j \in \tau_G$, $A_j = \cup_{j\in J} B_j$, where $B_j \in B_G$, $B_j = \cap_{j=1}^{n} S_j$, where $S_j \in S_G$ and $S_j = v_j R$, $v \in A_j$, then $A_j = \cup_{j\in J} (\cap_{j=1}^{n} v_j R)$, where $v_j R = \{deg_G(b_r)_{\text{br}}, r \in I\}$ and $A_j = \cup_{j\in J} (\cap_{j=1}^{n} deg_G(b_r))$, then $\cup_{j\in I} (\cap_{j=1}^{n} deg_G(b_r)) \in \tau_G$, where b_r are all the not neighborhood from v_j and $\cup_{j\in J} A_j \in \tau_G$.

(iii) Let $A_j, C_j \in \tau_G$, $A_i = \cup_{j\in J} B_j$, where $B_j \in B_G$, $B_i = \cap_{j=1}^{n} S_j$, where $S_j \in S_G$, $S_j = v_j R$, $v \in A_j$ then $A_j = \cup_{j\in J} (\cap_{j=1}^{n} v_j R)$. Then, $C_j = \cup_{j\in J} (\cap_{j=1}^{n} v_j R)$, there are two cases :

Case I: If there are no elements in intersection, i.e, $A_j \cap C_j = \emptyset$, since $\emptyset \in \tau_G$, then $A_j \cap C_j \in \tau_G$.

Case II: If there exist elements in intersection $A_j \cap C_j$, then we denote it $\{y_n : n \in N\}$. So $\{y_n\} \in v_j R$, $A_j = \cup_{j\in J} (\cap_{j=1}^{n} v_j R)$, and $C_j = \cup_{j\in J} (\cap_{j=1}^{n} v_j R)$.

So $\{y_n : n \in N\}$ one of these classes. Therefore, $\{y_n : n \in N\}$.

Let us give some examples to explain the above theorem.

Example 2.3 : Suppose that $G = (M(G), N(G))$ be a "simple graph" (see Figure 1). We build topological space as follows:

$m_1 R = \{m_3\}$,
$m_2 R = \{m_4, m_5\}$,
$m_3 R = \{m_1, m_5\}$,
$m_4 R = \{m_2\}$,
$m_5 R = \{m_2, m_3\}$. Then a subbase of a topology is $S_G = \{\{m_3\}, \{m_4, m_5\}, \{m_1, m_5\}, \{m_2\}, \{m_2, m_3\}\}$.
The base is
$B_G = \{M(G), \emptyset, \{m_3\}, \{m_5\}, \{m_2\}, \{m_1, m_5\}, \{m_2, m_3\}, \{m_4, m_5\}\}$.
Therefore, the topological graph on G will be $\tau_G = \{M(G), \emptyset, \{m_3\}, \{m_5\}, \{m_2\}, \{m_1, m_5\}, \{m_2, m_3\}, \{m_4, m_5\}, \{v_3, m_5\}, \{m_2, m_3\}, \{m_1, m_3, m_5\}, \{m_2, m_5\}, \{m_3, m_4, m_5\}, \{m_2, m_3, m_5\}, \{m_1, m_2, m_5\}, \{m_2, m_4, v_5\}, \{m_2, m_3, m_4, m_5\}, \{m_1, m_4, m_5\}, \{m_1, m_2, m_3, m_5\}\}$

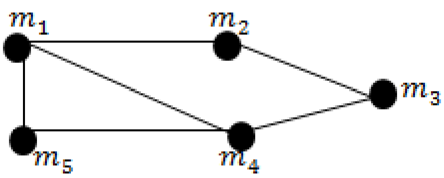

Fig. 1

Example 2.4: Suppose that $G = (M(G), N(G))$ be a (non-simple graph)which has double edges and loops (see Figure 2).

$m_1 R = \{m_3\}$,
$m_2 R = \{m_4, m_5\}$,
$m_3 R = \{m_1, m_5\}$,
$m_4 R = \{m_2\}$,
$m_5 R = \{m_2, m_3\}$. Then a subbase of a topology is $S_G = \{\{m_3\}, \{m_4, m_5\}, \{m_1, m_5\}, \{m_2\}, \{m_2, m_3\}\}$.
The base is
$B_G = \{M(G), \emptyset, \{m_3\}, \{m_5\}, \{m_2\}, \{m_1, m_5\}, \{m_2, m_3\}, \{m_4, m_5\}\}$
. Therefore, the topological graph on G will be
$[\tau_G = \{M(G), \emptyset, \{m_3\}, \{m_5\}, \{m_2\}, \{m_1, m_5\}, \{m_2, m_3\}, \{m_4, m_5\}, \{m_3, m_5\}, \{m_2, m_3\}, \{m_1, m_3, m_5\}, \{m_2, m_5\}, \{m_3, m_4, m_5\}, \{m_2, m_3, m_5\}, \{m_1, m_2, m_5\}, \{m_2, m_4, m_5\}, \{m_2, m_3, m_4, m_5\}, \{m_1, m_4, m_5\}, \{m_1, m_2, m_3, m_5\}\}$.

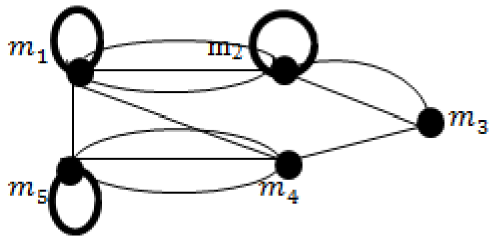

Fig. 2

Remark 2.5 : The complete graph is an indiscrete topology.

Definition 2.6 : Suppose that $G = (M(G), N(G))$ be a graph, U be a subgraph from G. Then the graph closure of $M(U)$ has the shape
$Cl_G(M(U)) = M(U) \cup \{m \in M(G) : mR \cap M(U) \neq \emptyset\}$.

Theorem 2.7 : Suppose that $G = (M(G), N(G))$ be a graph that contains a topological graph $(M(G), \tau_G)$. If U, W are subgraphs from G; then:
(i) $M(U) \subseteq Cl_G(M(U))$.
(ii) If $U \subseteq W$, then $Cl_G(M(U)) \subseteq Cl_G(M(W))$.
(iii) $Cl_G(Cl_G(M(U))) \neq Cl_G(M(U))$.
(iv) $Cl_G(M(U) \cup M(W)) = Cl_G(M(U) \cup Cl_G(M(W))$.
(v) $Cl_G(M(U) \cap M(W)) \subseteq Cl_G(M(U) \cap Cl_G(M(W))$.

Proof: (i) Suppose that $m \in M(U)$, by definition 2.6. $Cl_G(M(U)) = M(U) \cup \{m \in M(G) : mR \cap M(U) \neq \emptyset\}$. Then $M(U) \subseteq Cl_G(M(U))$.
(ii) From (i), $M(U) \subseteq Cl_G(M(U))$ and $M(W) \subseteq Cl_G(M(W))$. Since $U \subseteq W$, then $M(U) \subseteq M(W)$. Therefore, $Cl_G(M(U)) \subseteq Cl_G(M(W))$.
(iii) Suppose that $m \in M(U)$, by definition 2.6. $Cl_G(M(U)) = M(U) \cup \{m \in M(G) : mR \cap M(U) \neq \emptyset\}$.Then $M(U) \in Cl_G(M(U))$, since $Cl_G(M(U)) \subseteq Cl_G(Cl_G(M(U)))$, then $Cl_G(Cl_G(M(U))) \nsubseteq Cl_G(M(U))$. Therefore, $Cl_G(Cl_G(M(U))) \neq Cl_G(M(U))$.
(iv) From Theorem 2.7, it is obvious that $M(U) \cup M(W) \subseteq Cl_G(M(U)) \cup M(W))$. If $m \in Cl_G(M(U)) \cup M(W))$, then by definition 2.6, $m \in M(U) \cup M(W)$ or $mR \cap (M(U) \cup M(W)) \neq \emptyset$. Then
$m \in M(U)$ or $m \in M(W)$ or $vR \cap M(U) \neq \emptyset$ or $mR \cap M(W) \neq \emptyset$. Thus

$(m \in M(U)$ or $mR \cap M(U) \neq \emptyset)$ or $(m \in M(W)$ or $mR \cap M(W) \neq \emptyset)$. Then $m \in Cl_G(M(U))$ or $m \in Cl_G(M(W))$. Therefore, $m \in Cl_G(M(U) \cup Cl_G(M(W))$. Hence, $Cl_G(M(U) \cup M(W)) \subseteq Cl_G(M(U) \cup Cl_G(M(W))$.

(v) $M(U) \subseteq Cl_G(M(U)), M(W) \subseteq Cl_G(M(W))$. Since $M(U) \cap M(W) \subseteq M(U)$, $M(U) \cap M(W) \subseteq M(W)$. Then $Cl_G(M(U) \cap M(W)) \subseteq Cl_G(M(U))$, $Cl_G(M(U) \cap M(W)) \subseteq Cl_G(M(W))$. Therefore, $Cl_G(M(U) \cap M(W)) \subseteq Cl_G(M(U) \cap Cl_G(M(W))$.

Example 2.8 : Via Example 2.3. Suppose that U, W are subgraphs of G with vertices $M(U) = \{m_1, m_3\}$, $M(W) = \{m_1, m_3, m_5\}, M(U) \cup M(W) = \{m_1, m_3, m_5\}$. Then $m_2R = \{m_4, m_5\}, m_4R = \{m_2\}, m_5R = \{m_2, m_3\}$,. So $m_2R \cap M(U) = \emptyset, m_4R \cap M(U) = \emptyset, m_5R \cap M(U) \neq \emptyset$. Then $Cl_G(M(U)) = M(U) \cup \{m_5\} = \{m_1, m_3, m_5\}$, $Cl_G(M(W)) = M(W) \cup \{m_2\} = \{m_1, m_2, m_3, m_5\}$, $Cl_G(Cl_G(M(U))) = \{m_1, m_2, m_3, m_5\}$, $Cl_G(M(U) \cup M(W)) = \{m_1, m_2, m_3, m_5\}$. Since $Cl_G(Cl_G(M(U))) \neq Cl_G(M(U))$. Then $Cl_G(M(U) \cup V(W)) = Cl_G(M(U) \cup Cl_G(M(W))$.

Definition 2.9 : Suppose that $G = (M(G), N(G))$ be a graph, U be a subgraph from G. Then the graph interior of $M(U)$ has the shape $Int_G(M(U)) = \{m \in M(G): mR \subseteq M(U)\}$.

Theorem 2.10 : Suppose that $G = (M(G), N(G))$ be a graph that contains a topological graph $(M(G), \tau_G)$. If U, W are subgraphs from G; then:

(i) If $U \subseteq G$, then $Int_G(M(U)) \subseteq M(G)$.

(ii) If $U \subseteq W$, then $Int_G(M(U)) \subseteq Int_G(M(W))$.

(iii) $Int_G(Int_G(M(U))) \neq Int_G(M(U))$.

(iv) $Int_G(M(U) \cap M(W)) = Int_G(M(U)) \cap Int_G(M(W))$.

(v) $Int_G(M(U)) \cup Int_G(M(W)) \subseteq Int_G(M(U) \cup M(W))$.

Proof : (i) Since $M(U) \subseteq M(G)$, then by definition 2.9, $Int_G(M(U)) = \{m \in M(G): vR \subseteq M(U)\}$.This means that $Int_G(M(U)) \subseteq M(G)$. (ii) Since $M(U) \subseteq M(W)$, then by definition 2.9 ,$Int_G(M(U)) = \{m \in M(G): mR \subseteq M(U)\} \subseteq \{m \in M(G): mR \subseteq M(U) \subseteq M(W)\} \subseteq Int_G(M(W))$. Therefore, $Int_G(M(U)) \subseteq Int_G(M(W))$.

(iii) Suppose that $m \in M(U)$, by definition 2.9. $Int_G(M(U)) = \{m \in M(G): mR \subseteq M(U)\}$. Then $M(U) \nsubseteq Int_G(M(U))$, since $Int_G(M(U)) \nsubseteq Int_G(Int_G(M(U)))$, then $Int_G(Int_G(M(U))) \nsubseteq Int_G(M(U))$. Therefore, $Int_G(Int_G(M(U))) \neq Int_G(M(U))$. (iv) From (i), it is obvious that $Int_G(M(H) \cap M(W)) \subseteq M(U) \cap M(W)$. Then by definition 2.9, if $m \in Int_G(M(U) \cap M(W))$, then $m \in M(G)$, such that $mR \subseteq M(U) \cap M(W)$. Then $vR \subseteq M(U)$ and $mR \subseteq M(W)$. Therefore, $m \in Int_G(M(U))$ and $m \in Int_G(M(W))$. Then $m \in Int_G(M(U) \cap M(W))$.

Conversely $Int_G(M(U)) \cap Int_G(M(W)) \subseteq Int_G(M(U) \cap M(W))$. Let $m \in Int_G(M(U)) \cap Int_G(M(W))$. Then $m \in Int_G(M(U))$ and $m \in Int_G(M(W))$, by definition 2.9, for all $m \in M(G)$ such that $mR \subseteq M(U)$ and for all $m \in M(G)$ such that $mR \subseteq M(W)$. Then for all $m \in M(G)$ such that $mR \subseteq M(U) \cap M(W)$. Therefore, $m \in Int_G(M(U) \cap M(W))$. The proof is complete.

(v) Suppose that $M(U), M(W) \subseteq M(G)$, since $M(U) \subseteq M(U) \cup M(W), M(W) \subseteq M(U) \cup M(W)$. Then $Int_G(M(U)) \subseteq Int_G(M(U) \cup M(W)), Int_G(M(W)) \subseteq Int_G(M(U) \cup M(W))$. Therefore, $Int_G(M(U)) \cup Int_G(M(W)) \subseteq Int_G(M(U) \cup M(W))$.

Example 2.11 : Via Example 2.3. Suppose that U, W are subgraphs from G with vertices $(U) = \{m_1, m_3\}, M(W) = \{m_1, m_3, m_5\}, M(U) \cap M(W) = \{m_1, m_3\}, Int_G(M(U)) = \{m_1\}, Int_G(M(W)) = \{m_1, m_3\}, Int_G(Int_G(M(W))) = \emptyset, Int_G(M(U) \cap M(W) = \{m_1\}$. Since $Int_G(Int_G(M(U))) \neq Int_G(M(U))$. Then $Int_G(M(U) \cap M(W)) = Int_G(M(U)) \cap Int_G(M(W))$.

Definition 2.12 : Suppose that $G = (M(G), N(G))$ be a graph, U be a subgraph from G. Then the graph exterior from $M(U)$ has the shape $Ext_G(M(U)) = \{m \in M(G): mR \cap M(U) = \emptyset\}$.

Theorem 2.13 : Suppose that $G = (M(G), N(G))$ be a graph that contains a topological graph $(M(G), \tau_G)$. If H, W are subgraphs from G; then:

(i) $Ext_G(M(U)) = Int_G(C(M(U)))$.

(ii) $Ext_G(M(U)) \cap M(U) = \emptyset$.

(iii) $Ext_G(M(U)) = C(Cl_G(M(U)))$.

(iv) If $U \subseteq W$, then $Ext_G(M(W)) \subseteq Ext_G(M(U))$.

(v)$Ext_G(M(U) \cup M(W)) \subseteq Ext_G(M(U)) \cap Ext_G(M(W))$.

Proof: (i) Suppose that $m \in Ext_G(M(U))$, then $mR \subset C(M(U))$, if and only if $m \in Int_G(C(M(U)))$. Therefore, $Ext_G(M(U)) = Int_G(C(M(U)))$.

(ii) Suppose that $m \in Ext_G(M(U))$, then $Ext_G(M(U)) \cap M(U) \subset C(M(U)) \cap M(U))$, $Ext_G(M(U)) \cap M(U) \subset \emptyset$. Therefore, $Ext_G(M(U)) \cap M(U) = \emptyset$.

(iii) Suppose that $m \in Ext_G(M(U))$, from (i) $Ext_G(M(U)) = Int_G(C(M(U)))$. $C(Ext_G(M(U))) = C(Int_G(C(M(U))))$. $C(Ext_G(M(U))) = Cl_G(M(U))$, by taking complement both sides, $Ext_G(M(U)) = C(Cl_G(M(U)))$.

(iv) $C(M(U)) \subset M(W)), C(M(W)) \subset C(M(U))$, by taking interior both sides, $Int_G(C(M(W))) \subset Int_G(C(M(U)))$,from (i). Then $Ext_G(M(U)) = Int_G(C(M(U)))$. Therefore, $Ext_G(M(W)) \subseteq Ext_G(M(U))$.

(v) Suppose that $m \in Ext_G(U \cup W)$, from (i)

$Ext_G(U \cup W) = Int_G(C(M(U) \cup M(W)))$

$=Int_G(C(M(U)) \cap C(M(W)))$

$= Int_G(C(M(U))) \cap Int_G(C(M(W)))$

$= Ext_G(M(U)) \cap Ext_G(M(W))$.

Example 2.14 : Via Example 2.3, if U be a subgraph from G.
$M(U)=\{m_1,m_2\}, m_1R\cap M(U)\neq\emptyset, m_2R\cap M(U)=\emptyset, m_3R\cap M(U)\neq\emptyset, m_4R\cap M(U)=\emptyset, m_5R\cap M(U)\neq\emptyset$. Then $Ext_G(M(U))=\{m_2,m_4\}, C(M(U))=\{m_2,m_4,m_5\}$, $Int_G(C(M(U)))=\{m_2,m_4\}$. Therefore, $Ext_G(M(U))=Int_G(C(M(U)))$.

Definition 2.15 : Suppose that $G=(M(G),N(G))$ be a graph and U be a subgraph from G. Then the graph boundary from $M(U)$ has the shape $\dot{B}_G(M(U))=\{m\in M(G): Cl_G(M(U))-Int_G(M(U))\}$.

Theorem 2.16 : Suppose that $G=(M(G),E(G))$ be a graph that contains a topological graph $(M(G),\tau_G)$. If U be a subgraph from G; then:
(i) $\dot{B}_G(M(U))=Cl_G(M(U))\cap Cl_G(C(M(U)))$.
(ii) $\dot{B}_G(M(U))\subseteq B_G(C(M(U)))$.
(iii) $\dot{B}_G(M(G))=\emptyset$.
(iv) $\dot{B}_G(M(U))=Cl_G(M(U))-Int_G(M(U))$.
(v) $Cl_G(M(U)=M(U)\cup\dot{B}_G(M(U))$.

Proof: (i) Suppose that $m\in M(U), M(U)\in Cl_G(M(U))$, then $m\in Cl_G(M(U))$ and $m\in Cl_G(C(M(U)))$, then $m\in Cl_G(M(U))\cap Cl_G(C(M(U)))$.
$\dot{B}_G(M(U))=Cl_G(M(U))\cap Cl_G(C(M(U)))$.
(ii) Suppose that $m\in M(U)$, by definition 2.15.
$\dot{B}_G(M(U))=Cl_G(M(U))-Int_G(M(U))\}$, from (i) $B_G(M(U))=Cl_G(M(U))\cap Cl_G(C(M(U)))$. Since $Cl_G(C(M(U)))=Cl_G(C(M(U))\cap Cl_G(C(C(M(U))))$. Therefore, $\dot{B}_G(M(U))\subseteq\dot{B}_G(C(M(U)))$.
(iii) From (i), $B_G(M(G))=Cl_G(M(G))\cap Cl_G(C(M(G)))=M(G)\cap Cl_G\emptyset$ $=M(G)\cap\emptyset=\emptyset$. Therefore, $\dot{B}_G(M(U))=\emptyset$.
(iv) Suppose that $m\in M(U)$, from (i) $\dot{B}_G(M(U))=Cl_G(M(U))\cap Cl_G(C(M(U)))$ $=Cl_G(M(U))-C(Cl_G(C(M(U))))=Cl_G(M(U))-Int_G(M(U))$.
(v) Suppose that $v\in M(U), B_G(M(U))\cup Int_G(M(U))=(Cl_G(M(U))-Int_G(M(U)))\cup Int_G(M(U))$. Therefore, $Cl_G(M(U)=M(U)\cup\dot{B}_G(M(U))$.

Theorem 2.17 : Suppose that $G=(M(G),E(G))$ be a graph that contains a topological graph $(M(G),\tau_G)$. If U,W are subgraphs from G.
Then $\dot{B}_G(M(U)\cup M(W))\subseteq\dot{B}_G(M(U))\cup\dot{B}_G(M(W))$.

Proof: Suppose that $m\in\dot{B}_G(M(U)\cup M(W))$, then by definition 2.16,
$mR\cap(M(U)\cup M(W))\neq\emptyset\ and\ mR\cap C(M(U)\cup M(W))$
$\rightarrow(mR\cap M(U))\cup(mR\cap M(W))\neq\emptyset\ and\ mR\cap(C(M(U))\cap C(M(W)))\neq\emptyset\ \rightarrow(mR\cap M(U))\cup(mR\cap M(W))\neq\emptyset\ and\ (mR\cap C(M(U)))\cup(mR\cap C(M(W)))\neq\emptyset$
$\rightarrow[mR\cap M(U)\neq\emptyset\ or\ mR\cap M(W)\neq\emptyset\,]\ and\ [mR\cap C(M(U))\neq\emptyset\ and\ mR\cap C(M(W))\neq\emptyset\,]$
$\rightarrow[mR\cap M(U)\neq\emptyset\ and\ mR\cap M(W)\neq\emptyset\,]\ or\ [mR\cap C(M(U))\neq\emptyset\ and\ mR\cap C(M(W))\neq\emptyset\,]$, such that
$m\in\dot{B}_G(M(U))\ or\ m\in\dot{B}_G(M(W)), m\in\dot{B}_G(M(U))\cup\dot{B}_G(M(W))$. Therefore, $\dot{B}_G(M(U)\cup M(W))\subseteq\dot{B}_G(M(U))\cup\dot{B}_G(M(W))$.

Example 2.18 : Via Example 2.3, Suppose that U be a subgraph from $G$with vertices $M(U)=\{m_1,m_3\}$, $C(M(U))=\{m_2,m_4,m_5\}, Cl_G(M(U))=\{m_1,m_3,m_5\}$,
$Int_G(M(U)))=\{m_1\}, Cl_G(C(M(U)))=\{m_2,m_3,m_4,m_5\}, \dot{B}_G(M(U))=\{m_3,m_5\}, Cl_G(M(U))\cap Cl_G(C(M(U)))=\{m_3,m_5\}$. Therefore,
$\dot{B}_G(M(U))=Cl_G(M(U))\cap Cl_G(C(M(U)))$.

Theorem 2.19 : Suppose that $G=(M(G),N(G))$ be a graph that contains a topological graph $(M(G),\tau_G)$. If U be a subgraph from G; then
(i) $\dot{B}_G(M(U))\cap Int_G(M(U))=\emptyset$.
(ii) $\dot{B}_G(M(U))\cap Ext_G(M(U))=\emptyset$.
(iii) $Int_G(M(U))\cap Ext_G(M(U))=\emptyset$.
(iv) $Int_G(M(U))\cup Ext_G(M(U))\cup\dot{B}_G(M(U))=G$.

Proof: (i) Suppose that $m\in M(G), M(U)\subseteq M(G)$, by definition 2.15,
$\dot{B}_G(M(U))\cap Int_G(M(U))=(Cl_G(M(U))-Int_G(M(U)))\cap Int_G(M(U))$, by distributing intersection,
$\rightarrow(Cl_G(M(U))\cap Int_G(M(U)))-(Int_G(M(U))\cap Int_G(M(U)))$
$=Int_G(M(U))-Int_G(M(U))=\emptyset$.
(ii) Suppose that $M(U)\subseteq M(G), \dot{B}_G(M(U))\cap Ext_G(M(U))$, by theorem 2.13
$\rightarrow\dot{B}_G(C(M(U)))\cap Int_G(C(M(U)))$
$=\dot{B}_G(G-M(U))\cap Int_G(C(G-M(U)))=\emptyset$.
(iii) Suppose that $M(U)\subseteq M(G), Int_G(M(U))\cap Ext_G(M(U))$
$=\ Int_G(M(U))\cap Int_G(C(M(U)))=Int_G(M(U))\cap(C(Cl_G(M(U)))$
$=Int_G(M(U))\cap(G-Cl_G(V(U)))$, by distributing intersection,
$\rightarrow(Int_G(M(U))\cap G)-(Int_G(M(U))\cap Cl_G(M(U)))$
$=Int_G(M(U))-Int_G(M(U))=\emptyset$.
(iv) Suppose that $M(U)\subseteq M(G), Int_G(M(U))\cup Ext_G(M(U))\cup\dot{B}_G(M(U))$
$=Cl_G(M(U))\cup Int_G(C(M(U)))=Cl_G(M(U))\cup C(Cl_G(M(U)))=G$.

Theorem 2.20 : Suppose that $G=(M(G),N(G))$ be a graph that contains a topological graph $(M(G),\tau_G)$. If $Cl_G(M(U))\cap Cl_G(M(W))=\emptyset$, then
$Int_G(M(U))\cup Int_G(M(W))=Int_G(M(U)\cup M(W))$.

Proof: It`s clear by theorem 2.10, $Int_G(M(U))\cup Int_G(M(W))\subseteq Int_G(M(U)\cup M(W))$. To prove that $Int_G(M(U)\cup M(W))\subseteq Int_G(M(U))\cup Int_G(M(W))$.
Assume that $m\notin Int_G(M(U))\cup Int_G(M(W))$ and suppose that $m\in Int_G(M(U)\cup M(W))$, there exist a post stage A, so that $m\in A\subseteq M(U)\cup M(W)$, since

$A \subseteq M(U)\ or\ A \subseteq M(W)\ or\ (A \not\subset M(U)\ \&\ A \not\subset M(W))$.

(i) If $A \subseteq M(U)$, then $m \in Int_G(M(U))$, Therefore $m \in Int_G(M(W)) \cup Int_G(M(U))$.

(ii) If $A \subseteq M(W)$, then $m \in Int_G(M(W))$, Therefore $m \in Int_G(M(W)) \cup Int_G(M(U))$.

(iii) If $A \not\subset M(U)\ \&\ A \not\subset M(W)$, then $A \cap M(U) \neq \emptyset, A \cap M(W) \neq \emptyset$. For every $W \in M(G), m \in W$, we get $m \in Cl_G(M(U)), m \in Cl_G(M(W))$. This is contraction, because $Cl_G(M(U)) \cap Cl_G(M(W)) = \emptyset$. Therefore, (i) and (ii) true. Then $Int_G(M(U) \cup M(W)) \subseteq Int_G(M(U)) \cup Int_G(M(W))$.

Example 2.21 : Via example 2.3, suppose that U be a subgraph from G.

$M(U) = \{m_1\}, M(W) = \{m_2\}, M(U) \cup M(W) = \{m_1, m_2\}, Cl_G(M(U)) = \{m_1, m_3\}, Cl_G(M(W)) = \{m_2, m_4, m_5\}$, then $Cl_G(M(U)) \cap Cl_G(M(W)) = \emptyset$, $Int_G(M(U)) = \emptyset, Int_G(M(W)) = \{m_4\}, Int_G(M(H) \cup M(W)) = \{m_4\}$. Therefore, $Int_G(M(U)) \cup Int_G(M(W)) = Int_G(M(U) \cup M(W))$.

Definition 2.22 : Suppose that $G = (M(G), N(G))$ be a graph and U be a subgraph from G. It is said that the vertex $m \in M(U)$ is limit point to $M(U)$ if each set mR contain m. Also contain a point other than m shape:

$\dot{d}_G = \{m \in M(G): mR \cap M(U) - \{m\} \neq \emptyset\}$.

Theorem 2.23 : Suppose that $G = (M(G), N(G))$ be a graph that contains a topological graph $(M(G), \tau_G)$. If U, W are subgraphs from G; then

(i) If $U \subseteq W$, then $\dot{d}_G(M(U)) \subseteq \dot{d}_G(M(W))$.

(ii) $\dot{d}_G(M(U) \cup M(W)) = \dot{d}_G(M(U)) \cup \dot{d}_G(M(W))$.

(iii) $\dot{d}_G(M(U) \cap M(W)) \subseteq \dot{d}_G(M(U)) \cap \dot{d}_G(M(W))$.

Proof: (i) Suppose that $m \in M(U)$, then $m \in mR\ and\ mR \cap M(U) - \{m\} \neq \emptyset$, $m \in mR\ and\ mR \cap M(W) - \{m\} \neq \emptyset$, since $M(U) \subseteq M(W)$, by definition 2.22, $m \in M(W)$. Therefore, $\dot{d}_G(M(U)) \subseteq \dot{d}_G(M(W))$.

(ii) To prove that $\dot{d}_G(M(U) \cup M(W)) = \dot{d}_G(M(U)) \cup \dot{d}_G(M(W))$.

$M(U) \subseteq (M(U) \cup M(W))$, by definition 2.22, $\dot{d}_G(M(U)) \subseteq \dot{d}_G(M(U) \cup M(W))$, $\dot{d}_G(M(W)) \subseteq \dot{d}_G((M(U) \cup M(W))$. Therefore,

$\dot{d}_G(M(U)) \cup \dot{d}_G(V(W)) \subseteq \dot{d}_G(M(U) \cup M(W))$...... 1)

Conversely

Suppose that $m \notin \dot{d}_G(M(U) \cup M(W))$, then $m \notin \dot{d}_G(M(U)$ and $v \notin \dot{d}_G(M(W)$, by definition 2.21, $m \in xR\ and\ xR \cap M(U) - \{m\} = \emptyset\ and\ m \in yR\ and\ yR \cap M(U) - \{m\} = \emptyset$, $m \in xR \cap yR\ and\ (xR \cap yR) \cap (V(U) \cup M(W)) - \{m\} = \emptyset$, since $m \notin \dot{d}_G(M(U) \cup V(W))$. Therefore, $\dot{d}_G(M(U) \cup M(W)) \subseteq \dot{d}_G(M(U)) \cup \dot{d}_G(M(W))$............(2)

From (1) and (2) we have $\dot{d}_G(M(U) \cup M(W)) = \dot{d}_G(M(U)) \cup \dot{d}_G(M(W))$.

(iii) To prove $\dot{d}_G(M(U) \cap M(W)) \subseteq \dot{d}_G(M(U)) \cap \dot{d}_G(M(W))$.

Let $m \in \dot{d}_G(M(U) \cap M(W))$, by definition 2.22, $m \in mR\ and\ mR \cap (M(U) \cap M(W)) - \{m\} \neq \emptyset$, then $m \in mR\ and\ [(mR \cap M(U) - \{m\}) \cap (mR \cap M(W) - \{m\})] \neq \emptyset$, by definition 2.22, $(mR \cap M(U) - \{m\}) \neq \emptyset\ and\ (mR \cap M(W) - \{m\}) \neq \emptyset$, then $m \in \dot{d}_G(M(U)), m \in \dot{d}_G(M(W))$, $m \in \dot{d}_G(M(U) \cap M(W))$. Therefore, $\dot{d}_G(M(U) \cap M(W)) \subseteq \dot{d}_G(M(U)) \cap \dot{d}_G(M(W))$.

Example 2.24: Via Example 2.3, suppose that U, W are subgraphs from G.

$M(U) = \{m_1, m_3\}, M(W) = \{m_1, m_3, m_5\}$, Then $\dot{d}_G\big(M(U)\big) = \{m_1, m_4\}, \dot{d}_G(M(W)) = \{m_1, m_4\}$. Therefore, $\dot{d}_G(M(U)) \subseteq \dot{d}_G(M(W))$.

Conclusion

In this paper, we were can topological construct of any graph by using the definition of topological graph, we studied the graph closure, graph exterior, graph interior, graph boundary, and graph limit point with some result.

References

[1] Engelking, R.(1976). "General Topology", PWN-Polish Scientific Publishers.

[2] Munkres, J.(2000). "Topology", 2nd; Prentice-Hall: Upper Saddle River, NJ, USA.

[3] Kelley J. L. (1975). "General Topology", Nostrand; Springer-Verlag, New York.

[4] Jung, S.-M. (2016). "Interiors and closures of sets and applications". Int. J. Pure Math. 3, 41–45.

[5] Bondy. J.A and Murty. U.S.R. (2008). "Graph Theory", Springer, Berli.

[6] Chartrand. G, Lesniak. L, Zhang. P. (2016). Textbooks in Mathematics "Graphs and Digraphs", Taylor and Francis Group, LLC.

[7] Diestel. R. (2000). "Graph Theory", Springer-Verlag Heidelberg Press, New York.

بعض المفاهيم التوبولوجية بواسطة نظرية البيان

طه حميد جاسم[1] ، اياد ابراهيم عواد[2]

[1] قسم الرياضيات , كلية علوم الحاسوب والرياضيات , جامعة تكريت ، تكريت ، العراق

[2] قسم الرياضيات, كلية علوم الحاسوب والرياضيات , جامعة الموصل ، الموصل ، العراق

الملخص

نظرا لصعوبة ايجاد تطبيقات على الفضاءات التوبولوجية والتي تعتبر من فروع الرياضيات البحتة تأتي اهمية هذا البحث لايجاد تطبيقات في نظرية البيان. من خلال بعض مفاهيم الفضاء التوبولوجي فاننا عممنا على نظرية البيان (دواخل البيان, انغلاق البيان, خوارج البيان, حدود البيان, نقطة غاية البيان) وتمت دراسة العلاقات بينهم ثم اثبات نظريات عديدة على الاقل كتوصيف وبعض الامثلة المقدمة لشرح الموضوع.