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Some New Results on Near Fields

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ABSTRACT
In this paper, we considered investigating some properties of near fields and new results are obtained. In particular, we investigated some conditions under which some near rings become near fields.

1 Introduction
The study of near rings and near fields is dated back to the beginning of the twentieth century. The idea of a near field is introduced in 1905 where the American mathematician L. Dickson examined it for the first time and presented a first example of a near field. Also, He proved the basic properties of near fields. Dickson started his work by modifying the multiplication of division rings and keeping the addition unchanged and these near fields are referred to as Dickson near fields [3].
The subject of near rings has received a great deal of interest in varied aspects. Beidleman [1] studied the algebraic theory for the near ring modules. In [2] the distributively generated near rings with descending chain condition are studied by the same author. The connection between specific finite groups and near rings is examined in [4]. Ligh considered the study of near rings in a series of papers in 1960s and 1970s [11, 14, 12, 13, 15, 16]. In [11] the author examined the distributively generated near rings. In [14] the division near rings (near fields) are studied. The near rings with descending chain condition are studied in [12]. In addition, Ligh investigated the Boolean near rings in [13]. The commutative near rings are considered in [15, 16].
Heatherly in 1978 [8] examined the additive groups in finite near field and showed that the additive group of a finite near field is abelian. In 1982 Roberts [19], considered the generalised distributivity in near rings. In [3] some important findings on near rings and near fields are obtained. Dheena in [5] investigated some characteristics of some near fields. A significant number of research papers about near rings and near fields are presented in the conferences held in 1985 and 1997 in [3,7]. The generalisation of division near rings is investigated by Rasovic and Dasic in [18]. Some important properties of zero near rings are studied in [10]. The zero near rings and near fields are studied and some new results are proved in [9].
In this study, we considered studying some important properties of near fields and examined the relationship between them and near rings. This paper is organised as follows. In section 2 we give the important and relevant definitions of the topic. Also, the new theorems and corollaries are presented in section 2. The conclusions are given in section 3.

2 Basic Definitions, Examples and Theorems on Near Fields
In this section, we give some important definitions on near fields and present and prove some important theorems and corollaries on them.

Definition 2.1 (Near Ring) *[20] An algebraic system $(N,+,.)$ where $N \neq \phi$ with two binary operations $(+)$ and $(.)$ is said to be a left near ring if the following conditions are satisfied:*

- *$(N,+)$ satisfies all conditions of a group.*
- *$(N,.)$ satisfies all conditions of a semigroup.*
- *$n_3(n_1 + n_2) = n_3n_1 + n_3n_2, \forall\, n_1, n_2, n_3 \in N$ (left distributive law).*

In this paper, we consider the left near rings and left near fields and the results for the right near rings and right near fields are analogous. Also, we follow the common and widely used notation in the near ring and near field theory.

Definition 2.2 (Zero-Symmetric Near Ring) *[17] A left near ring N is said to be zero-symmetric if $0n = 0$, for all n in N, i.e., the left distributive law results in $n0 = 0$. The set of all zero-symmetric elements of N is denoted by $N_0 = \{x \in N: 0x = 0\}$ is referred to as the zero-symmetric part of N. If $N = N_0$, then N is said to be zero-symmetric. The zero-symmetric near ring is sometimes named as C-ring.*

Remark 2.3 *Every ring and near field are C-rings.*

Definition 2.4 (Regular Near Ring) *[21] If for each element x of a near ring N, there is an element y in N such that $x = xyx$, then N is called a regular near ring. The regular rings are generalisations of division rings and notice that not every regular ring is a division ring.*

Definition 2.5 (Near Field) *[6] A near field **(division near ring)** is a nonempty set N together with two binary operations $(+)$ (called addition) and $(.)$ (called multiplication) and constants 0 and 1 such that it satisfies the following conditions:*

- *$(N,+)$ is an abelian group.*
- *$(N^* = N\backslash\{0\},.)$ is a group.*
- *$c(a+b) = ca + cb, \forall\, a, b, c \in N$ (The left distributive law).*

Note that the near field is a near ring with identity such that each non-zero element has an inverse. Hence, each near field is a near ring but the converse is not true. For example, $(Z,+,.)$ where Z is the set of integers with usual addition $(+)$ and usual multiplication $(.)$ is a near ring but it is not a near field. The symbols 0 and 1 will be used for the additive and multiplicative identities, respectively. In a near ring or a near field $N, 1 \neq 0$. If $1 = 0$, then for all x we have $x = x1 = x0 = 0$, so $N = \{0\}$, contradicting the assumption that N has at least two elements [6].

Remark 2.6 *All fields are near fields and also any division ring is a near field.*

Theorem 2.7 *[6] A near field N has exactly two ideals, (0) and N.*

Proof. Let N is a near field. Clearly (0) is an ideal of N. If K is an ideal of N, then $NK \subseteq K$. Suppose that $K \neq (0)$, and let k be a non-zero element of K. Now, k has a multiplicative inverse h, so $hk = 1 \in K$. So, $n.1 = n \in K, \forall\, n \in N$, and hence $K = N$.

Note that the additive group $(N,+)$ sometimes is referred to as N^+ for brevity.

Lemma 2.8 *[11] Let N be a near ring with identity 1. then, $(-1)(-1) = 1$. In addition, if $(-1)x = x(-1), \forall\, x \in N$, then N^+ is abelian.*

Proof. For proof see [11].

Theorem 2.9 (Zassenhaus) *[11] The additive group N^+ of a near field N is abelian.*

Proof. For proof see the reference [11].

Corollary 2.10 *[11] Let N be a near field with identity 1 such that $1x = x1, \forall\, x \in N$. Then, $(-1)x = x(-1), \forall\, x \in N$.*

Proof. Assume that N is a near field with identity 1 such that $1x = x1, \forall\, x \in N$ and suppose that there exists an element $w \in N$ sucht that $(-1)w = w(-1) + y, y \neq 0$. Then, $y = w + (-1)w = (-1)[(-1)w + w] =$
$(-1)[w + (-1)w] = (-1)y$. So, $1 = -1$ and this implies that $w = w + y$ and finally results in $y = 0$ which is a contradiction.

In what follows, we consider and present some important results and examples on near fields.

Theorem 2.11 *Let N be a near field with three or more elements. If 1 is the identity of the multiplicative group then 1 is the identity of N.*

Proof. Assume that N is a near field with three or more elements. Let $0 \neq 1 \in N$, since $1.0 = 0$ is true for any near ring. Now, we need to show that $0.1 = 0$. Suppose that $0.1 \neq 0$. Then there exists an element $0 \neq y \in N$ such that $(0.1)y = y(0.1) = 1$. But this implies that $1 = y(0.1) = (y0)1 = 0.1$. Let $0 \neq x \in N$ such that $x = 1x = (0.1)x = 0(1x) = 0x$. Therefore, $1 = x^{-1}x = x^{-1}(0x) = (x^{-1}0)x = 0x = x$, which contradicts the fact that N has at least three elements. So, $0.1 = 0$ and 1 is the identity of N.

We know that any field is a near field and the only field of two elements is the field of integers modulo 2. For example, the only near field N for which 1 is not a multiplicative identity is defined as follows. Assume $N = \{0,1\}$ where $(+)$ and $(.)$ are defined below:

Table 1: Addition Table

+	0	1
0	0	1
1	1	0

Table 2: Multiplication Table

.	0	1
0	0	1
1	0	1

Remark 2.12 *[8] In general, if a near ring has an identity 1, then the additive inverse of 1 i.e., -1 need not to be commute with all the elements in the near ring. For example, if G is the additive group of order three, then the set of mappings defined on G is a near ring whose additive group is abelian. But $(-1)f \neq f(-1)$ where 1 is the identity function and f is a*

non-zero constant function. Nevertheless, in a near field, $(-1)x = x(-1)$ is true for each element x.
In the following, we extend some results in near rings to near fields.

Theorem 2.13 *[14] A near ring N is a near field if and only if it contains a right distributive element $r \neq 0$ and $\forall\, 0 \neq a \in N, aN = N$.*

Proof. For proof see the reference [14].

Theorem 2.14 *Let N be a near ring that contains an identity element $1 \neq 0$, then N is a near field if and only if N contains no proper N-subgroups.*

Proof. Suppose that N is a near field and let $1 \neq 0$ denotes the identity for the multiplication group of non-zero elements in N. Let B be a non-zero N-subgroup and let b be a non-zero element of B. Since N is a near field, there is an element $b' \in N$ such that $bb' = 1$ and so $1 \in B$. Hence, $1.r = r \in B, \forall r \in N$. This shows that $B = N$ so, N has no proper N-subgroups. Conversely, assume that N is a near ring with no proper N-subgroups, we first show that 1 is a left identity for N. Since $1 \neq 0$. Let N be any non-zero element of N. Then, $1(1r - r) = 1^2 r + 1(-r) = 1r - 1r = 0$, thus, $1r - r = 0$. This implies that 1 is a left identity of N. Since $1r = r \neq 0$, it is obvious that $xN = N$ and so there is an element $r' \in N$ such that $rr' = 1$. Similarly, there is an element $r'' \in N$ such that $r'r'' = 1$. From this we have $r'r = (r'r)(r'r'') = r'(rr')r'' = r'r'' = 1$ and so, N is a near field.

In the above theorem the assumption that N contains an identity element is essential. For example, let N be a group with at least three elements define for each $0 \neq a \in N, ab = b, \forall\, b \in N$ and $0b = 0, \forall\, b \in N$. Then, it is clear that $xN = N$ for each $0 \neq x$ in N. Every non-zero element of N is a left identity. But N has no identity and hence N cannot be a near field.

Proposition 2.15 *[14] A near ring N is a near field if and only if N contains a non-zero right distributive element and $\forall\, 0 \neq x \in N, \exists\, y \in N$, probably depending on x, such that $xy \neq 0$ and N has no proper N-subgroups.*

Proof. If N is a near field then the first assumption is clear. Assume that N is a near ring contains a non-zero right distributive element and $\forall\, 0 \neq x \in N, \exists\, y \in N$, probably depending on x, such that $xy \neq 0$ and N has no proper N-subgroups. For each $0 \neq x \in N, xN$ is an N-subgroup of N. Since $\exists\, y$ in N such that $xy \neq 0$ and N has no proper subgroups, we conclude that $xN = N$. So, by Theorem 2.13, we have N is a near field.

Note that the existence of a non-zero right distributive element in Proposition 2.16 is essential. As example, let N be a group with at least three elements define for each $0 \neq a \in N, ab = b, \forall\, b \in N$ and $0b = 0, \forall\, b \in N$. Then, it is clear that $xN = N$ for each $0 \neq x$ in N. Every non-zero element of N is a left identity. But N has no identity and hence N cannot be a near field. This example showed that despite that a near ring K contains no proper K-subgroups but it is not a near field.

Theorem 2.16 *[14] Let N be a finite near ring. Then N is a near field if and only if N contains a right distributive element $r \neq 0$ and for each $x \neq 0$ in N there is $y \in N$ such that $xy \neq 0$, and the N-module $(N, +)$ is simple.*

Proof. The first assumption is clearly obvious. Suppose that N be a finite near ring contains a right distributive element $r \neq 0$ and for each $x \neq 0$ in N there is $y \in N$ such that $xy \neq 0$, and the N-module $(N, +)$ is simple. For each $0 \neq x$ in N, we define $T(x) = \{y \in N: xy = 0\}$. It is clear that $T(x)$ is a submodule of $(N, +)$. Since there is $y \in N$ such that $xy \neq 0$, it follows that $T(x) = 0$. This shows that the set of non-zero elements of N is closed under multiplication. Consider the map $f_x: N \to xN$ defined by $f_x(a) = xa, \forall\, a \in N$. We see that f_x is one to one $(1-1)$ or injective map. Since N is finite, we conclude that $xN = N$. By Theorem 2.13, we deduce that N is a near field.

Theorem 2.17 *[12] A near ring N with a non-zero right distributive element is a near field if and only if N has no zero divisors and the d.c.c. is satisfied by the principal N-subgroups of N.*

Proof. The first direction is clear since the near field has no zero divisors and the d.c.c is satisfied by the principal N-subgroups of N. Assume that N is a near ring and $x \neq 0$ be a right distributive element in N and the d.c.c. is satisfied by the principal N-subgroups of N. Since $N \supseteq x^2N \supseteq x^3N \supseteq \cdots$ the d.c.c. means that there must exist a positive integer n such that $x^nN = x^{n+1}N = \cdots$ so, $x^n x = x^{n+1}1$ for some $1 \in N$. This implies that $x^n(x - x1) = 0$, thus, $x = x1$ and this implies that $x(1x - x) = 0$, hence, 1 is a two-sided identity for x. Let w be any element in N. Then, $x(1w - w) = 0$ and this results in 1 is a left identity for w. Analogously, $(w1 - w)x = 0$ this implies that 1 is a right identity for w and hence 1 is an identity for N. It remains to show that each non-zero element $y \in N$ has a multiplicative inverse. For each $0 \neq y \in N$, there is a positive integer n such that $y^nN = y^{n+1}N, y^n 1 = y^{n+1}z$ for some $z \in N$. This means that $y^n(1 - yz) = 0$, so, $1 = yz$ and this means that each non-zero element has a right inverse. Therefore, N is a near field.

Since any finite near ring satisfies the d.c.c. on N-subgroups, then we have the following corollary.

Corollary 2.18 *[12] A finite near ring N is a near field if and only if N contains a non-zero right distributive element and N has no zero divisors.*

The following theorem shows that a regular near ring is a near field.

Theorem 2.19 Let N be a *near ring, then N is a near field if and only if N is regular and contains a non-zero right distributive element.*

Proof. The first direction is obvious since every near field is a regular near ring. To prove the second direction, suppose that N is a regular near ring contains a non-zero right distributive element and let $a \neq 0$ and $c \neq 0$ are elements in N. Then, $ac \neq 0$, if $ac = 0$, we have $a(b + c)a = (ab + ac)a = aba$, which is contrary to the uniqueness of b. Thus, N has no zero divisors. Since N is regular, we have $\forall\, 0 \neq a \in N, aba = a$ implies that $a(bab - b) = ab - ab = 0$, thus $bab = b$. Let $r \neq 0$ be a right distributive element. Then, there is w such that $rwr = r$, so $r(wrr - r) = 0$. This means that $wr = 1$ is a two-sided identity for r. Let d be any element of N. Then, $r(1d - d) = 0$ and this implies that $1d = d$. Also, $(d1 - d)r = 0$. Thus, 1 is a two-sided identity for N. There is $d' \in N$ such that $dd'd = d = d1$. Therefore, $d'd = 1$. Now, $d'(dd' - 1) = d' - d' = 0$, this leads to $dd' = d'd = 1$. Hence, N *is a near field.*

3 Conclusions

In this paper, we studied the structure of some near rings and the connection between them and near fields. In particular, some new results about near fields are obtained and proved under specific conditions.

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بعض النتائج الجديدة في الحقول القريبة

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الملخص

في هذه البحث تم دراسة بعض خواص الحقل القريب وتم الحصول على بعض النتائج الجديدة في هذا الخصوص. وبشكل خاص تم دراسة بعض الشروط التي عندها الحلقات القريبة تصبح حقول قريبة.