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Double Intuitionistic Continuous Function in Double Intuitionistic Topological Spaces

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ABSTRACT

The aim of this paper is to introduce a new classis of Double intuitionistic continuous function, Double intuitionistic (open and closed) functions in Double intuitionistic topological spaces, we investigate the properties of the relationships among these types.

1- Introduction

A. Ghareeb [1, 2] investigated Normality of double fuzzy topological spaces and week forms of continuity in I double gradation fuzzy to topological spaces. Ali Saad Kandil, Osama Abd El-Hamed, Sobhy A. Ali, Salama H. Ali shaliel [3] presented Double connected spaces. A. Kandil, O. A. E. Tantawy and M. Wafaie [4] explain on flou intuitionistic topological spaces. A. S. Farrag and S. E. Abbas [5] introduced General topology step by step. A.Z. Ozcelik and S. Narli [6] introduced the concept on submaximality intuitionistic topological spaces. C. L. Chang [7] studied fuzzy topological spaces. The concept was used to define intuitionistic sets and on intuitionistic gradation of openness by Coker [8,9]. Jeon, J. K., Jun, Y. B. and Park, J. H. [10] researched intuitionistic fuzzy alpha-continuity and intuitionistic fuzzy pre-continuity. K. Atanassov and S. Stoeva [11] studied Intuitionistic fuzzy sets. The concept of fuzzy set which was presented by Zadeh in his classical paper1965 [12]. Seok Jong Lee and Eun Pyo Lee [13] investigated the category of intuitionistic fuzzy topological spaces. S.K. Samanta and T.K. Mondal, [14] examined on intuitionistic gradation of openness. In this paper, we give a new class of functions namely, Double I continuous function and Double I (open and closed) functions in Double intuitionistic topological space. After that, we presented several examples of each type and concluded that there are relationships to each other that were presented through theorems.

2- Preliminaries

We recall the following definitions which we need it in our work.

Let $X \neq \emptyset$, and let $\mathfrak{P}$ and $\mathfrak{Q}$ be IS having the form $\mathfrak{P} = \langle x, \mathfrak{P}_1, \mathfrak{P}_2 \rangle, \mathfrak{Q} = \langle x, \mathfrak{Q}_1, \mathfrak{Q}_2 \rangle$ respectively. Also, $\{\mathfrak{P}_i : i \in I\}$ be an arbitrary family of IS in X, where $\mathfrak{P}_i = \langle x, \mathfrak{P}_i^{(1)}, \mathfrak{P}_i^{(2)} \rangle$, afterward:

1) $\widetilde{\emptyset} = \langle x, \emptyset, X \rangle$; $\widetilde{X} = \langle x, X, \emptyset \rangle$.

2) $\mathfrak{P} \subseteq \mathfrak{Q}$ iff $\mathfrak{P}_1 \subseteq \mathfrak{Q}_1$ and $\mathfrak{Q}_2 \supseteq \mathfrak{P}_2$.

3) $\mathfrak{P}^{c} = \langle x, \mathfrak{P}_2, \mathfrak{P}_1 \rangle$.

4) $\cup \mathfrak{P}_i = \langle x, \cup \mathfrak{P}_i^{(1)}, \cap \mathfrak{P}_i^{(2)} \rangle$, $\cap \mathfrak{P}_i = \langle x, \cap \mathfrak{P}_i^{(1)}, \cup \mathfrak{P}_i^{(2)} \rangle$[9]. Let X be a non-empty set, an intuitionistic set $\mathfrak{P}$ (IS, for short) is an object having g the form $\mathfrak{P}$

$= \langle x, \mathfrak{P}_1, \mathfrak{P}_2 \rangle$ where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are disjoint subset of X. Then $\mathfrak{P}_1$ is called set of members of $\mathfrak{P}$, while $\mathfrak{P}_2$ is called set of nonmembers of $\mathfrak{P}$ [9]. An intuitionistic topology (IT, for short) on a non-empty set X, is a family T of IS in X containing $\widetilde{\emptyset}, \widetilde{X}$ and closed under arbitrary unions and finitely intersections. The pair (X, T) is called ITS [6]. Let $X \neq \emptyset$.

1) A Double-set (D- set, for short) $\underline{\mathfrak{U}}$ is an ordered pair $(\mathfrak{U}_1, \mathfrak{U}_2) \in \wp(X) \times \wp(X)$ such that $\mathfrak{U}_1 \subseteq \mathfrak{U}_2$.

2) D (X) = $\{(\mathfrak{U}_1, \mathfrak{U}_2) \in \wp(X) \times \wp(X), \mathfrak{U}_1 \subseteq \mathfrak{U}_2\}$ is the family of all D-sets on X.

3) The D-set $\underline{X}$ = (X,X) is called the universal D-set, and the D-set $\underline{\emptyset} = (\emptyset,\emptyset)$ is called the empty D-set.

4) Let $\mathfrak{U} = (\mathfrak{U}_1, \mathfrak{U}_2)$; $\underline{\vartheta} = (\vartheta_1; \vartheta_2) \in D(X)$:

1) $(\underline{\mathfrak{U}}^c) = (\mathfrak{U}^c_2, \mathfrak{U}^c_1)$ where $\underline{\mathfrak{U}}^c$ is the complement of $\underline{\mathfrak{U}}$.

2) $\underline{\mathfrak{U}} - \underline{\vartheta} = (\mathfrak{U}_1 - \vartheta_2, \mathfrak{U}_2 - \vartheta_1)$ [4].

Let X be a non-empty set. The family η of D-sets in X is called a double topology on X if it satisfies the following axioms:

a) $\underline{\emptyset}, \underline{X} \in \eta$.

b) If $\underline{\mathfrak{U}}, \underline{\vartheta} \in \eta$, then $\underline{\mathfrak{U}} \cap \underline{\vartheta} \in \eta$,

c) If $\{\underline{\mathfrak{U}z}: z \in Z\} \subseteq \eta$, then $\cup_{z\in Z} \underline{\mathfrak{U}z} \in \eta$. The pair (X, η) is called a DTS. Each element of η is called an open D-set in X. The complement of open D-set is called closed D-set [4].

Let X be a non-empty set defined by:

1) IN (X) = $\{\underline{\emptyset}, \underline{X}\}$, then IN is a Double topology on X and is called indiscrete Double topology. (X, IN) is called indiscrete Double space.

2) dis (X) = $\wp(X) \times \wp(X)$ (power set of X), then dis is a Double topology on X and is called discrete Double topology. (X, dis) is called discrete Double space [4].

Let (X, η) be a DTS and $\underline{\mathfrak{U}} \in D(X)$. The double closure and interior of $\mathfrak{U}$, denoted by cl $(\underline{\mathfrak{U}})$, int $(\underline{\mathfrak{U}})$ defined by: cl $(\underline{\mathfrak{U}}) = \cap \{\underline{\vartheta} : \underline{\vartheta} \in \eta^c$ and $\mathfrak{U} \subseteq \vartheta\}$, int $(\underline{\mathfrak{U}}) = \cup \{\underline{G_i} : \underline{G_i} \in \eta$ and $G \subseteq \mathfrak{U}\}$ [4, 7].

Let (X,τ) be a topological space, β be a subfamily from τ. We called β is a basis or base for τ, if every element in τ is a union of elements of β, i.e., β is a basis or base for τ ⇔ (1) β ⊆ τ.

(2) ∀ U ∈ τ ; U = ∪i Bi; Bi ∈ β ∀ i [5]. Let (X, τ) be a topological space and β be a basis for τ and $\mathcal{S}$ be a subfamily from τ. We called $\mathcal{S}$ is a sub basis for τ if every element of β equal finite intersection of elements of $\mathcal{S}$ [5]. Let (X, η) be a DTS and let $\underline{\mathfrak{y}}, \underline{\mathfrak{h}} \in D(X)$: $\underline{\mathfrak{y}}, \underline{\mathfrak{h}}$ are said to be separated double sets (separated D-sets, for short) if clη $(\underline{\mathfrak{y}}) \cap \underline{\mathfrak{h}} = \emptyset$, and clη $(\underline{\mathfrak{h}}) \cap \underline{\mathfrak{y}} = \emptyset$ [3].

Let (X,η) be a DTS, and any a nonempty subset of X. If there exist two non-empty separated D-sets $\underline{\mathfrak{y}}, \underline{\mathfrak{h}} \in D(X)$ such that $\underline{\mathfrak{y}} \cup \underline{\mathfrak{h}} = \underline{\mathfrak{N}}$. Then the D-sets $\underline{\mathfrak{y}}$ and $\underline{\mathfrak{h}}$ form a D-separation of $\mathfrak{N}$ and it is said to be double disconnected set (D-disconnected set, for short). Otherwise, $\mathfrak{N}$ is said to be double connected set [3]. Consider two ordinary sets X and Y, let f be a mapping from X into Y. The image of a D-set $\mathfrak{U}$ in D (X) defined by: f $(\mathfrak{U})$ = (f $(\mathfrak{U}_1)$, f $(\mathfrak{U}_2)$). Also, the inverse image of a D-set $\underline{\vartheta} \in D(Y)$ defined by: $f^{-1}(\vartheta) = (f^{-1}(\vartheta_1), f^{-1}(\vartheta_2))$ [4]. Let f: X → Y be a mapping and let (X, η), (Y, η^*) be DTS. Then, f is called a D-continuous if $f^{-1}(\vartheta) \in \eta$, whenever $\underline{\vartheta} \in \eta^*$. Let (X, τ) and (Y, τ^*) be two DTS and let f: X → Y be a mapping and $\underline{\mathfrak{U}} \in D(X)$:

1) f is called D-open if f $(\underline{\mathfrak{U}}) \in \tau^*$, ∀ $\underline{\mathfrak{U}} \in \tau$.

2) f is called D-closed if f $(\underline{\mathfrak{U}}) \in \tau^{*c}$, ∀ $\underline{\mathfrak{U}} \in \tau^c$ [5].

3- Double Intuitionistic Continuous Function in DITS

In this section, we define a new class of function forms of Double intuitionistic topological spaces namely Double intuitionistic continuous function (Double I- continuous). And we introduce a new kind of functions called Double intuitionistic open (resp. Double intuitionistic closed), we also explain the relationships between these types in Double intuitionistic topological spaces.

<u>Definition 3.1</u> Let Ӿ be a non-empty set.

1) A Double intuitionistic set (Double I-set, for short) is an ordered pair $(\mathcal{Q},\mathcal{D}) = (\langle x, \mathcal{Q}_1, \mathcal{Q}_2\rangle, \langle x, \mathcal{D}_1, \mathcal{D}_2\rangle)$ $\in pI(Ӿ) \times pI(Ӿ)$ *such that* $\mathcal{Q} \subseteq \mathcal{D}$.

2) Double I (Ӿ) = $\{(\mathcal{Q}, \mathcal{D}) \in pI(Ӿ) \times pI(Ӿ), \mathcal{Q} \subseteq \mathcal{D}\}$ is the family of all Double I-sets on Ӿ.

3) The Double I-set $(\langle x, Ӿ, \emptyset\rangle, \langle x, Ӿ, \emptyset\rangle) = (\widetilde{Ӿ}, \widetilde{Ӿ})$ is called the universal Double I-set, and the Double I-set $(\widetilde{\emptyset}, \widetilde{\emptyset}) = (\langle x, \emptyset, Ӿ\rangle, \langle x, \emptyset, Ӿ\rangle)$ is called the empty Double I-set.

4) Let $(\mathcal{Q}, \mathcal{D}) \in$ Double I (Ӿ), $(\mathcal{Q}, \mathcal{D})^c = (\mathcal{D}^c, \mathcal{Q}^c)$ $= (\langle x, \mathcal{D}_1, \mathcal{D}_2\rangle^c, \langle x, \mathcal{Q}_1, \mathcal{Q}_2\rangle^c) =$ $(\langle x, \mathcal{D}_2, \mathcal{D}_1\rangle, \langle x, \mathcal{Q}_2, \mathcal{Q}_1\rangle)$. Each element of Ψ is called a Double intuitionistic open set (DIOS, for short) in Ӿ. The complement of DIOS is called Double intuitionistic closed set (DICS, for short).

Now, we want to introduce the next important theorem to construct the Double intuitionistic topological Spaces**.**

<u>Theorem 3.2</u> Let $Ӿ \neq \emptyset$, then the family T of all Double intuitionistic open sets in Ӿ is Double Intuitionistic topological spaces (DITS)**.**

<u>Proof</u> Let $(Ӿ, T)$ be intuitionistic topological spaces (ITS), then:

1) $\widetilde{\emptyset} = \langle x, \emptyset, Ӿ\rangle$, $\widetilde{Ӿ} = \langle x, Ӿ, \emptyset\rangle \in I\!T \to (\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{Ӿ},\widetilde{Ӿ})$ $\in DI\!TS$.

2) Let $(\mathcal{Q}, \mathcal{D}), (\mathcal{C}, \mathcal{G}) \in DI\!T \to \mathcal{Q}, \mathcal{D}, \mathcal{C}, \mathcal{G} \in I\!T$. Since IT is intuitionistic topology, then $\mathcal{Q} \cap \mathcal{D} \in I\!T$ and $\mathcal{C} \cap \mathcal{G} \in IT$. Now, let $\mathcal{K} = (\mathcal{Q}, \mathcal{D})$ and $\mathcal{W} = (\mathcal{C}, \mathcal{G}) \to$ $(\mathcal{K}, \mathcal{W}) = ((\mathcal{Q}, \mathcal{C}), (\mathcal{D}, \mathcal{G})) \in DI\!TS$.

3) Let $(\mathcal{Q}_s, \mathcal{D}_s)$ be a family of IS and s ∈ S and $(\mathcal{Q}_s, \mathcal{D}_s) \in DI\!T \to \mathcal{Q}_s, \mathcal{D}_s \in I\!TS$, since $I\!T$ is intuitionistic topology, then $\cup_{s\in S} \mathcal{Q}_s \in I\!T$ *and* $\cup_{s\in S} \mathcal{D}_s \in I\!T$. Thus $\cup_{s\in S} (\mathcal{Q}_s, \mathcal{D}_s) \in DI\!T$. Therefore, $(Ӿ, T)$ is Double intuitionistic topological spaces.

<u>Definition 3.3</u> Let $(Ӿ,\Psi)$ be a DITS and $(\mathcal{Q}, \mathcal{D}) \in$ Double I(Ӿ), then the Double interior of $(\mathcal{Q}, \mathcal{D})$ is the Double I-set such that int $(\mathcal{Q}, \mathcal{D}) = \cup \{(\mathcal{C}, \mathcal{G}) : (\mathcal{C}, \mathcal{G}) \in$ Ψ and $(\mathcal{C}, \mathcal{G}) \subseteq (\mathcal{Q}, \mathcal{D})\}$.

<u>Definition 3.4</u> Let $(Ӿ, \Psi)$ be a DITS and $(\mathcal{Q}, \mathcal{D}) \in$ Double I(Ӿ), the Double closure of $(\mathcal{Q}, \mathcal{D})$ denoted by *cl* $(\mathcal{Q}, \mathcal{D})$ *or* $\overline{(\mathcal{Q},\mathcal{D})}$ is the Double I-set such that cl $(\mathcal{Q}, \mathcal{D}) = \cap \{(\mathcal{C}, \mathcal{G}): (\mathcal{C}, \mathcal{G}) \in \Psi^c$ and $(\mathcal{Q}, \mathcal{D}) \subseteq (\mathcal{C}, \mathcal{G})\}$.

Definition 3.5 Let (X, Ψ) be a DITS, $(\mathcal{Q}, \mathcal{D}), (\mathcal{C}, \mathcal{G}) \in$ Double I(Ẋ), then $(\mathcal{Q}, \mathcal{D}), (\mathcal{C}, \mathcal{G})$ are said to be separated Double intuitionistic sets (separated Double I-sets, for short) if cl $(\mathcal{Q}, \mathcal{D}) \cap (\mathcal{C}, \mathcal{G}) = (\tilde{\emptyset}, \tilde{\emptyset})$, and $(\mathcal{Q}, \mathcal{D}) \cap cl\ (\mathcal{C}, \mathcal{G}) = (\tilde{\emptyset}, \tilde{\emptyset})$. Or, $(cl\ (\mathcal{Q}, \mathcal{D}) \cap (\mathcal{C}, \mathcal{G})) \cup ((\mathcal{Q}, \mathcal{D}) \cap cl\ (\mathcal{C}, \mathcal{G})) = \overline{(\tilde{\emptyset}, \tilde{\emptyset})}$.

Definition 3.6 Let (X, Ψ) be a DITS, if there exist two non-empty separated Double I-sets then $(\mathcal{Q}, \mathcal{D}), (\mathcal{C}, \mathcal{G}) \in$ Double *I(X)* such that $(\mathcal{Q}, \mathcal{D}) \cup (\mathcal{C}, \mathcal{G}) = (\tilde{X}, \tilde{X})$, then $(\mathcal{Q}, \mathcal{D})$ and $(\mathcal{C}, \mathcal{G})$ are said to be Double I- division for DITS (X, Ψ). (X, Ψ) is said to be Double intuitionistic disconnected space (Double I-disconnected space, for short), if (X, Ψ) has a Double I- division. Otherwise, (X, Ψ) is said to be Double intuitionistic connected space (Double I-connected space, for short).

Definition 3.7 Let (X, Ψ) be a DITS and β be a subfamily from Ψ. We called β is a Double intuitionistic basis (Double I-basis, for short) for $\Psi \Leftrightarrow$ (1) $\beta \subseteq \Psi$ (2) $\forall (\mathcal{U}, \mathcal{V}) \in \Psi; (\mathcal{U}, \mathcal{V}) = \cup_j (\mathcal{Q}_j, \mathcal{D}_j)$; $(\mathcal{Q}_j, \mathcal{D}_j) \in \beta; \forall j$.

Definition 3.8 Let (X, Ψ) be a DITS and β be a Double I-basis for Ψ and $\mathcal{S}$ be a subfamily from Ψ. We called $\mathcal{S}$ is a Double I-sub basis for $\Psi \Leftrightarrow \forall (\mathcal{Q}, \mathcal{D}) \in \beta \rightarrow (\mathcal{C}_h, \mathcal{G}_h) \in \mathcal{S} \in \cap_{h=1}^{n} (\mathcal{C}_h, \mathcal{G}_h) = (\mathcal{Q}, \mathcal{D})$; $\forall h = 1, 2, \ldots h$.

Definition 3.9 Let (X, Ψ) and (Y, Ψ^*) be two DITS, and f: $(X, \Psi) \rightarrow (Y, \Psi^*)$:

a) If $(\mathcal{Q}, \mathcal{D}) = (\langle y, \mathcal{Q}_1, \mathcal{Q}_2 \rangle, \langle y, \mathcal{D}_1, \mathcal{D}_2 \rangle)$ is Double I-set in Y, then the inverse image of $(\mathcal{Q}, \mathcal{D})$ under f is denoted by $f^{-1}(\mathcal{Q}, \mathcal{D})$ is Double I-set in Ẋ defined by $f^{-1}(\mathcal{Q}, \mathcal{D}) = (\langle x, f^{-1}(\mathcal{Q}_1), f^{-1}(\mathcal{Q}_2) \rangle, \langle x, f^{-1}(\mathcal{D}_1), f^{-1}(\mathcal{D}_2) \rangle)$.

b) If $(\mathcal{C}, \mathcal{G}) = (\langle x, \mathcal{C}_1, \mathcal{C}_2 \rangle, \langle x, \mathcal{G}_1, \mathcal{G}_2 \rangle)$ is a Double I-set in Ẋ, then the image of $(\mathcal{C}, \mathcal{G})$ under f is denoted by f $(\mathcal{C}, \mathcal{G})$ is a Double I-set in Ỵ defined by $f(\mathcal{C}, \mathcal{G}) = (\langle y, f(\mathcal{C}_1), \underline{f}(\mathcal{C}_2) \rangle, \langle y, f(\mathcal{G}_1), \underline{f}(\mathcal{G}_2) \rangle)$. where $\underline{f}(\mathcal{C}_2) = (f(\mathcal{C}_2^c))^c$ and $\underline{f}(\mathcal{G}_2) = (f(\mathcal{G}_2^c))^c$.

Definition 3.10 Let (X, Ψ) and (Y, Ψ^*) be two DITS, and let f: $(X, \Psi) \rightarrow (Y, \Psi^*)$. The function f is called Double intuitionistic continuous (Double I-continuous, for short) if the inverse image for any Double I-open set in Ỵ is a Double I-open set in Ẋ, i.e., f:$(X, \Psi) \rightarrow (Y, \Psi^*)$ is Double I-continuous $\Leftrightarrow f^{-1}(\mathcal{Q}, \mathcal{D}) \in \Psi; \forall (\mathcal{Q}, \mathcal{D}) \in \Psi^*$. And the function f is called Double I-discontinuous $\Leftrightarrow \exists (\mathcal{Q}, \mathcal{D}) \in \Psi^* \wedge f^{-1}(\mathcal{Q}, \mathcal{D}) \notin \Psi$.

The following two examples show the Double I-continuous and the Double I-discontinuous.

Example 3.11 Let $X = \{\alpha, \pi, \sigma\}$; $\Psi = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{X}, \tilde{X}), (J_1, J_2), (J_3, J_4), (J_1, J_4), (J_4, \tilde{X}), (\tilde{\emptyset}, J_1), (\tilde{\emptyset}, J_4), (J_1, \tilde{X}), (J_1, J_1), (J_4, J_4)\}$ where $(J_1, J_2) = (\langle x, \{\pi\}, \{\alpha, \sigma\} \rangle, \langle x, \{\alpha, \pi\}, \{\sigma\} \rangle)$, $(J_3, J_4) = \langle x, \{\sigma\}, \{\alpha, \pi\} \rangle, \langle x, \{\pi, \sigma\}, \{\alpha\} \rangle)$, $(J_1, J_4) = (\langle x, \{\pi\}, \{\alpha, \sigma\} \rangle, \langle x, \{\pi, \sigma\}, \{\alpha\} \rangle)$, $(J_4, \tilde{X}) = (\langle x, \{\pi, \sigma\}, \{\alpha\} \rangle, \langle x, Ẋ, \emptyset \rangle)$, $(\tilde{\emptyset}, J_1) = (\langle x, \emptyset, Ẋ \rangle, \langle x, \{\pi\}, \{\alpha, \sigma\} \rangle)$, $(\tilde{\emptyset}, J_4) = (\langle x, \emptyset, Ẋ \rangle, \langle x, \{\pi, \sigma\}, \{\alpha\} \rangle)$, $(J_1, \tilde{X}) = (\langle x, \{\pi\}, \{\alpha, \sigma\} \rangle, \langle x, Ẋ, \emptyset \rangle)$, $(J_1, J_1) = (\langle x, \{\pi\}, \{\alpha, \sigma\} \rangle, \langle x, \{\pi\}, \{\alpha, \sigma\} \rangle)$, $(J_4, J_4) = (\langle x, \{\pi, \sigma\}, \{\alpha\} \rangle, \langle x, \{\pi, \sigma\}, \{\alpha\} \rangle)$. *And let* $Y = \{1,2,3\}$; $\Psi^* = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{Y}, \tilde{Y}), (Z_1, Z_2), (Z_3, Z_3)\}$ *where* $(Z_1, Z_2) = (\langle y, \{1\}, \{3\} \rangle, \langle y, \{1,2\}, \{3\} \rangle)$ *and* $(Z_3, Z_3) = (\langle y, \{1\}, \{2,3\} \rangle, \langle y, \{1\}, \{2,3\} \rangle)$.

Define a function f:(Ẋ,Ψ) → (Ỵ,Ψ*) by f({π}, {α, σ}) = ({1},{3}) and f ({α, σ}, {π}) = ({3},{1}). f is Double I-continuous, the Double I-open set in Ỵ are $(\tilde{\emptyset}, \tilde{\emptyset})$, $(\tilde{Y}, \tilde{Y})$, (Z_1, Z_2), (Z_3, Z_3). Now take the inverse image of this I-sets $(\tilde{\emptyset}, \tilde{\emptyset}) \in \Psi^* \rightarrow f^{-1}(\tilde{\emptyset}, \tilde{\emptyset}) \in \Psi$, the I-set of all element in $(\tilde{\emptyset}, \tilde{\emptyset})$ their image in $(\tilde{\emptyset}, \tilde{\emptyset})$. $(\tilde{Y}, \tilde{Y}) \in \Psi^* \rightarrow f^{-1}(\tilde{Y}, \tilde{Y}) = (\tilde{X}, \tilde{X}) \in \Psi$, the I-set of all elements in $(\tilde{X}, \tilde{X})$ their image in $(\tilde{Y}, \tilde{Y})$, $(Z_1, Z_2) \in \Psi^* \rightarrow f^{-1}(Z_1, Z_2) = (J_1, J_1) \in \Psi$, $(Z_3, Z_3) \in \Psi^* \rightarrow f^{-1}(Z_3, Z_3) = (J_1, J_1) \in \Psi$, the I-sets of all elements in Ẋ their images in (Z_1, Z_1) and (Z_3, Z_3). Therefore, the inverse image of every element in Ψ^* is element in Ψ.

Example 3.12 Let Ẋ = $\{t, w, u\}$; $\Psi = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{X}, \tilde{X}), (\mathcal{K}_1, \mathcal{K}_1), (\mathcal{K}_1^c, \mathcal{K}_1^c)\}$ where $(\mathcal{K}_1, \mathcal{K}_1) = (\langle x, \{t\}, \{w, u\} \rangle, \langle x, \{t\}, \{w, u\} \rangle)$ and $(\mathcal{K}_1^c, \mathcal{K}_1^c) = (\langle x, \{w, u\}, \{t\} \rangle, \langle x, \{w, u\}, \{t\} \rangle)$. And let Ỵ = $\{10,11,12\}$; $\Psi^* = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{Y}, \tilde{Y}), (R, Q)\}$ where $(R, Q) = (\langle y, \{11\}, \emptyset \rangle, \langle y, \{11,12\}, \emptyset \rangle)$. Define a function h: $(X, \Psi) \rightarrow (Y, \Psi^*)$ by h$(\{t\}, \{w, u\}) = (\{11\}, \{12\})$ and h $(\{w, u\}, \{t\}) = (\{12\}, \{11\})$. Since $(R, Q) \in \Psi^* \rightarrow h^{-1}(R, Q) = (\langle x, \{t\}, \emptyset \rangle, \langle x, Ẋ, \emptyset \rangle) \notin \Psi^*$.

Therefore, h is Double I-discontinuous.

Remark 3.13 There are special cases of Double I-continuous function:

1) If $\Psi^* = IN(X)$, then the function f: $(X, \Psi) \rightarrow (Y, IN)$ is Double I-continuous for any I-set Ỵ and any Double intuitionistic topological spaces (X, Ψ). *i.e.*, $IN = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{Y}, \tilde{Y})\}$ and $f^{-1}(\tilde{Y}, \tilde{Y}) = (\tilde{X}, \tilde{X}) \in \Psi$, $f^{-1}(\tilde{\emptyset}, \tilde{\emptyset}) = (\tilde{\emptyset}, \tilde{\emptyset}) \in \Psi$. Also, f: $(X, IN) \rightarrow (Y, IN)$ Double I-continuous, and the function f: $(X, IN) \rightarrow (Y, \Psi^*)$; $\Psi^* \neq IN$ is Double I-discontinuous for example:

Example 3.14 Let $X = \{a, s, d\}$; $IN = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{X}, \tilde{X})\}$ and *let* $Y = \{20,21,22\}$; $\Psi^* = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{Y}, \tilde{Y}), (\gamma_1, \gamma_2), (\gamma_3, \gamma_2), (\gamma_2, \gamma_2), (\gamma_4, \gamma_2)\}$ where $(\gamma_1, \gamma_2) = (\langle y, \{20\}, \{21\} \rangle, \langle y, \{20,22\}, \{21\} \rangle)$, $(\gamma_3, \gamma_2) = \langle y, \{22\}, \{21\} \rangle, \langle y, \{20,22\}, \{21\} \rangle)$, $((\gamma_2, \gamma_2) = (\langle y, \{20,22\}, \{21\} \rangle, \langle y, \{20,22\}, \{21\} \rangle)$ and $(\gamma_4, \gamma_2) = (\langle y, \emptyset, \{21\} \rangle, \langle y, \{20,22\}, \{21\} \rangle)$.

Define a function f:$(X, IN) \rightarrow (Y, \Psi^*)$ *by* f$(\{a\}, \{d\}) = (\{20\}, \{21\})$, f$(\{s\}, \{a\}) = (\{22\}, \{20\})$ and f$(\{d\}, \{a, s\}) = (\{21\}, \{22\})$. f is Double I-discontinuous. Since $f^{-1}(\gamma_1, \gamma_2) = (\langle x, \{a\}, \{d\} \rangle, \langle x, \{a, s\}, \{d\} \rangle) \notin IN$.

2) If Ψ = dis, then the function g: $(X, dis) \rightarrow (Y, \Psi^*)$ Double I-continuous for any I-set

of $\mathbb{X}$ and any Double intuitionistic topological spaces $(\mathbb{Y},\Psi^*)$ and for any function g, since $(\mathcal{Q},\mathcal{D})\in\Psi^*$, then $g^{-1}(\mathcal{Q},\mathcal{D})\subseteq\mathbb{X}$ this means $g^{-1}(\mathcal{Q},\mathcal{D})\in$ dis. Therefore, g is Double I-continuous. Also, g: $(\mathbb{X},dis)\rightarrow(\mathbb{Y},\ dis)$ is Double I-continuous, and the function $g{:}\ (\mathbb{X},\Psi)\rightarrow(\mathbb{Y},\ dis)$; $\Psi\neq$ dis is Double I-discontinuous function for example.

Example 3.15

Let $\mathbb{X}=\{8,9,10\}$; $\Psi=\{(\tilde{\emptyset},\tilde{\emptyset}),(\tilde{\mathbb{X}},\tilde{\mathbb{X}}),(\theta_1,\theta_3),(\theta_2,\theta_3),(\theta_3,\theta_3),(\tilde{\emptyset},\theta_3)\}$ where $(\theta_1,\theta_3)=(\langle x,\{8\},\{9,10\}\rangle,\langle x,\{8,9\},\emptyset\rangle)$, $(\theta_2,\theta_3)=(\langle x,\{9\},\{8\}\rangle,\langle x,\{8,9\},\emptyset\rangle)$, $(\theta_3,\theta_3)=(\langle x,\{8,9\},\emptyset\rangle,\langle x,\{8,9\},\emptyset\rangle)$ and $(\tilde{\emptyset},\theta_3)=(\langle x,\emptyset,\mathbb{X}\rangle,\langle x,\{8,9\},\emptyset\rangle)$. Let $\mathbb{Y}=\{i,j\}$; $dis=\{(\tilde{\emptyset},\tilde{\emptyset}),(\tilde{\mathbb{Y}},\tilde{\mathbb{Y}}),(\tilde{\emptyset},\mathcal{D}_1),(\tilde{\emptyset},\mathcal{D}_2),(\tilde{\emptyset},\mathcal{D}_3),(\tilde{\emptyset},\mathcal{D}_3^c),(\tilde{\emptyset},\mathcal{D}_1^c),(\tilde{\emptyset},\mathcal{D}_2^c),(\tilde{\emptyset},\mathcal{D}_4),(\mathcal{D}_1,\tilde{\mathbb{Y}}),(\mathcal{D}_3,\tilde{\mathbb{Y}}),(\mathcal{D}_2,\tilde{\mathbb{Y}}),(\mathcal{D}_3^c,\tilde{\mathbb{Y}}),(\mathcal{D}_1^c,\tilde{\mathbb{Y}}),(\mathcal{D}_2^c,\tilde{\mathbb{Y}}),(\mathcal{D}_4,\tilde{\mathbb{Y}}),(\mathcal{D}_3,\mathcal{D}_1),(\mathcal{D}_1,\mathcal{D}_1),(\mathcal{D}_3,\mathcal{D}_3),(\mathcal{D}_2,\mathcal{D}_2),(\mathcal{D}_3^c,\mathcal{D}_3^c),(\mathcal{D}_1^c,\mathcal{D}_1^c),(\mathcal{D}_2^c,\mathcal{D}_2^c),(\mathcal{D}_4,\mathcal{D}_4),(\mathcal{D}_3^c,\mathcal{D}_2),(\mathcal{D}_1^c,\mathcal{D}_1),(\mathcal{D}_1^c,\mathcal{D}_2),(\mathcal{D}_1^c,\mathcal{D}_3^c),(\mathcal{D}_1^c,\mathcal{D}_4),(\mathcal{D}_2^c,\mathcal{D}_1),(\mathcal{D}_2^c,\mathcal{D}_3),(\mathcal{D}_2^c,\mathcal{D}_2),(\mathcal{D}_2^c,\mathcal{D}_4),(\mathcal{D}_4,\mathcal{D}_1),(\mathcal{D}_4,\mathcal{D}_2),(\tilde{\emptyset},\mathbb{Y})\}$ where $(\tilde{\emptyset},\mathcal{D}_1)=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\{i\},\emptyset\rangle)$, $(\tilde{\emptyset},\mathcal{D}_2)=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\{j\},\emptyset\rangle)$, $(\tilde{\emptyset},\mathcal{D}_3)=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\{i\},\{j\}\rangle)$, $(\tilde{\emptyset},\mathcal{D}_3^c)=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\{j\},\{i\}\rangle)$, $(\tilde{\emptyset},\mathcal{D}_1^c)=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\emptyset,\{i\}\rangle)$, $(\tilde{\emptyset},\mathcal{D}_2^c)=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\emptyset,\{j\}\rangle)$, $(\tilde{\emptyset},\mathcal{D}_4)=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\emptyset,\emptyset\rangle)$, $(\mathcal{D}_1,\tilde{\mathbb{Y}})=(\langle y,\{i\},\emptyset\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$, $(\mathcal{D}_3,\tilde{\mathbb{Y}})=(\langle y,\{i\},\{j\}\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$, $(\mathcal{D}_2,\tilde{\mathbb{Y}})=(\langle y,\{j\},\emptyset\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$, $(\mathcal{D}_3^c,\tilde{\mathbb{Y}})=(\langle y,\{j\},\{i\}\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$, $(\mathcal{D}_1^c,\tilde{\mathbb{Y}})=(\langle y,\emptyset,\{i\}\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$, $(\mathcal{D}_2^c,\tilde{\mathbb{Y}})=(\langle y,\emptyset,\{j\}\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$, $(\mathcal{D}_4,\tilde{\mathbb{Y}})=(\langle y,\emptyset,\emptyset\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$, $(\mathcal{D}_3,\mathcal{D}_1)=(\langle y,\{i\},\{j\}\rangle,\langle y,\{i\},\emptyset\rangle)$, $(\mathcal{D}_1,\mathcal{D}_1)=(\langle y,\{i\},\emptyset\rangle,\langle y,\{i\},\emptyset\rangle)$, $(\mathcal{D}_3,\mathcal{D}_3)=(\langle y,\{i\},\{j\}\rangle,\langle y,\{i\},\{j\}\rangle)$, $(\mathcal{D}_2,\mathcal{D}_2)=(\langle y,\{j\},\emptyset\rangle,\langle y,\{j\},\emptyset\rangle)$, $(\mathcal{D}_3^c,\mathcal{D}_3^c)=(\langle y,\{j\},\{i\}\rangle,\langle y,\{j\},\{i\}\rangle)$, $(\mathcal{D}_1^c,\mathcal{D}_1^c)=(\langle y,\emptyset,\{i\}\rangle,\langle y,\emptyset,\{i\}\rangle)$, $(\mathcal{D}_2^c,\mathcal{D}_2^c)=(\langle y,\emptyset,\{j\}\rangle,\langle y,\emptyset,\{j\}\rangle)$, $(\mathcal{D}_4,\mathcal{D}_4)=(\langle y,\emptyset,\emptyset\rangle,\langle y,\emptyset,\emptyset\rangle)$, $(\mathcal{D}_3^c,\mathcal{D}_2)=(\langle y,\{j\},\{i\}\rangle,\langle y,\{j\},\emptyset\rangle)$, $(\mathcal{D}_1^c,\mathcal{D}_1)=(\langle y,\emptyset,\{i\}\rangle,\langle y,\{i\},\emptyset\rangle)$, $(\mathcal{D}_1^c,\mathcal{D}_2)=(\langle y,\emptyset,\{i\}\rangle,\langle y,\{j\},\emptyset\rangle)$, $(\mathcal{D}_1^c,\mathcal{D}_3^c)=(\langle y,\emptyset,\{i\}\rangle,\langle y,\{j\},\{i\}\rangle)$, $(\mathcal{D}_1^c,\mathcal{D}_4)=(\langle y,\emptyset,\{i\}\rangle,\langle y,\emptyset,\emptyset\rangle)$, $(\mathcal{D}_2^c,\mathcal{D}_1)=(\langle y,\emptyset,\{j\}\rangle,\langle y,\{i\},\emptyset\rangle)$, $(\mathcal{D}_2^c,\mathcal{D}_3)=(\langle y,\emptyset,\{j\}\rangle,\langle y,\{i\},\{j\}\rangle)$, $(\mathcal{D}_2^c,\mathcal{D}_2)=(\langle y,\emptyset,\{j\}\rangle,\langle y,\{j\},\emptyset\rangle)$, $(\mathcal{D}_2^c,\mathcal{D}_4)=(\langle y,\emptyset,\{j\}\rangle,\langle y,\emptyset,\emptyset\rangle)$, $(\mathcal{D}_4,\mathcal{D}_1)=(\langle y,\emptyset,\emptyset\rangle,\langle y,\{i\},\emptyset\rangle)$, $(\mathcal{D}_4,\mathcal{D}_2)=(\langle y,\emptyset,\emptyset\rangle,\langle y,\{j\},\emptyset\rangle)$ and $(\tilde{\emptyset},\tilde{\mathbb{Y}})=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$. Define a function f:$(\mathbb{X},\Psi)\rightarrow(\mathbb{Y},\ dis)$ by f $(\{9\},\ \{8,10\})=(\{i\},\ \{j\})$ and f $(\{8,10\},\{9\})=(\{j\},\ \{i\})$, f is Double I-discontinuous, since $(\mathcal{D}_3,\mathcal{D}_1)\in$ *dis and* $f^{-1}(\mathcal{D}_3,\mathcal{D}_1)=(\langle x,\{9\},\{8,10\}\rangle,\langle x,\{9\},\emptyset\rangle)\notin\Psi$. Notes that the function f: $(\mathbb{X},\ dis)\rightarrow(\mathbb{Y},\ IN)$ is Double I-continuous always for any I-set $\mathbb{X}$ and any I-set $\mathbb{Y}$. Since it's added the remark (1), (2) such that Ψ=dis and Ψ^*=IN.

3) Every identity function from Double I-space to the same Double I-space is Double I-continuous, i.e.

f: $(\mathbb{X},\Psi)\rightarrow(\mathbb{X},\Psi)$ is Double I-continuous function.

The following theorem proved that the composition are both Double I-continuous functions is Double I-continuous.

Theorem 3.16 Let f:$(\mathbb{X},\ \Psi)\rightarrow(\mathbb{Y},\Psi^*)$ and g: $(\mathbb{Y},\Psi^*)\rightarrow(\mathcal{Z},\Psi^{**})$ are both Double I-continuous functions, then the composition g∘ f: $(\mathbb{X},\ \Psi)\rightarrow(\mathcal{Z},\Psi^{**})$ is Double I-continuous.

Proof Let $(\mathcal{Q},\mathcal{D})$ be Double I-open in $\mathcal{Z}$, we have to show that $(g\circ f)^{-1}(\mathcal{Q},\mathcal{D})$is Double I-open in $\mathbb{X}$. $(g\circ f)^{-1}(\mathcal{Q},\mathcal{D})=(f^{-1}\circ g^{-1})(\mathcal{Q},\mathcal{D})=f^{-1}(g^{-1}(\mathcal{Q},\mathcal{D}))$. Since g is Double I-continuous$\rightarrow g^{-1}(\mathcal{Q},\mathcal{D})$ is Double I-open in $\mathbb{Y}$. Since f is Double I-continuous$\rightarrow f^{-1}(g^{-1}(\mathcal{Q},\mathcal{D}))$ is Double I-open in $\mathbb{X}$, $(g\circ f)^{-1}(\mathcal{Q},\mathcal{D})=f^{-1}(g^{-1}(\mathcal{Q},\mathcal{D}))\rightarrow(g\circ f)^{-1}(\mathcal{Q},\mathcal{D})$ is Double I-open in $\mathbb{X}$. Hances g∘ f is Double I-continuous.

Definition 3.17 Let $(\mathbb{X},\Psi)$ and $(\mathbb{Y},\ \Psi^*)$ be two DITS, and let f: $(\mathbb{X},\ \Psi)\rightarrow(\mathbb{Y},\ \Psi^*)$ be a function, and $(\mathcal{Q},\mathcal{D})\in$ Double I ($\mathbb{X}$):

1) The function f is called Double intuitionistic open (Double I-open, for short), if the direct image for any Double I-open set in $\mathbb{X}$ is a Double I-open set in $\mathbb{Y}$, i.e., f is Double I-open function $\Leftrightarrow\forall(\mathcal{Q},\ \mathcal{D})\in\Psi\rightarrow f(\mathcal{Q},\mathcal{D})\in\Psi^*$.

2) The function f is called Double intuitionistic closed (Double I-closed, for short), if the direct image for any Double I-closed set in $\mathbb{X}$ is a Double I-closed set in $\mathbb{Y}$, i.e., f is Double I-closed function$\Leftrightarrow\forall(\mathcal{Q},\ \mathcal{D})^c\in\Psi^c\rightarrow(f(\mathcal{Q},\mathcal{D}))^c\in\Psi^{*c}$.

Example 3.18 Let $\mathbb{X}=\{p,q,r\}$; $\Psi=\{(\tilde{\emptyset},\tilde{\emptyset}),\ (\tilde{\mathbb{X}},\tilde{\mathbb{X}}),(\mathbb{F}_1,\mathbb{F}_2)\}$ where $(\mathbb{F}_1,\mathbb{F}_2)=(\langle x,\{q\},\{p\}\rangle,\langle x,\{p,q\},\emptyset\rangle)$, $\Psi^c=\{(\tilde{\emptyset},\tilde{\emptyset}),(\tilde{\mathbb{X}},\tilde{\mathbb{X}}),(\mathbb{F}_2^c,\mathbb{F}_1^c)\}$ where $(\mathbb{F}_2^c,\mathbb{F}_1^c)=(\langle x,\emptyset,\{p,q\}\rangle,\langle x,\{p\},\{q\}\rangle)$. Let $\mathbb{Y}=\{1,2,3\}$; $\Psi^*=\{(\tilde{\emptyset},\tilde{\emptyset}),\ (\tilde{\mathbb{Y}},\tilde{\mathbb{Y}}),(U_1,U_2),(U_2,\tilde{\mathbb{Y}}),(U_3,U_2)\}$, where $(U_1,U_2)=(\langle y,\{1,3\},\{2\}\rangle,\langle y,\{1,3\},\emptyset\rangle)$,$(U_2,\tilde{\mathbb{Y}})$ $\langle y,\{1,3\},\emptyset\rangle,\langle y,\mathbb{Y},\emptyset\rangle)$ and$(U_3,U_2)=(\langle y,\{1\},\{2\}\rangle,\langle y,\{1,3\},\emptyset\rangle)$. $\Psi^{*c}=\{(\tilde{\emptyset},\tilde{\emptyset}),(\tilde{\mathbb{Y}},\tilde{\mathbb{Y}}),(U_2^c,U_1^c),(\tilde{\emptyset},U_2^c),(U_2^c,U_3^c)\}$ *where* $(U_2^c,U_1^c)=(\langle y,\emptyset,\{1,3\}\rangle,\langle y,\{2\},\{1,3\}\rangle)$, $(\tilde{\emptyset},U_2^c)=(\langle y,\emptyset,\mathbb{Y}\rangle,\langle y,\emptyset,\{1,3\}\rangle)$, $(U_2^c,U_3^c)=(\langle y,\emptyset,\{1,3\}\rangle,\langle y,\{2\},\{1\}\rangle)$. Define a function f: $(\mathbb{X},\Psi)\rightarrow(\mathbb{Y},\Psi^*)$ by $f(\{p\},\{q\})=(\{2\},\{1\})$ and $f(\{q,r\},\{p,r\})=(\{1,3\},\ \{2,3\})$. We can see f is Double I-open, since $f(\mathbb{F}_1,\mathbb{F}_2)=(U_2,\tilde{\mathbb{Y}})$

$\in \Psi^*$. But not Double I-closed, since there exist $(\mathbb{F}_2^c, \mathbb{F}_1^c) \in \Psi^c$, then $f(\mathbb{F}_2^c, \mathbb{F}_1^c) \notin \Psi^{*c}$.

Example 3.19 Let Ӿ= $\{h, \ell, s\}$; $\Psi = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{Ӿ},\widetilde{Ӿ}), (\mathbb{F}_1, \widetilde{Ӿ})\}$ where $(\mathbb{F}_1, \widetilde{Ӿ}) = (\langle x, \{h\}, \{\ell\}\rangle, \langle x, Ӿ, \emptyset\rangle)$, $\Psi^c = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{Ӿ},\widetilde{Ӿ}), (\widetilde{\emptyset}, \mathbb{F}_1^c)\}$ where $(\widetilde{\emptyset}, \mathbb{F}_1^c) = (\langle x, \emptyset, Ӿ\rangle, \langle x, \{\ell\}, \{h\}\rangle)$. Let Ɏ $= \{1,2,3\}$; $\Psi^* = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{Ɏ},\widetilde{Ɏ}), (W_1, W_2), (W_3, W_4), (W_1, W_4), (W_4, \widetilde{Ɏ}), (\widetilde{\emptyset}, W_1), (W_1, \widetilde{Ɏ}), (W_1, W_1), (\widetilde{\emptyset}, W_4), (W_4, W_4)\}$ where $(W_1, W_2) = (\langle y, \{2\}, \{1,3\}\rangle, \langle y, \{1,2\}, \{3\}\rangle)$, $(W_3, W_4) = (\langle y, \{3\}, \{1,2\}\rangle, \langle y, \{2,3\}, \{1\}\rangle)$, $(W_1, W_4) = (\langle y, \{2\}, \{1,3\}\rangle, \langle y, \{2,3\}, \{1\}\rangle)$, $(W_4, \widetilde{Ɏ}) = (\langle y, \{2,3\}, \{1\}\rangle, \langle y, Ɏ, \emptyset\rangle)$, $(\widetilde{\emptyset}, W_1) = (\langle y, \emptyset, Ɏ\rangle, \langle y, \{2\}, \{1,3\}\rangle)$, $(W_1, \widetilde{Ɏ}) = (\langle y, \{2\}, \{1,3\}\rangle, \langle y, Ɏ, \emptyset\rangle)$, $(W_1, W_1) = (\langle y, \{2\}, \{1,3\}\rangle, \langle y, \{2\}, \{1,3\}\rangle)$, $(\widetilde{\emptyset}, W_4) = (\langle y, \emptyset, Ɏ\rangle, \langle y, \{2,3\}, \{1\}\rangle)$ and $(W_4, W_4) = (\langle y, \{2,3\}, \{1\}\rangle, \langle y, \{2,3\}, \{1\}\rangle)$. $\Psi^{*c} = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{Ɏ},\widetilde{Ɏ}), (W_2^c, W_1^c), (W_4^c, W_3^c), (W_4^c, W_1^c), (\widetilde{\emptyset}, W_4^c), (W_1^c, \widetilde{Ɏ}), (\widetilde{\emptyset}, W_1^c), (W_1^c, W_1^c), (W_4^c, \widetilde{Ɏ}), (W_4^c, W_4^c)\}$ where $(W_2^c, W_1^c) = (\langle y, \{3\}, \{1,2\}\rangle, \langle y, \{1,3\}, \{2\}\rangle)$, $(W_4^c, W_3^c) = (\langle y, \{1\}, \{2,3\}\rangle, \langle y, \{1,2\}, \{3\}\rangle)$, $(W_4^c, W_1^c) = (\langle y, \{1\}, \{2,3\}\rangle, \langle y, \{1,3\}, \{2\}\rangle)$, $(\widetilde{\emptyset}, W_4^c) = (\langle y, \emptyset, Ɏ\rangle, \langle y, \{1\}, \{2,3\}\rangle)$, $(W_1^c, \widetilde{Ɏ}) = (\langle y, \{1,3\}, \{2\}\rangle, \langle y, Ɏ, \emptyset\rangle)$, $(\widetilde{\emptyset}, W_1^c) = (\langle y, \emptyset, Ɏ\}\rangle, \langle y, \{1,3\}, \{2\}\rangle)$, $(W_1^c, W_1^c) = (\langle y, \{1,3\}, \{2\}\rangle, \langle y, \{1,3\}, \{2\}\rangle)$, $(W_4^c, \widetilde{Ɏ}) = (\langle y, \{1\}, \{2,3\}\rangle, \langle y, Ɏ, \emptyset\rangle)$ and $(W_4^c, W_4^c) = (\langle y, \{1\}, \{2.3\}\rangle, \langle y, \{1\}, \{2.3\}\rangle)$. Define a function $f: (Ӿ, \Psi) \to (Ɏ, \Psi^*)$ by $f(\{\ell\}, \{h\}) = (\{2\}, \{2,3\})$, $f(\{h\}, \{\ell\}) = (\{3\}, \{1\})$ and $f(\{s\}, \{h, s\}) = (\{1\}, \{2\})$. It is easy to see that f is Double I-closed, since $(\widetilde{\emptyset}, \mathbb{F}_1^c) \in \Psi^c$, then $f(\widetilde{\emptyset}, \mathbb{F}_1^c) = (\widetilde{\emptyset}, W_4^c) \in \Psi^{*c}$. However, not Double I-open, since there exist $(\mathbb{F}_1, \widetilde{Ӿ}) \in \Psi$, $f(\mathbb{F}_1, \widetilde{Ӿ}) \notin \Psi^*$.

There are several characterizations of Double I-continuous functions, hence, that any one of them may be used to show Double I-continuity of function. These are given in the next theorem.

Theorem 3.20 Let $f: (Ӿ, \Psi) \to (Ɏ, \Psi^*)$ be a function, then f is Double I-continuous if and only if it satisfies one of the following properties:

1) The inverse image of every Double I-closed in Ɏ is Double I-closed in Ӿ.

2) The inverse image of every element in any Double I-basis for Ψ^* is Double I-open set in Ӿ.

3) The inverse image of every element in any Double I-sub basis for Ψ^* is Double I-open set in Ӿ.

4) $cl((f^{-1}(Q, \mathcal{D})) \subseteq f^{-1}(cl(Q, \mathcal{D}))$; $\forall (Q, \mathcal{D}) \subseteq$ Double I (Ɏ).

5) $f^{-1}(int(Q, \mathcal{D})) \subseteq int(f^{-1}(Q, \mathcal{D}))$; $\forall (Q, \mathcal{D}) \subseteq$ Double I (Ɏ).

Proof 1) To prove f is Double I-continuous $\Leftrightarrow$ Ӿ - $f^{-1}(Q, \mathcal{D}) \in \Psi\ \forall$ Ɏ $- (Q, \mathcal{D}) \in \Psi^*$.

($\Rightarrow$) Let f is Double I-continuous and $(Q, \mathcal{D})$ be the Double I-closed set in Ɏ $\to$ Ɏ $- (Q, \mathcal{D})$ is Double I-open set in Ɏ. Since f is Double I-continuous, $f^{-1}(Ɏ - (Q, \mathcal{D}))$ is Double I-open set in Ӿ. But $f^{-1}(Ɏ - (Q, \mathcal{D})) = f^{-1}(Ɏ) - f^{-1}(Q, \mathcal{D}) = Ӿ - f^{-1}(Q, \mathcal{D})$ (since $f^{-1}(Ɏ) = Ӿ$). Since $f^{-1}(Ɏ - (Q, \mathcal{D})) \in \Psi \to (Ӿ - f^{-1}(Q, \mathcal{D}) \in \Psi$.

($\Leftarrow$) Let $(\mathcal{C}, \mathcal{G})$ is Double I-open set in Ɏ $\to$ Ɏ $- (\mathcal{C}, \mathcal{G})$ is Double I-closed set in Ɏ $\to f^{-1}(Ɏ - (\mathcal{C}, \mathcal{G}))$ is Double I-closed set in Ӿ $\to (Ӿ - f^{-1}(Ɏ - (\mathcal{C}, \mathcal{G}))) \in \Psi$. But, $Ӿ - f^{-1}(Ɏ - (\mathcal{C}, \mathcal{G})) = Ӿ - (f^{-1}(Ɏ) - f^{-1}(\mathcal{C}, \mathcal{G})) = Ӿ - (Ӿ - f^{-1}(\mathcal{C}, \mathcal{G})) = f^{-1}(\mathcal{C}, \mathcal{G}) \to f^{-1}(\mathcal{C}, \mathcal{G}) \in \Psi$.

Therefore, f is Double I-continuous.

2) To prove f is Double I-continuous $\Leftrightarrow$ $f^{-1}(Q, \mathcal{D}) \in \Psi, \forall (Q, \mathcal{D}) \in \beta$, β is Double I-basis for Ψ^*.

($\Rightarrow$) Let β is Double I-basis for Ψ^* and $\beta \subseteq \Psi^*$, so $(Q, \mathcal{D}) \in \beta \subseteq \Psi^* \to \beta \in \Psi^*$. Then $(Q, \mathcal{D})$ is Double I-open set in Ɏ. Since f is Double I-continuous, $f^{-1}(Q, \mathcal{D})$ is Double I-open set in Ӿ, i.e., $f^{-1}(Q, \mathcal{D}) \in \Psi\ \forall (Q, \mathcal{D}) \in \beta$

($\Leftarrow$) Let $(\mathcal{C}, \mathcal{G})$ is Double I-open set in Ɏ, to prove $f^{-1}(\mathcal{C}, \mathcal{G})$ is Double I-open set in Ӿ, i.e., $(\mathcal{C}, \mathcal{G}) \in \Psi^*$ and β is Double I-basis for Ψ. Then $(\mathcal{C}, \mathcal{G}) = \cup_{\alpha \in \Lambda} (\mathcal{C}_\alpha, \mathcal{G}_\alpha)$ where $(\mathcal{C}_\alpha, \mathcal{G}_\alpha) \in \beta, \forall \alpha \in \Lambda$. Now, for each $(\mathcal{C}_\alpha, \mathcal{G}_\alpha) \in \beta \to f^{-1}(\mathcal{C}, \mathcal{G}) = f^{-1}(\cup_{\alpha \in \Lambda} (\mathcal{C}_\alpha, \mathcal{G}_\alpha)) = \cup_{\alpha \in \Lambda} f^{-1}(\mathcal{C}_\alpha, \mathcal{G}_\alpha) \to \cup_{\alpha \in \Lambda} f^{-1}(\mathcal{C}_\alpha, \mathcal{G}_\alpha) \in \Psi$ (by theorem 3.2) $\to f^{-1}(\mathcal{C}, \mathcal{G}) \in \Psi$, so $f^{-1}(\mathcal{C}, \mathcal{G})$ is Double I-open set in Ӿ. Hence f is Double I-continuous.

3) To prove f is Double I-continuous $\Leftrightarrow$ $f^{-1}(Q, \mathcal{D}) \in \Psi\ \forall (Q, \mathcal{D}) \in \mathcal{S}$, $\mathcal{S}$ is Double I-sub basis for Ψ^*.

($\Rightarrow$) Let $\mathcal{S}$ is Double I-sub basis for Ψ^* and $(Q, \mathcal{D}) \in \mathcal{S}$, then $(Q, \mathcal{D}) \subseteq \beta \subseteq \Psi^* \to (Q, \mathcal{D}) \subseteq \Psi^* \to (Q, \mathcal{D}) \in \Psi^*$ (since $\mathcal{S} \subseteq \Psi^*$), since $(Q, \mathcal{D})$ is Double I-open set in Ɏ and f is Double I-continuous $\to f^{-1}(Q, \mathcal{D})$ is Double I-open set in Ӿ, i.e., $f^{-1}(Q, \mathcal{D}) \in \Psi\ \forall (Q, \mathcal{D}) \in \mathcal{S}$.

($\Leftarrow$) Let $(\mathcal{C}, \mathcal{G})$ is Double I-open set in Ɏ, i.e., $(\mathcal{C}, \mathcal{G}) \in \Psi^*$. Since β is Double I-basis for $\Psi^* \to (\mathcal{C}, \mathcal{G}) = \cup_{\alpha \in \Lambda} (\mathcal{C}_\alpha, \mathcal{G}_\alpha)$ where $(\mathcal{C}_\alpha, \mathcal{G}_\alpha) \in \beta$, $\forall \alpha \in \Lambda$. Now each $(\mathcal{C}_\alpha, \mathcal{G}_\alpha) \in \beta$ is finite intersection members of $\mathcal{S}$. So $(\mathcal{C}_\alpha, \mathcal{G}_\alpha) = (\mathcal{C}_1, \mathcal{G}_1) \cap (\mathcal{C}_2, \mathcal{G}_2) \cap \ldots \cap (\mathcal{C}_n, \mathcal{G}_n)$ for some $(\mathcal{C}_1, \mathcal{G}_1), (\mathcal{C}_2, \mathcal{G}_2), \ldots, (\mathcal{C}_n, \mathcal{G}_n) \in \mathcal{S} \to f^{-1}(\mathcal{C}_\alpha, \mathcal{G}_\alpha) = f^{-1}((\mathcal{C}_1, \mathcal{G}_1) \cap (\mathcal{C}_2, \mathcal{G}_2) \cap \ldots \cap (\mathcal{C}_n, \mathcal{G}_n)) = f^{-1}(\mathcal{C}_1, \mathcal{G}_1) \cap f^{-1}(\mathcal{C}_2, \mathcal{G}_2) \cap \ldots \cap f^{-1}(\mathcal{C}_n, \mathcal{G}_n) = f^{-1}(\cap_{j=1}^n (\mathcal{C}_j, \mathcal{G}_j))$.

Now $f^{-1}(\mathcal{C}, \mathcal{G}) = f^{-1}(\cup_{\alpha \in \Lambda} (\mathcal{C}_\alpha, \mathcal{G}_\alpha)) = \cup_{\alpha \in \Lambda} (f^{-1}(\mathcal{C}_\alpha, \mathcal{G}_\alpha)) = \cup_{\alpha \in \Lambda} (f^{-1}(\cap_{j=1}^n (\mathcal{C}_j, \mathcal{G}_j))) = \cup_{\alpha \in \Lambda} (\cap_{j=1}^n f^{-1}(\mathcal{C}_j, \mathcal{G}_j))$. Since $f^{-1}(\mathcal{C}_j, \mathcal{G}_j) \in \Psi \to f^{-1}(\mathcal{C}_j, \mathcal{G}_j)$ is Double I-open set in Ӿ $\to (\cap_{j=1}^n f^{-1}(\mathcal{C}_j, \mathcal{G}_j) \in \Psi \cdots$ (1) (by theorem 3.2) $\to \cup_{\alpha \in \Lambda} (\cap_{j=1}^n f^{-1}(\mathcal{C}_j, \mathcal{G}_j)) \in \Psi$ (by (1)). Hence $f^{-1}(\mathcal{C}, \mathcal{G})$ is

Double I-open set in Ẋ. i.e., $f^{-1}(\mathcal{C},\mathcal{G}) \in \Psi$. Therefore f is Double I-continuous.

4) To prove f is Double I-continuous $\Leftrightarrow$ cl $(f^{-1}(\mathcal{Q},\mathcal{D})) \subseteq f^{-1}(cl(\mathcal{Q},\mathcal{D}))$; $\forall$ $(\mathcal{Q},\mathcal{D}) \subseteq$ Double I (Ɏ).

($\Rightarrow$) Let f is Double I-continuous, cl$(\mathcal{Q},\mathcal{D})$ is Double I-closed set in Ɏ (by part (1) in this theorem), $(f^{-1}(cl(\mathcal{Q},\mathcal{D})))$ is Double I-closed set in Ẋ. cl $(f^{-1}(cl(\mathcal{Q},\mathcal{D})))= (f^{-1}(cl(\mathcal{Q},\mathcal{D}))\ldots$ (1). Now $(\mathcal{Q},\mathcal{D}) \subseteq cl(\mathcal{Q},\mathcal{D}) \rightarrow f^{-1}(\mathcal{Q},\mathcal{D}) \subseteq f^{-1}(cl(\mathcal{Q},\mathcal{D})) \rightarrow$ $cl(f^{-1}(\mathcal{Q},\mathcal{D})) \subseteq cl(f^{-1}(cl(\mathcal{Q},\mathcal{D}))) \rightarrow$ $cl(f^{-1}(\mathcal{Q},\mathcal{D})) \subseteq f^{-1}(cl(\mathcal{Q},\mathcal{D}))$ (by part (1)).

($\Leftarrow$) Let $(\mathcal{C},\mathcal{G})$ is Double I-closed set in Ɏ, to prove $f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-closed set in Ẋ, i.e., $cl(f^{-1}(\mathcal{C},\mathcal{G}))=f^{-1}(\mathcal{C},\mathcal{G})$. Since $(\mathcal{C},\mathcal{G}) \subseteq cl(\mathcal{C},\mathcal{G}) \rightarrow f^{-1}(\mathcal{C},\mathcal{G}) \subseteq cl(f^{-1}(\mathcal{C},\mathcal{G}))\ldots(1)$. Since $(\mathcal{C},\mathcal{G})$ is Double I-closed $\rightarrow (\mathcal{C},\mathcal{G}) = cl(\mathcal{C},\mathcal{G}) \rightarrow f^{-1}(\mathcal{C},\mathcal{G}) = f^{-1}(cl(\mathcal{C},\mathcal{G})) \rightarrow cl(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(cl(\mathcal{C},\mathcal{G})) = f^{-1}(\mathcal{C},\mathcal{G}) \rightarrow cl(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(\mathcal{C},\mathcal{G})$ ...(2) from (1) and (2), we have $cl(f^{-1}(\mathcal{C},\mathcal{G})) = f^{-1}(\mathcal{C},\mathcal{G})$, so $f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-closed set in Ẋ. Therefore, f is Double I-continuous.

5) To prove f is Double I-continuous $\Leftrightarrow f^{-1}(int(\mathcal{Q},\mathcal{D})) \subseteq int(f^{-1}(\mathcal{Q},\mathcal{D}))$; $\forall$ $(\mathcal{Q},\mathcal{D}) \subseteq$ Double I(Ɏ).

($\Rightarrow$) Let $(\mathcal{Q},\mathcal{D}) \subseteq$ Ɏ, since $int(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D}) \rightarrow f^{-1}(int(\mathcal{Q},\mathcal{D})) \subseteq f^{-1}(\mathcal{Q},\mathcal{D})$, so $int(\mathcal{Q},\mathcal{D})$ is Double I- open set in Ɏ, f is Double I-continuous $\rightarrow f^{-1}(int(\mathcal{Q},\mathcal{D}))$ Double I-open set in Ẋ. Since $\rightarrow f^{-1}(int(\mathcal{Q},\mathcal{D}))$ is union of all Double I-open set that contain in $f^{-1}(\mathcal{Q},\mathcal{D})$ and $f^{-1}(int(\mathcal{Q},\mathcal{D}))$ is one of the Double I-open set that contain in $f^{-1}(\mathcal{Q},\mathcal{D})$, then $f^{-1}(int(\mathcal{Q},\mathcal{D})) \subseteq int(f^{-1}(\mathcal{Q},\mathcal{D}))$.

($\Leftarrow$) Let $(\mathcal{C},\mathcal{G})$ is Double I-open set in Ɏ, to prove $f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-open in Ẋ, i.e., $f^{-1}(\mathcal{C},\mathcal{G})=$ int $(f^{-1}(\mathcal{C},\mathcal{G}))$. Since $int(\mathcal{C},\mathcal{G}) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow int(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(\mathcal{C},\mathcal{G})\ldots(1)$. Since $(\mathcal{C},\mathcal{G})$ is Double I-open $\rightarrow (\mathcal{C},\mathcal{G}) = int(\mathcal{C},\mathcal{G}) \rightarrow f^{-1}(\mathcal{C},\mathcal{G}) = f^{-1}(int(\mathcal{C},\mathcal{G})) \rightarrow f^{-1}(\mathcal{C},\mathcal{G}) = f^{-1}(int(\mathcal{C},\mathcal{G})) \subseteq int(f^{-1}(\mathcal{C},\mathcal{G})) \rightarrow f^{-1}(\mathcal{C},\mathcal{G}) \subseteq int(f^{-1}(\mathcal{C},\mathcal{G}))$ ...(2) from (1) and (2), we have $f^{-1}(\mathcal{C},\mathcal{G}) = int(f^{-1}(\mathcal{C},\mathcal{G}))$. Hence $f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-open in Ẋ. Therefore, f is Double I-continuous.

Theorem 3.21 Let $(\mathcal{X},\Psi)$ and (Ɏ, Ψ^*) be two DITS, then f: $(\mathcal{X}, \Psi) \rightarrow (Ɏ, \Psi^*)$ is Double I-continuous if and only if $f(cl(\mathcal{Q},\mathcal{D})) \subseteq cl(f(\mathcal{Q},\mathcal{D}))$; $\forall$ $(\mathcal{Q},\mathcal{D}) \subseteq$ Double I(Ẋ).

Proof Suppose that f is Double I-continuous, to prove $f(cl(\mathcal{Q},\mathcal{D})) \subseteq cl(f(\mathcal{Q},\mathcal{D}))$;$\forall$ $(\mathcal{Q},\mathcal{D}) \subseteq$ Double I (Ẋ). Let $cl(f(\mathcal{Q},\mathcal{D}))$ be Double I-closed set in Ɏ, then $f^{-1}(cl(f(\mathcal{Q},\mathcal{D})))$ is Double I-closed in Ẋ (since f is Double I-continuous) $\rightarrow$ cl $(f^{-1}(cl(f(\mathcal{Q},\mathcal{D}))))= f^{-1}(cl(f(\mathcal{Q},\mathcal{D})))\ldots$ (1) Now, f $(\mathcal{Q},\mathcal{D}) \subseteq cl(f(\mathcal{Q},\mathcal{D})) \rightarrow (\mathcal{Q},\mathcal{D}) \subseteq f^{-1}(cl(f(\mathcal{Q},\mathcal{D}))) \rightarrow cl(\mathcal{Q},\mathcal{D}) \subseteq cl(f^{-1}(cl(f(\mathcal{Q},\mathcal{D}))))$ from (1) $\rightarrow cl(\mathcal{Q},\mathcal{D}) \subseteq f^{-1}(cl(f(\mathcal{Q},\mathcal{D}))) \rightarrow f(cl(\mathcal{Q},\mathcal{D})) \subseteq f(f^{-1}(cl(f(\mathcal{Q},\mathcal{D}))) \rightarrow f(cl(\mathcal{Q},\mathcal{D})) \subseteq cl(f(\mathcal{Q},\mathcal{D}))\ldots$ (2)

Conversely: Let $(\mathcal{C},\mathcal{G})$ be Double I-closed set in Ɏ, $f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-closed set in Ẋ. (Since f is Double I-continuous) $\rightarrow f(cl(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq cl(f(f^{-1}(\mathcal{C},\mathcal{G})))$ from (2) $\rightarrow f(cl(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq cl(\mathcal{C},\mathcal{G}) = (\mathcal{C},\mathcal{G})$ (since $(\mathcal{C},\mathcal{G})$ is Double I-closed) $\rightarrow f(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow cl(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(\mathcal{C},\mathcal{G})$. Also $f^{-1}(\mathcal{C},\mathcal{G}) \subseteq cl(f^{-1}(\mathcal{C},\mathcal{G}))$, so $cl(f^{-1}(\mathcal{C},\mathcal{G})) = f^{-1}(\mathcal{C},\mathcal{G})$. Hence $f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-closed set in Ẋ.

Therefore, f is Double I-continuous.

Theorem 3.22 Let (Ẋ, Ψ) and (Ɏ, Ψ^*) be two DITS, then f: $(\mathcal{X}, \Psi) \rightarrow (Ɏ, \Psi^*)$ is Double I-continuous if and only if $int(f(\mathcal{Q},\mathcal{D})) \subseteq f(int(\mathcal{Q},\mathcal{D}))$; $\forall (\mathcal{Q},\mathcal{D}) \subseteq$ Double I (Ẋ).

Proof Suppose that f is Double I-continuous, to prove $int(f(\mathcal{Q},\mathcal{D})) \subseteq f(int(\mathcal{Q},\mathcal{D}))$; $\forall$ $(\mathcal{Q},\mathcal{D}) \subseteq$ Double I (Ẋ). Let $int(f(\mathcal{Q},\mathcal{D}))$ be Double I-open set in Ɏ, then $f^{-1}(int(f(\mathcal{Q},\mathcal{D})))$ is Double I-open in Ẋ (since f is Double I-continuous) $\rightarrow$ int $(f^{-1}(int(f(\mathcal{Q},\mathcal{D}))))= f^{-1}(int(f(\mathcal{Q},\mathcal{D})))\ldots$ (1). Now, int $(f(\mathcal{Q},\mathcal{D})) \subseteq f(\mathcal{Q},\mathcal{D}) \rightarrow f^{-1}(int(f(\mathcal{Q},\mathcal{D}))) \subseteq (\mathcal{Q},\mathcal{D}) \rightarrow int(f^{-1}(int(f(\mathcal{Q},\mathcal{D})))) \subseteq int(\mathcal{Q},\mathcal{D})$ from (1)$\rightarrow f^{-1}(int(f(\mathcal{Q},\mathcal{D}))) \subseteq int(\mathcal{Q},\mathcal{D}) \rightarrow f(f^{-1}(int(f(\mathcal{Q},\mathcal{D})))) \subseteq f(int(\mathcal{Q},\mathcal{D})) \rightarrow int(f(\mathcal{Q},\mathcal{D})) \subseteq f(int(\mathcal{Q},\mathcal{D}))\ldots$ (2) .

Conversely: Let $(\mathcal{C},\mathcal{G})$ be Double I-open set in Ɏ ,$f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-open set in Ẋ. (since f is Double I- continuous)$\rightarrow int(f(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq f(int(f^{-1}(\mathcal{C},\mathcal{G})))$ from (2)$\rightarrow$ $(\mathcal{C},\mathcal{G}) = int(\mathcal{C},\mathcal{G}) \subseteq f(int(f^{-1}(\mathcal{C},\mathcal{G})))$ (since $(\mathcal{C},\mathcal{G})$ is Double I-open) $\rightarrow (\mathcal{C},\mathcal{G}) \subseteq f(f^{-1}(\mathcal{C},\mathcal{G}) \rightarrow f^{-1}(\mathcal{C},\mathcal{G}) \subseteq int(f^{-1}(\mathcal{C},\mathcal{G}))$. Also $int(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(\mathcal{C},\mathcal{G})$, so $f^{-1}(\mathcal{C},\mathcal{G}) = int(f^{-1}(\mathcal{C},\mathcal{G}))$. Hence $f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-open set in Ẋ. Therefore, f is Double I-continuous.

Remark 3.23 If h: $(\mathcal{X}, \Psi) \rightarrow (Ɏ, \Psi^*)$ is Double I-continuous function, then it's not necessary that the direct image of Double I-open set in Ẋ is Double I-open set in Ɏ. i.e., $(\mathcal{Q},\mathcal{D}) \in \Psi \nRightarrow h(\mathcal{Q},\mathcal{D}) \notin \Psi^*$.

Example 3.24 Let $\mathcal{X} = \{v,n,m\}$; $\Psi = \{(\widetilde{\emptyset},\widetilde{\emptyset})$, $(\widetilde{Ẋ},\widetilde{Ẋ})$, (K_1,K_2), (K_3,K_4), $(K_5,\widetilde{Ẋ})$, $(K_6,K_7)\}$ where $(K_1,K_2)(\langle x,\{v\},\{n\}\rangle,\langle x,\{v,m\},\{n\}\rangle)$,$(K_3,K_4)=(\langle x,\{m\},\{v\}\rangle,\langle x,\{n,m\},\{v\}\rangle)$,$(K_5,\widetilde{Ẋ})=(\langle x,\{v,m\},\emptyset\rangle, \langle x,Ẋ,\emptyset\rangle)$,$(K_6,K_7)=(\langle x,\emptyset,\{v,n\}\rangle,\langle x,\{m\},\{v,n\}\rangle)$.

Let Ɏ = {4,5,6}; IN= $\{(\widetilde{\emptyset},\widetilde{\emptyset}),(\widetilde{Ɏ},\widetilde{Ɏ})\}$, define a function h: $(\mathcal{X},\Psi) \rightarrow (Ɏ, IN)$ by h $(\{v\},\{m\}) = (\{5\},\{6\})$, h$(\{m\},\{n\}) = (\{6\},\{4\})$ and h$(\{n\},\{v\}) = (\{4\}\{5\})$. h is Double I-continuous (remark 3.13(1)). Let (K_1,K_2) (be Double I-open set in $(\mathcal{X}, \Psi) \rightarrow$h $(K_1,K_2)=(\langle y,\{5\},\{4\}\rangle,\langle y,\{5,6\},\{4\}\rangle)$ is not Double I-open in (Ɏ, IN). Therefore, we show that $(K_1,K_2) \in \Psi \wedge h(K_1,K_2) \notin$ IN.

Remark 3.25 If h: (Ẋ, Ψ)$\rightarrow$ (Ɏ, Ψ^*) is Double I-continuous function, then it's not necessary that the direct image of Double I-closed set in Ẋ is Double I-closed set in Ɏ, i.e., $(\mathcal{Q},\mathcal{D})^c \in \Psi^c \nRightarrow h(\mathcal{Q},\mathcal{D})^c \notin \Psi^{*c}$.

Example 3.26 In the previous example h: $(Ӿ,\Psi) \to (Ɏ, IN)$, h is Double I-continuous. Let $(K_2^c, K_1^c) = (\langle x, \{n\}, \{v, m\}\rangle, \langle x, \{n\}, \{v\}\rangle)$ be Double I-closed set in $(Ӿ, \Psi^c) \to h(K_2^c, K_1^c)$ $= (\langle y, \{4\}, \{5,6\}\rangle, \langle y, \{4\}, \{5\}\rangle)$ is not Double I-closed in (Ɏ, IN). Therefore, we show that $(K_2^c, K_1^c) \in \Psi^c \wedge h(K_2^c, K_1^c) \notin IN$.

Theorem 3.27 Let $(Ӿ, \Psi)$ and $(Ɏ, \Psi^*)$ be two DITS, f: $(Ӿ, \Psi) \to (Ɏ, \Psi^*)$ be a function. Then the following statements are equivalent:

1) f is Double I - closed function.

2) $cl(f(\mathcal{Q},\mathcal{D})) \subseteq f(cl(\mathcal{Q},\mathcal{D}))$; $\forall (\mathcal{Q},\mathcal{D}) \in$ Double I(Ӿ).

Proof 1→ 2 Let f be Double I-closed function, $(\mathcal{Q},\mathcal{D})$ be any Double I(Ӿ). Clearly, $cl(\mathcal{Q},\mathcal{D})$ is Double I- closed set in Ӿ. Since f is Double I-closed, f $(cl(\mathcal{Q},\mathcal{D}))$ is Double I-closed in Ɏ. Thus, we have $cl(f(\mathcal{Q},\mathcal{D})) \subseteq cl(f(cl(\mathcal{Q},\mathcal{D}))) = f(cl(\mathcal{Q},\mathcal{D}))) \to cl(f(\mathcal{Q},\mathcal{D})) \subseteq f(cl(\mathcal{Q},\mathcal{D}))$.

2→ 1 Let $(\mathcal{Q},\mathcal{D})$ be any Double I-closed set in Ӿ, then $cl(\mathcal{Q},\mathcal{D})=(\mathcal{Q},\mathcal{D})$ (by (2)),$cl(f(\mathcal{Q},\mathcal{D})) \subseteq f(cl(\mathcal{Q},\mathcal{D}))= f(\mathcal{Q},\mathcal{D}) \subseteq cl(f(\mathcal{Q},\mathcal{D}))$. Thus $f(\mathcal{Q},\mathcal{D}) = cl(f(\mathcal{Q},\mathcal{D}))$ and hence $f(\mathcal{Q},\mathcal{D})$ is Double I-closed in Ɏ. Therefore, f is Double I-closed function.

Theorem 3.28 Let $(Ӿ, \Psi)$, $(Ɏ, \Psi^*)$ be two DITS, f: $(Ӿ, \Psi) \to (Ɏ, \Psi^*)$ be a function. Then the following statements are equivalent:

1) f is Double I-open function.

2) $f(int(\mathcal{Q},\mathcal{D})) \subseteq int(f(\mathcal{Q},\mathcal{D}))$; $\forall(\mathcal{Q},\mathcal{D}) \in$ Double I(Ӿ).

3) $int(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(int(\mathcal{C},\mathcal{G}))$; $\forall(\mathcal{C},\mathcal{G}) \in$ Double I(Ɏ).

Proof 1→ 2 Let f be Double I-open and, $(\mathcal{Q},\mathcal{D}) \in$ DI(Ӿ). Since $int(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D})$, then $f(int(\mathcal{Q},\mathcal{D})) \subseteq f(\mathcal{Q},\mathcal{D})$ ($int(\mathcal{Q},\mathcal{D})$ is Double I-open → $f(int(\mathcal{Q},\mathcal{D}))$ is Double I-open). Also, f is Double I-open function, $f(int(\mathcal{Q},\mathcal{D}))$ is Double I-open in Ɏ that contained in $f(\mathcal{Q},\mathcal{D})$ and this implies that $f(int(\mathcal{Q},\mathcal{D}) = int(f(int(\mathcal{Q},\mathcal{D}))) \subseteq int(f(\mathcal{Q},\mathcal{D}))$. i.e., $f(int(\mathcal{Q},\mathcal{D})) \subseteq int(f(\mathcal{Q},\mathcal{D}))$.

2→ 3 Let $(\mathcal{C},\mathcal{G})$ be any Double I-open in Ɏ, then $f^{-1}(\mathcal{C},\mathcal{G})$ is Double I-open set in Ӿ (by(2)) $f(int(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq int(f(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq int(\mathcal{C},\mathcal{G}) \to f(int(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq int(\mathcal{C},\mathcal{G})$. Thus, we have $int(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(f(int(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq f^{-1}(int(\mathcal{C},\mathcal{G})) \to int(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(int(\mathcal{C},\mathcal{G}))$.

3→ 1 Let $(\mathcal{Q},\mathcal{D})$ be any Double I-open in Ӿ, then $int(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D})$ and $f(\mathcal{Q},\mathcal{D})$ is Double I-open in Ɏ

(by (3)),$(\mathcal{Q},\mathcal{D}) = int(\mathcal{Q},\mathcal{D}) \subseteq int(f^{-1}(f(\mathcal{Q},\mathcal{D})) \subseteq f^{-1}(int(f(\mathcal{Q},\mathcal{D})))$. Hence, we have $f(\mathcal{Q},\mathcal{D}) \subseteq f(f^{-1}(int(f(\mathcal{Q},\mathcal{D})))) \subseteq int(f(\mathcal{Q},\mathcal{D})) \subseteq f(\mathcal{Q},\mathcal{D})$. Thus $f(\mathcal{Q},\mathcal{D}) = int(f(\mathcal{Q},\mathcal{D})$, so $f(\mathcal{Q},\mathcal{D})$ is Double I-open in Ɏ. Therefore, f is Double I-open function.

Theorem 3.29 Let $(Ӿ, \Psi)$ and $(Ɏ, \Psi^*)$ be two DITS, f: $(Ӿ, \Psi) \to (Ɏ, \Psi^*)$ be a function, then the following condition are equivalent:

1) $f(cl(\mathcal{Q},\mathcal{D})) \subseteq cl(f(\mathcal{Q},\mathcal{D}))$; $\forall (\mathcal{Q},\mathcal{D}) \in$ Double I(Ӿ).

2) $cl(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(cl(\mathcal{C},\mathcal{G}))$; $\forall (\mathcal{C},\mathcal{G}) \in$ Double I(Ɏ).

Proof 1→ 2 Suppose that $(\mathcal{Q},\mathcal{D})$ is any DI(Ӿ) and $(\mathcal{C},\mathcal{G})$ is any Double I(Ɏ), $cl(\mathcal{C},\mathcal{G})$ Double I-closed set in Ɏ. Since $(\mathcal{C},\mathcal{G}) \subseteq cl(\mathcal{C},\mathcal{G})$, then $f^{-1}(\mathcal{C},\mathcal{G}) \subseteq f^{-1}(cl(\mathcal{C},\mathcal{G}))$, we get from hypothesis that $f(cl(\mathcal{Q},\mathcal{D})) \subseteq cl(f(\mathcal{Q},\mathcal{D}))$, then $f(cl(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq cl(f(f^{-1}(\mathcal{C},\mathcal{G}))) \subseteq cl(\mathcal{C},\mathcal{G})$ (by replacing $(\mathcal{Q},\mathcal{D})$ by $f^{-1}(\mathcal{C},\mathcal{G})$). So $cl(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(cl(\mathcal{C},\mathcal{G}))$.

2→ 1 Suppose that $(\mathcal{C},\mathcal{G})$ is any Double I (Ɏ), and $(\mathcal{C},\mathcal{G}) = f(\mathcal{Q},\mathcal{D})$, where $(\mathcal{Q},\mathcal{D}) \in$ Double I (Ӿ). Since from hypothesis $cl(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}(cl(\mathcal{C},\mathcal{G}))$, then $cl(f^{-1}(f(\mathcal{Q},\mathcal{D}))) \subseteq cl(f^{-1}(\mathcal{C},\mathcal{G})) \subseteq f^{-1}\left(cl(f(\mathcal{Q},\mathcal{D}))\right)$. So $cl(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D}) \subseteq f^{-1}(cl(f(\mathcal{Q},\mathcal{D})))$, therefore $f(cl(\mathcal{Q},\mathcal{D})) \subseteq cl(f(\mathcal{Q},\mathcal{D}))$.

Theorem 3.30 Let $(Ӿ, \Psi)$ and $(Ɏ, \Psi^*)$ be two DITS, f:$(Ӿ, \Psi) \to (Ɏ, \Psi^*)$ be Double I-continuous and Double I-open function, then $f^{-1}(cl(\mathcal{Q},\mathcal{D})) = cl(f^{-1}(\mathcal{Q},\mathcal{D}))$; $\forall (\mathcal{Q},\mathcal{D}) \in$ Double I(Ɏ).

Proof Suppose that $(\mathcal{Q},\mathcal{D})$ be any Double I(Ɏ), we have $(\mathcal{Q},\mathcal{D}) \subseteq cl(\mathcal{Q},\mathcal{D})$. So $f^{-1}(\mathcal{Q},\mathcal{D}) \subseteq f^{-1}(cl(\mathcal{Q},\mathcal{D}))$. Since f is Double I-continuous, cl $(f^{-1}(\mathcal{Q},\mathcal{D})) \subseteq cl(f^{-1}(cl(\mathcal{Q},\mathcal{D}))) = f^{-1}(cl(\mathcal{Q},\mathcal{D})) \to$ cl $(f^{-1}(\mathcal{Q},\mathcal{D})) \subseteq f^{-1}(cl(\mathcal{Q},\mathcal{D}))$... (1). Now we know that int $(f(cl(\mathcal{Q},\mathcal{D}))) \subseteq f^{-1}(cl(\mathcal{Q},\mathcal{D}))$, then f (int $(f^{-1}(cl(\mathcal{Q},\mathcal{D}))) \subseteq f (f^{-1}(cl(\mathcal{Q},\mathcal{D}))) \subseteq cl(\mathcal{Q},\mathcal{D})$. Since f is Double I-open, f (int $(f^{-1}(cl(\mathcal{Q},\mathcal{D}))))$=int (f (int($f^{-1}(cl(\mathcal{Q},\mathcal{D})))))) \subseteq int(cl(\mathcal{Q},\mathcal{D}))$. This implies that int $(f^{-1}(cl(\mathcal{Q},\mathcal{D}))) \subseteq f^{-1}(f(int(f^{-1}(cl(\mathcal{Q},\mathcal{D}))))) \subseteq f^{-1}(int(cl(\mathcal{Q},\mathcal{D})))$. Then int (cl $(f^{-1}(\mathcal{Q},\mathcal{D}))) \subseteq f^{-1}(cl(cl(\mathcal{Q},\mathcal{D})))$. Therefore cl $(cl(f^{-1}(\mathcal{Q},\mathcal{D}))) \subseteq cl(f^{-1}(cl(\mathcal{Q},\mathcal{D})))$. Hence, $f^{-1}(cl(\mathcal{Q},\mathcal{D})) \subseteq cl(f^{-1}(\mathcal{Q},\mathcal{D}))$… (2), from (1) and (2) we get the result.

Theorem 3.31 The image of Double I-connected under a Double I-continuous are Double I-connected

Proof Let f (E_1, E_2) is Double I-disconnected, there exist two nonempty, disjoint and separated Double I-sets $(\mathcal{Q},\mathcal{D})$, $(\mathcal{C},\mathcal{G})$ such that f $(E_1, E_2) = (\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G})$... (1). Let $(\mathcal{A}_1, \mathcal{A}_2) = f^{-1}(\mathcal{Q},\mathcal{D}) \cap (E_1, E_2)$, $(\mathcal{B}_1, \mathcal{B}_2) = f^{-1}(\mathcal{C},\mathcal{G}) \cap (E_1, E_2)$.

$(\mathcal{A}_1, \mathcal{A}_2) \cup (\mathcal{B}_1, \mathcal{B}_2) = (f^{-1}(\mathcal{Q},\mathcal{D}) \cap (E_1, E_2)) \cup (f^{-1}(\mathcal{C},\mathcal{G}) \cap$
$(E_1, E_2)) = f^{-1}(\mathcal{Q},\mathcal{D}) \cup$
$f^{-1}(\mathcal{C},\mathcal{G}) \cap (E_1, E_2) = f^{-1}((\mathcal{Q},\mathcal{D}) \cup$
$(\mathcal{C},\mathcal{G})) \cap (E_1, E_2) = (f^{-1}(f(E_1, E_2))) \cap$
$(E_1, E_2) = (E_1, E_2)) \cap (E_1, E_2) = (E_1, E_2) \to (\mathcal{A}_1, \mathcal{A}_2) \cup$
$(\mathcal{B}_1, \mathcal{B}_2) = (E_1, E_2)$ to show that $cl(\mathcal{A}_1, \mathcal{A}_2) \cap$
$(\mathcal{B}_1, \mathcal{B}_2) = (\tilde{\emptyset}, \tilde{\emptyset}) \to$
$cl(\mathcal{A}_1, \mathcal{A}_2) \cap (\mathcal{B}_1, \mathcal{B}_2) = cl(f^{-1}(\mathcal{Q},\mathcal{D}) \cap (E_1, E_2)) \cap$
$(f^{-1}(\mathcal{C},\mathcal{G}) \cap (E_1, E_2)) \subseteq (cl(f^{-1}(\mathcal{Q},\mathcal{D})) \cap$
$cl(E_1, E_2)) \cap (f^{-1}(\mathcal{C},\mathcal{G}) \cap$
$(E_1, E_2)) = (cl(f^{-1}(\mathcal{Q},\mathcal{D})) \cap f^{-1}(\mathcal{C},\mathcal{G})) \cap$

$(cl(E_1, E_2) \cap$ $(E_1, E_2))$= $(\mathrm{cl}(\mathrm{f}^{-1}(\mathcal{Q}, \mathcal{D}))) \cap \mathrm{f}^{-1}(\mathcal{C}, \mathcal{G}) \cap (E_1, E_2) \subseteq \mathrm{f}^{-1}(\mathrm{cl}(\mathcal{Q}, \mathcal{D})) \cap \mathrm{f}^{-1}(\mathcal{C}, \mathcal{G}) \cap (E_1, E_2)$ (by theorem 3.20 (4)) $= (\mathrm{f}^{-1}(\mathrm{cl}(\mathcal{Q}, \mathcal{D})) \cap (\mathcal{C}, \mathcal{G}))) \cap (E_1, E_2)$ $(\mathrm{cl}(\mathcal{Q}, \mathcal{D}) \cap (\mathcal{C}, \mathcal{G}) = (\tilde{\emptyset}, \tilde{\emptyset})$ as $(\mathcal{Q}, \mathcal{D}), (\mathcal{C}, \mathcal{G})$ are separated Double I-sets) $= \mathrm{f}^{-1}(\tilde{\emptyset}, \tilde{\emptyset}) \cap (E_1, E_2) = (\tilde{\emptyset}, \tilde{\emptyset})$ $\rightarrow \mathrm{cl}(\mathcal{Q}, \mathcal{D}) \cap (\mathcal{C}, \mathcal{G}) = (\tilde{\emptyset}, \tilde{\emptyset})$, and $(\mathcal{Q}, \mathcal{D}) \cap cl(\mathcal{C}, \mathcal{G}) = (\tilde{\emptyset}, \tilde{\emptyset}) \rightarrow$ $(\mathcal{Q}, \mathcal{D})$ and $(\mathcal{C}, \mathcal{G})$ are separated Double I-sets, also $(\mathcal{Q}, \mathcal{D}) \cup (\mathcal{C}, \mathcal{G}) = (E_1, E_2) \rightarrow (E_1, E_2)$ is Double I-disconnected. But (E_1, E_2) is Double I-connected which a contradiction. Therefore f $(E_1, E_2)$is Double I- connected.

4 – Conclusion

In this paper, we got the next results. We have presented a new set of the following concepts; Double intuitionistic set (DIS) (resp., Double intuitionistic topological spaces (DITS), Double interior I-set, Double closure I-set, Double I- (basis and sub basis) sets, Double connected I-set, separated Double I-sets, Double I-continuous function and Double I-(open and closed) functions in DITS. And study the basic characteristics and qualities related to these types and the relationships between them.

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الدوال الحدسية المزدوجة المستمرة في الفضاءات التبولوجية الحدسية المزدوجة

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الملخص

الهدف من هذا البحث هو تقديم فئات جديدة من الدوال الحدسية المزدوجة المستمرة، الدوال الحدسية مزدوجة (مفتوحة ومغلقة) في الفضاءات التبولوجية الحدسية المزدوجة، كما ندرس خصائص العلاقات بين هذه الأنواع.