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Inferring the Eigenvalues and Eigenfunctions Asymptotically for the Eighth Order Boundary Value Problems

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ABSTRACT

In the present paper, we consider an eigenvalue problem generated by eight- order differential equations with suitable boundary conditions, that containing a spectral parameter. New accurate asymptotic expressions for the 8^{th} linearly independent solutions are computed. Then, new asymptotic formulas for the eigenvalues and eigenfunctions of this boundary value problem are obtained.

Introduction

In this paper, an eight-order linear differential operator is generated by the differential equation and boundary conditions of the form:

$$l(y) = y^{(8)}(x) + q(x)y(x) = \lambda^8 y(x), \quad x \in [0, a] \quad \ldots . (1)$$

$$U_j\big(y(x)\big) = \begin{cases} y^{(j)}(0) = 0, j = 0,1,2,3 \\ \sum_{k=1}^{8} \left(iw_j\lambda\right)^{k-1} y^{(8-k)}(a,\lambda) = 0, j = 4,5,6,7 \end{cases} \ldots . (2)$$

where λ is the spectral parameter and $q(x)$ is an arbitrary complex-valued function such that $q(x) \in C^2[0, a]$.

And also satisfies:

$q'(0) = q'(a) = 0, \int_0^a q(x)dx = 0$, provided $q(a) \neq 0$.

Many authors have studied the spectral properties of eigenvalues and eigenfunctions of differential equations such as [1- 12].

The differential equation of second order has been studied by [2, 5, 6, 7, 9, 10] and an eigenvalue problem generated by fourth-order differential equation has been investigated by [1, 3, 4], and got asymptotic formulas for eigenvalues and eigenfunctions.

While [8] have studied eigenvalue problem generated by 6^{th} order differential equations and also got asymptotic formulas for eigenvalues and eigenfunctions.

In this paper, a new expression for the 8^{th} linearly independent solutions and asymptotic formulas of the eigenvalues and eigenfunctions of equations (1) and (2) is generated the auxiliary results needed are proven in section 2.

2. Expressions of Fundamental Solutions

In this section, we find a new asymptotic expression for the fundamental solutions of (1).

Theorem 1: If we have the differential equation (1) .where, $q(x) \in C^{n-1}[0, a]$, then for $\lambda \in T_0$, where $T_0 = \left\{\lambda : \arg \lambda \in \left[0, \frac{\pi}{8}\right]\right\}$ and $w_k, k = 0:7$ are eight root of unity, then eight linearly independent solutions and their derivatives can be expressed as

$$y_k^{(s)}(x,\lambda) = (i\lambda w_k)^s e^{i\lambda w_k x} \left[A_{0sk}(x) + \frac{A_{1sk}(x)}{\lambda} + \frac{A_{2sk}(x)}{\lambda^2} + \frac{A_{3sk}(x)}{\lambda^3} + \frac{A_{4sk}(x)}{\lambda^4} + \frac{A_{5sk}(x)}{\lambda^5} + \frac{A_{6sk}(x)}{\lambda^6} + \cdots + \frac{A_{nsk}(x)}{\lambda^n} + O\left(\frac{1}{\lambda^{n+1}}\right)\right],$$

where

$A_{1sk} = A_{1k}(x), A_{2sk} = A_{2k}(x), A_{3sk} = A_{3k}(x), A_{4sk} = A_{4k}(x), A_{5sk} = A_{5k}(x), A_{6sk} = A_{6k}(x),$

$A_{7sk} = A_{7k}(x),$

$A_{8sk} = A_{8k}(x) - \binom{S}{1} iw_k^7 A'_{7k}(x),$

$A_{9sk} = A_{9k}(x) - \binom{S}{1} iw_k^7 A'_{8k}(x) - \binom{S}{2} w_k^6 A''_{7k}(x),$

$A_{10sk} = A_{10k}(x) - \binom{S}{1} iw_k^7 A'_{9k}(x) - \binom{S}{2} w_k^6 A''_{8k}(x) + \binom{S}{3} iw_k^5 A'''_{7k}(x),$

$A_{11sk} = A_{11k}(x) - \binom{S}{1} iw_k^7 A'_{10k}(x) - \binom{S}{2} w_k^6 A''_{9k}(x) + \binom{S}{3} iw_k^5 A'''_{8k}(x) + \binom{S}{4} w_k^4 A^{(4)}_{7k}(x),$

$A_{12sk} = A_{12k}(x) - \binom{S}{1} iw_k^7 A'_{11k}(x) - \binom{S}{2} w_k^6 A''_{10k}(x) + \binom{S}{3} iw_k^5 A'''_{9k}(x) + \binom{S}{4} w_k^4 A^{(4)}_{8k}(x) - \binom{S}{5} iw_k^3 A^{(5)}_{7k}(x),$

$A_{13sk} = A_{13k}(x) - \binom{S}{1} iw_k^7 A'_{12k}(x) - \binom{S}{2} w_k^6 A''_{11k}(x) + \binom{S}{3} iw_k^5 A'''_{10k}(x) + \binom{S}{4} w_k^4 A^{(4)}_{9k}(x) - \binom{S}{5} iw_k^3 A^{(5)}_{8k}(x) - \binom{S}{6} w_k^2 A^{(6)}_{7k}(x),$

$A_{14sk} = A_{14k}(x) - \binom{S}{1} iw_k^7 A'_{13k}(x) - \binom{S}{2} w_k^6 A''_{12k}(x) + \binom{S}{3} iw_k^5 A'''_{11k}(x) + \binom{S}{4} w_k^4 A^{(4)}_{10k}(x) - \binom{S}{5} iw_k^3 A^{(5)}_{9k}(x) - \binom{S}{6} w_k^2 A^{(6)}_{8k}(x) + \binom{S}{7} iw_k A^{(7)}_{7k}(x),$

$A_{15sk} = A_{15k}(x) - \binom{S}{1} iw_k^7 A'_{14k}(x) - \binom{S}{2} w_k^6 A''_{13k}(x) + \binom{S}{3} iw_k^5 A'''_{12k}(x) + \binom{S}{4} w_k^4 A^{(4)}_{11k}(x) - \binom{S}{5} iw_k^3 A^{(5)}_{10k}(x) - \binom{S}{6} w_k^2 A^{(6)}_{9k}(x) + \binom{S}{7} iw_k A^{(7)}_{8k}(x) + \binom{S}{8} A^{(8)}_{7k}(x).$

$\vdots$

And for $n > 15$ we have

$A_{nsk} = A_{nk}(x) - \binom{S}{1} iw_k^7 A'_{n-1,k}(x) - \binom{S}{2} w_k^6 A''_{n-2,k}(x) + \binom{S}{3} iw_k^5 A'''_{n-3,k}(x) + \binom{S}{4} w_k^4 A^{(4)}_{n-4,k}(x) - \binom{S}{5} iw_k^3 A^{(5)}_{n-5,k}(x) - \binom{S}{6} w_k^2 A^{(6)}_{n-6,k}(x) + \binom{S}{7} iw_k A^{(7)}_{n-7,k}(x) + \binom{S}{8} A^{(8)}_{n-8,k}(x).$

And

$A_{0k}(x) = 1, A_{1k}(x) = 0, A_{2k}(x) = 0, A_{3k}(x) = 0, A_{4k}(x) = 0, A_{5k}(x) = 0, A_{6k}(x) = 0,$

$A_{7k}(x) = -\frac{iw_k}{8}\int_0^x q(t)A_{0,k}(t)dt,$

$A_{8k}(x) = -\frac{iw_k}{8}\int_0^x \left(-28w_k^6 A''_{7,k}(t) + q(t)A_{1,k}(t)\right)dt,$

$A_{9k}(x) = -\frac{iw_k}{8}\int_0^x \left(-28w_k^6 A''_{8,k}(t) + 56iw_k^5 A'''_{7,k}(t) + q(t)A_{2,k}(t)\right)dt,$

$A_{10k}(x) = -\frac{iw_k}{6}\int_0^x \left(-28w_k^6 A''_{9,k}(t) + 56iw_k^5 A'''_{8,k}(t) + 70w_k^4 A^{(4)}_{7,k}(t) + q(t)A_{3,k}(t)\right)dt,$

And for integer $n \geq 11$ we get that

$A_{nk}(x) = -\frac{iw_k}{8}\int_0^x \left(-28w_k^6 A''_{n-1,k}(x) + 56iw_k^5 A'''_{n-2,k}(x) + 70w_k^4 A^{(4)}_{n-3,k}(x) - 56iw_k^3 A^{(5)}_{n-4,k}(x) - 28w_k^2 A^{(6)}_{n-5,k}(x) + 8iw_k A^{(7)}_{n-6,k}(x) + A^{(8)}_{n-7,k}(x) + q(x)A_{n-7,k}(x)\right)dt.$

Proof: As we see in [1], the solution of the differential equation can be written in a power series of the form $y_k(x,\lambda) = e^{\lambda\int_0^x \phi_k dt}\sum_{j=0}^{\infty}\frac{A_j(x)}{\lambda^j}$, where $\phi_k(x) = iw_k\sqrt[8]{\rho(x)}$, but in our problem $\rho(x) = 1$, can be written as $y_k(x,\lambda) = e^{i\lambda w_k x}\sum_{j=0}^{\infty}\frac{A_j(x)}{\lambda^j}$.

We try to find $y'_k, y''_k, y'''_k, y^{(4)}_k, y^{(5)}_k, y^{(6)}_k, y^{(7)}_k, y^{(8)}_k$ and putting in the differential equation (1).

$y_k(x,\lambda) = e^{i\lambda w_k x}\left[A_0(x) + \frac{A_1(x)}{\lambda} + \cdots + \frac{A_n(x)}{\lambda^n} + O\left(\frac{1}{\lambda^{n+1}}\right)\right], \ldots\ldots (3)$

$y'_k(x,\lambda) = i\lambda w_k e^{i\lambda w_k x}\left[A_0(x) + \frac{1}{\lambda}\left(A_1(x) - iw_k^7 A'_0(x)\right) + \frac{1}{\lambda^2}\left(A_2(x) - iw_k^7 A'_1(x)\right) + \frac{1}{\lambda^3}\left(A_3(x) - iw_k^7 A'_2(x)\right) + \frac{1}{\lambda^4}\left(A_4(x) - iw_k^7 A'_3(x)\right) + \cdots + \frac{1}{\lambda^n}\left(A_n(x) - iw_k^7 A'_{n-1}(x)\right) + O\left(\frac{1}{\lambda^{n+1}}\right)\right], \ldots\ldots (4)$

$y''_k(x,\lambda) = (i\lambda w_k)^2 e^{i\lambda w_k x}\left[A_0(x) + \frac{1}{\lambda}\left(A_1(x) - 2iw_k^7 A'_0(x)\right) + \frac{1}{\lambda^2}\left(A_2(x) - 2iw_k^7 A'_1(x) - w_k^6 A''_0(x)\right) + \frac{1}{\lambda^3}\left(A_3(x) - 2iw_k^7 A'_2(x) - w_k^6 A''_1(x)\right) + \frac{1}{\lambda^4}\left(A_4(x) - 2iw_k^7 A'_3(x) - w_k^6 A''_2(x)\right) + \frac{1}{\lambda^5}\left(A_5(x) - 2iw_k^7 A'_4(x) - w_k^6 A''_3(x)\right) + \cdots + \frac{1}{\lambda^n}\left(A_n(x) - 2iw_k^7 A'_{n-1}(x) - w_k^6 A''_{n-2}(x)\right) + O\left(\frac{1}{\lambda^{n+1}}\right)\right], \ldots\ldots (5)$

$y'''_k(x,\lambda) = (i\lambda w_k)^3 e^{i\lambda w_k x}\left[A_0(x) + \frac{1}{\lambda}\left(A_1(x) - 3iw_k^7 A'_0(x)\right) + \frac{1}{\lambda^2}\left(A_2(x) - 3iw_k^7 A'_1(x) - 3w_k^6 A''_0(x)\right) + \frac{1}{\lambda^3}\left(A_3(x) - 3iw_k^7 A'_2(x) - 3w_k^6 A''_1(x) + iw_k^5 A'''_0(x)\right) + \frac{1}{\lambda^4}\left(A_4(x) - 3iw_k^7 A'_3(x) - 3w_k^6 A''_2(x) + iw_k A'''_1(x)\right) + \frac{1}{\lambda^5}\left(A_5(x) - 3iw_k^7 A'_4(x) - 3w_k^6 A''_3(x) + iw_k A'''_2(x)\right) + \cdots + \frac{1}{\lambda^n}\left(A_n(x) - 3iw_k^7 A'_{n-1}(x) - 3w_k^6 A''_{n-2}(x) + iw_k^5 A'''_{n-3}(x)\right) + O\left(\frac{1}{\lambda^{n+1}}\right)\right], \ldots\ldots (6)$

$$y_k^{(4)}(x,\lambda) = (i\lambda w_k)^4 e^{i\lambda w_k x}\Big[A_0(x) + \tfrac{1}{\lambda}\big(A_1(x) - 4iw_k^7 A_0'(x)\big) + \tfrac{1}{\lambda^2}\big(A_2(x) - 4iw_k^7 A_1'(x) - 6w_k^6 A_0''(x)\big) + \tfrac{1}{\lambda^3}\big(A_3(x) - 4iw_k^7 A_2'(x) - 6w_k^6 A_1''(x) + 4iw_k^5 A_0'''(x)\big) + \tfrac{1}{\lambda^4}\big(A_4(x) - 4iw_k^7 A_3'(x) - 6w_k^6 A_2''(x) + 4iw_k^5 A_1'''(x) + w_k^4 A_0^{(4)}(x)\big) + \tfrac{1}{\lambda^5}\big(A_5(x) - 4iw_k^7 A_4'(x) - 6w_k^6 A_3''(x) + 4iw_k^5 A_2'''(x) + w_k^4 A_1^{(4)}(x)\big) + \tfrac{1}{\lambda^6}\big(A_6(x) - 4iw_k^7 A_5'(x) - 6w_k^6 A_4''(x) + 4iw_k^5 A_3'''(x) + w_k^4 A_2^{(4)}(x)\big) + \cdots + \tfrac{1}{\lambda^n}\big(A_n(x) - 4iw_k^7 A_{n-1}'(x) - 6w_k^6 A_{n-2}''(x) + 4iw_k^5 A_{n-3}'''(x) + w_k^4 A_{n-4}^{(4)}(x)\big) + O\Big(\tfrac{1}{\lambda^{n+1}}\Big)\Big], \quad \ldots.(7)$$

$$y_k^{(5)}(x,\lambda) = (i\lambda w_k)^5 e^{i\lambda w_k x}\Big[A_0(x) + \tfrac{1}{\lambda}\big(A_1(x) - 5iw_k^7 A_0'(x)\big) + \tfrac{1}{\lambda^2}\big(A_2(x) - 5iw_k^7 A_1'(x) - 10w_k^6 A_0''(x)\big) + \tfrac{1}{\lambda^3}\big(A_3(x) - 5iw_k^7 A_2'(x) - 10w_k^6 A_1''(x) + 10iw_k^5 A_0'''(x)\big) + \tfrac{1}{\lambda^4}\big(A_4(x) - 5iw_k^7 A_3'(x) - 10w_k^6 A_2''(x) + 10iw_k^5 A_1'''(x) + 5w_k^4 A_0^{(4)}(x)\big) + \tfrac{1}{\lambda^5}\big(A_5(x) - 5iw_k^7 A_4'(x) - 10w_k^6 A_3''(x) + 10iw_k^5 A_2'''(x) + 5w_k^4 A_1^{(4)}(x) - iw_k^3 A_0^{(5)}(x)\big) + \cdots + \tfrac{1}{\lambda^n}\big(A_n(x) - 5iw_k^7 A_{n-1}'(x) - 10w_k^6 A_{n-2}''(x) + 10iw_k^5 A_{n-3}'''(x) + 5w_k^4 A_{n-4}^{(4)}(x) - iw_k^3 A_{n-5}^{(5)}(x)\big) + O\Big(\tfrac{1}{\lambda^{n+1}}\Big)\Big], \ldots\ldots (8)$$

$$y_k^{(6)}(x,\lambda) = (i\lambda w_k)^6 e^{i\lambda w_k x}\Big[A_0(x) + \tfrac{1}{\lambda}\big(A_1(x) - 6iw_k^7 A_0'(x)\big) + \tfrac{1}{\lambda^2}\big(A_2(x) - 6iw_k^7 A_1'(x) - 15w_k^6 A_0''(x)\big) + \tfrac{1}{\lambda^3}\big(A_3(x) - 6iw_k^7 A_2'(x) - 15w_k^6 A_1''(x) + 20iw_k^5 A_0'''(x)\big) + \tfrac{1}{\lambda^4}\big(A_4(x) - 6iw_k^7 A_3'(x) - 15w_k^6 A_2''(x) + 20iw_k^5 A_1'''(x) + 15w_k^4 A_0^{(4)}(x)\big) + \tfrac{1}{\lambda^5}\big(A_5(x) - 6iw_k^7 A_4'(x) - 15w_k^6 A_3''(x) + 20iw_k^5 A_2'''(x) + 15w_k^4 A_1^{(4)}(x) - 6iw_k^3 A_0^{(5)}(x)\big) + \tfrac{1}{\lambda^6}\big(A_6(x) - 6iw_k^7 A_5'(x) - 15w_k^6 A_4''(x) + 20iw_k^5 A_3'''(x) + 15w_k^4 A_2^{(4)}(x) - 6iw_k^3 A_1^{(5)}(x) - w_k^2 A_0^{(6)}(x)\big) + \cdots + \tfrac{1}{\lambda^n}\big(A_n(x) - 6iw_k^7 A_{n-1}'(x) - 15w_k^6 A_{n-2}''(x) + 20iw_k^5 A_{n-3}'''(x) + 15w_k^4 A_{n-4}^{(4)}(x) - 6iw_k^3 A_{n-5}^{(5)}(x) - w_k^2 A_{n-6}^{(6)}(x)\big) + O\Big(\tfrac{1}{\lambda^{n+1}}\Big)\Big], \ldots (9)$$

$$y_k^{(7)}(x,\lambda) = (i\lambda w_k)^7 e^{i\lambda w_k x}\Big[A_0(x) + \tfrac{1}{\lambda}\big(A_1(x) - 7iw_k^7 A_0'(x)\big) + \tfrac{1}{\lambda^2}\big(A_2(x) - 7iw_k^7 A_1'(x) - 21w_k^6 A_0''(x)\big) + \tfrac{1}{\lambda^3}\big(A_3(x) - 7iw_k^7 A_2'(x) - 21w_k^6 A_1''(x) + 35iw_k^5 A_0'''(x)\big) + \tfrac{1}{\lambda^4}\big(A_4(x) - 7iw_k^7 A_3'(x) - 21w_k^6 A_2''(x) + 35iw_k^5 A_1'''(x) + 35w_k^4 A_0^{(4)}(x)\big) + \tfrac{1}{\lambda^5}\big(A_5(x) - 7iw_k^7 A_4'(x) - 21w_k^6 A_3''(x) + 35iw_k^5 A_2'''(x) + 35w_k^4 A_1^{(4)}(x) - 21iw_k^3 A_0^{(5)}(x)\big) + \tfrac{1}{\lambda^6}\big(A_6(x) - 7iw_k^7 A_5'(x) - 21w_k^6 A_4''(x) + 35iw_k^5 A_3'''(x) + 35w_k^4 A_2^{(4)}(x) - 21iw_k^3 A_1^{(5)}(x) - 7w_k^2 A_0^{(6)}(x)\big) + \tfrac{1}{\lambda^7}\big(A_7(x) - 7iw_k^7 A_6'(x) - 21w_k^6 A_5''(x) + 35iw_k^5 A_4'''(x) + 35w_k^4 A_3^{(4)}(x) - 21iw_k^3 A_2^{(5)}(x) - 7w_k^2 A_1^{(6)}(x) + iw_k A_0^{(7)}(x)\big) + \cdots + \tfrac{1}{\lambda^n}\big(A_n(x) - 7iw_k^7 A_{n-1}'(x) - 21w_k^6 A_{n-2}''(x) + 35iw_k^5 A_{n-3}'''(x) + 35w_k^4 A_{n-4}^{(4)}(x) - 21iw_k^3 A_{n-5}^{(5)}(x) - 7w_k^2 A_{n-6}^{(6)}(x) + iw_k A_{n-7}^{(7)}(x)\big) + O\Big(\tfrac{1}{\lambda^{n+1}}\Big)\Big], \ldots\ldots (10)$$

$$y_k^{(8)}(x,\lambda) = (i\lambda w_k)^8 e^{i\lambda w_k x}\Big[A_0(x) + \tfrac{1}{\lambda}\big(A_1(x) - 8iw_k^7 A_0'(x)\big) + \tfrac{1}{\lambda^2}\big(A_2(x) - 8iw_k^7 A_1'(x) - 28w_k^6 A_0''(x)\big) + \tfrac{1}{\lambda^3}\big(A_3(x) - 8iw_k^7 A_2'(x) - 28w_k^6 A_1''(x) + 56iw_k^5 A_0'''(x)\big) + \tfrac{1}{\lambda^4}\big(A_4(x) - 8iw_k^7 A_3'(x) - 28w_k^6 A_2''(x) + 56iw_k^5 A_1'''(x) + 70w_k^4 A_0^{(4)}(x)\big) + \tfrac{1}{\lambda^5}\big(A_5(x) - 8iw_k^7 A_4'(x) - 28w_k^6 A_3''(x) + 56iw_k^5 A_2'''(x) + 70w_k^4 A_1^{(4)}(x) - 56iw_k^3 A_0^{(5)}(x)\big) + \tfrac{1}{\lambda^6}\big(A_6(x) - 8iw_k^7 A_5'(x) - 28w_k^6 A_4''(x) + 56iw_k^5 A_3'''(x) + 70w_k^4 A_2^{(4)}(x) - 56iw_k^3 A_1^{(5)}(x) - 28w_k^2 A_0^{(6)}(x)\big) + \tfrac{1}{\lambda^7}\big(A_7(x) - 8iw_k^7 A_6'(x) - 28w_k^6 A_5''(x) + 56iw_k^5 A_4'''(x) + 70w_k^4 A_3^{(4)}(x) - 56iw_k^3 A_2^{(5)}(x) - 28w_k^2 A_1^{(6)}(x) + 8iw_k A_0^{(7)}(x)\big) + \tfrac{1}{\lambda^8}\big(A_8(x) - 8iw_k^7 A_7'(x) - 28w_k^6 A_6''(x) + 56iw_k^5 A_5'''(x) + 70w_k^4 A_4^{(4)}(x) - 56iw_k^3 A_3^{(5)}(x) - 28w_k^2 A_2^{(6)}(x) + 8iw_k A_1^{(7)}(x) + A_0^{(8)}(x)\big) + \cdots + \tfrac{1}{\lambda^n}\big(A_n(x) - 8iw_k^7 A_{n-1}'(x) - 28w_k^6 A_{n-2}''(x) + 56iw_k^5 A_{n-3}'''(x) + 70w_k^4 A_{n-4}^{(4)}(x) - 56iw_k^3 A_{n-5}^{(5)}(x) - 28w_k^2 A_{n-6}^{(6)}(x) + 8iw_k A_{n-7}^{(7)}(x) + A_{n-8}^{(8)}(x)\big) + O\Big(\tfrac{1}{\lambda^{n+1}}\Big)\Big] \ldots..(11)$$

Putting (3) and (11) in (1) and after simplification we get:

$$\lambda^8 e^{i\lambda w_k x}\Big[\tfrac{1}{\lambda}\big(-8iw_k^7 A_0'(x)\big) + \tfrac{1}{\lambda^2}\big(-8iw_k^7 A_1'(x) - 28w_k^6 A_0''(x)\big) + \tfrac{1}{\lambda^3}\big(-8iw_k^7 A_2'(x) - 28w_k^6 A_1''(x) + 56iw_k^5 A_0'''(x)\big) + \tfrac{1}{\lambda^4}\big(-8iw_k^7 A_3'(x) - 28w_k^6 A_2''(x) + 56iw_k^5 A_1'''(x) + 70w_k^4 A_0^{(4)}(x)\big) + \tfrac{1}{\lambda^5}\big(-8iw_k^7 A_4'(x) - 28w_k^6 A_3''(x) + 56iw_k^5 A_2'''(x) + 70w_k^4 A_1^{(4)}(x) - 56iw_k^3 A_0^{(5)}(x)\big) +$$

$\frac{1}{\lambda^6}(-8iw_k^7A_5'(x) - 28w_k^6A_4''(x) + 56iw_k^5A_3'''(x) + 70w_k^4A_2^{(4)}(x) - 56iw_k^3A_1^{(5)}(x) - 28w_k^2A_0^{(6)}(x)) +$
$\frac{1}{\lambda^7}(-8iw_k^7A_6'(x) - 28w_k^6A_5''(x) + 56iw_k^5A_4'''(x) + 70w_k^4A_3^{(4)}(x) - 56iw_k^3A_2^{(5)}(x) - 28w_k^2A_1^{(6)}(x) + 8iw_kA_0^{(7)}(x)) +$
$\frac{1}{\lambda^8}(-8iw_k^7A_7'(x) - 28w_k^6A_6''(x) + 56iw_k^5A_5'''(x) + 70w_k^4A_4^{(4)}(x) - 56iw_k^3A_3^{(5)}(x) - 28w_k^2A_2^{(6)}(x) + 8iw_kA_1^{(7)}(x) + A_0^{(8)}(x) + q(x)A_0(x)) +$
$\frac{1}{\lambda^9}(-8iw_k^7A_8'(x) - 28w_k^6A_7''(x) + 56iw_k^5A_6'''(x) + 70w_k^4A_5^{(4)}(x) - 56iw_k^3A_4^{(5)}(x) - 28w_k^2A_3^{(6)}(x) + 8iw_kA_2^{(7)}(x) + A_1^{(8)}(x) + q(x)A_1(x)) +$
$\frac{1}{\lambda^{10}}(-8iw_k^7A_9'(x) - 28w_k^6A_8''(x) + 56iw_k^5A_7''(x) + 70w_k^4A_6^{(4)}(x) - 56iw_k^3A_5^{(5)}(x) - 28w_k^2A_4^{(6)}(x) + 8iw_kA_3^{(7)}(x) + A_2^{(8)}(x) + q(x)A_2(x)) + \cdots +$
$\frac{1}{\lambda^n}(-8iw_k^7A_{n-1}'(x) - 28w_k^6A_{n-2}''(x) + 56iw_k^5A_{n-3}'''(x) + 70w_k^4A_{n-4}^{(4)}(x) - 56iw_k^3A_{n-5}^{(5)}(x) - 28w_k^2A_{n-6}^{(6)}(x) + 8iw_kA_{n-7}^{(7)}(x) + A_{n-8}^{(8)}(x) + q(x)A_{n-8}(x)) + O\left(\frac{1}{\lambda^{n+1}}\right)] = 0.$

By equating the coefficients of the same power of $\frac{1}{\lambda}$, then we get the following relation:

$-8iw_k^7A_0'(x) = 0,$
$-8iw_k^7A_1'(x) - 28w_k^6A_0''(x) = 0,$
$-8iw_k^7A_2'(x) - 28w_k^6A_1''(x) + 56iw_k^5A_0'''(x) = 0,$
$-8iw_k^7A_3'(x) - 28w_k^6A_2''(x) + 56iw_k^5A_1'''(x) + 70w_k^4A_0^{(4)}(x) = 0,$
$-8iw_k^7A_4'(x) - 28w_k^6A_3''(x) + 56iw_k^5A_2'''(x) + 70w_k^4A_1^{(4)}(x) - 56iw_k^3A_0^{(5)}(x) = 0,$
$-8iw_k^7A_5'(x) - 28w_k^6A_4''(x) + 56iw_k^5A_3'''(x) + 70w_k^4A_2^{(4)}(x) - 56iw_k^3A_1^{(5)}(x) - 28w_k^2A_0^{(6)}(x) = 0,$
$-8iw_k^7A_6'(x) - 28w_k^6A_5''(x) + 56iw_k^5A_4'''(x) + 70w_k^4A_3^{(4)}(x) - 56iw_k^3A_2^{(5)}(x) - 28w_k^2A_1^{(6)}(x) + 8iw_kA_0^{(7)}(x) = 0,$
$-8iw_k^7A_7'(x) - 28w_k^6A_6''(x) + 56iw_k^5A_5'''(x) + 70w_k^4A_4^{(4)}(x) - 56iw_k^3A_3^{(5)}(x) - 28w_k^2A_2^{(6)}(x) + 8iw_kA_1^{(7)}(x) + A_0^{(8)}(x) + q(x)A_0(x) = 0,$
$-8iw_k^7A_8'(x) - 28w_k^6A_7''(x) + 56iw_k^5A_6'''(x) + 70w_k^4A_5^{(4)}(x) - 56iw_k^3A_4^{(5)}(x) - 28w_k^2A_3^{(6)}(x) + 8iw_kA_2^{(7)}(x) + A_1^{(8)}(x) + q(x)A_1(x) = 0,$
$-8iw_k^7A_9'(x) - 28w_k^6A_8''(x) + 56iw_k^5A_7'''(x) + 70w_k^4A_6^{(4)}(x) - 56iw_k^3A_5^{(5)}(x) - 28w_k^2A_4^{(6)}(x) + 8iw_kA_3^{(7)}(x) + A_2^{(8)}(x) + q(x)A_2(x) = 0,$
$\vdots$
$-8iw_k^7A_{n-1}'(x) - 28w_k^6A_{n-2}''(x) + 56iw_k^5A_{n-3}'''(x) + 70w_k^4A_{n-4}^{(4)}(x) - 56iw_k^3A_{n-5}^{(5)}(x) - 28w_k^2A_{n-6}^{(6)}(x) + 8iw_kA_{n-7}^{(7)}(x) + A_{n-8}^{(8)}(x) + q(x)A_{n-8}(x) = 0.$

by solving above equations, we get
$A_{0k}(x) = 1, A_{1k}(x) = 0, A_{2k}(x) = 0, A_{3k}(x) = 0, A_{4k}(x) = 0, A_{5k}(x) = 0, A_{6k}(x) = 0,$

$A_{7k}(x) = -\frac{iw_k}{8}\int_0^x q(t)A_{0,k}(t)dt,$
$A_{8k}(x) = -\frac{iw_k}{8}\int_0^x \left(-28w_k^6\, A_{7,k}''(t) + q(t)A_{1,k}(t)\right)dt,$
$A_{9k}(x) = -\frac{iw_k}{8}\int_0^x \left(-28w_k^6A_{8,k}''(t) + 56iw_k^5A_{7,k}'''(t) + q(t)A_{2,k}(t)\right)dt,$
$A_{10k}(x) = -\frac{iw_k}{6}\int_0^x \left(-28w_k^6A_{9,k}''(t) + 56iw_k^5A_{8,k}'''(t) + 70w_k^4A_{7,k}^{(4)}(t) + q(t)A_{3,k}(t)\right)dt,$

And for integer $n \geq 11$, we get that $A_{nk}(x) = -\frac{iw_k}{8}\int_0^x \left(-28w_k^6A_{n-1,k}''(x) + 56iw_k^5A_{n-2,k}'''(x) + 70w_k^4A_{n-3,k}^{(4)}(x) - 56iw_k^3A_{n-4,k}^{(5)}(x) - 28w_k^2A_{n-5,k}^{(6)}(x) + 8iw_kA_{n-6,k}^{(7)}(x) + A_{n-7,k}^{(8)}(x) + q(x)A_{n-7,k}(x)\right)dt.$

So, we obtain

$A_{1sk} = A_{1k}(x) - \binom{S}{1} iw_k^7A_{0k}'(x),$
$A_{2sk} = A_{2k}(x) - \binom{S}{1} iw_k^7A_{1k}'(x) - \binom{S}{2} w_k^7A_{0k}''(x),$
$A_{3sk} = A_{3k}(x) - \binom{S}{1} iw_k^7A_{2k}'(x) - \binom{S}{2} w_k^6A_{1k}''(x) + \binom{S}{3} iw_k^5A_{0k}'''(x),$
$A_{4sk} = A_{4k}(x) - \binom{S}{1} iw_k^7A_{3k}'(x) - \binom{S}{2} w_k^6A_{2k}''(x) + \binom{S}{3} iw_k^5A_{1k}'''(x) + \binom{S}{4} w_k^4A_{0k}^{(4)}(x),$

$A_{5sk} = A_{5k}(x) - \binom{S}{1} iw_k^7A_{4k}'(x) - \binom{S}{2} w_k^6A_{3k}''(x) + \binom{S}{3} w_k^5A_{2k}'''(x) + \binom{S}{4} w_k^4A_{1k}^{(4)}(x) - \binom{S}{5} iw_k^3A_{0k}^{(5)}(x),$
$A_{6sk} = A_{6k}(x) - \binom{S}{1} iw_k^7A_{5k}'(x) - \binom{S}{2} w_k^6A_{4k}''(x) + \binom{S}{3} iw_k^5A_{3k}'''(x) + \binom{S}{4} w_k^4A_{2k}^{(4)}(x) - \binom{S}{5} iw_k^3A_{1k}^{(5)}(x) - \binom{S}{6} w_k^2A_{0k}^{(6)}(x),$
$A_{7sk} = A_{7k}(x) - \binom{S}{1} iw_k^7A_{6k}'(x) - \binom{S}{2} w_k^6A_{5k}''(x) + \binom{S}{3} iw_k^5A_{4k}'''(x) + \binom{S}{4} w_k^4A_{3k}^{(4)}(x) - \binom{S}{5} iw_k^3A_{2k}^{(5)}(x) - \binom{S}{6} w_k^2A_{1k}^{(6)}(x) + \binom{S}{7} iw_kA_{0k}^{(7)}(x),$
$A_{8sk} = A_{8k}(x) - \binom{S}{1} iw_k^7A_{7k}'(x) - \binom{S}{2} w_k^6A_{6k}''(x) + \binom{S}{3} iw_k^5A_{5k}'''(x) + \binom{S}{4} w_k^4A_{4k}^{(4)}(x) - \binom{S}{5} iw_k^3A_{3k}^{(5)}(x) - \binom{S}{6} w_k^2A_{1k}^{(6)}(x) + \binom{S}{7} iw_kA_{1k}^{(7)}(x) + \binom{S}{8} A_{0k}^{(8)}(x),$

and so on. For $n > 8$ we have

$A_{nsk} = A_{nk}(x) - \binom{S}{1} iw_k^7A_{n-1k}'(x) - \binom{S}{2} w_k^6A_{n-2,k}''(x) + \binom{S}{3} iw_k^5A_{n-3,k}'''(x) + \binom{S}{4} w_k^4A_{n-4,k}^{(4)}(x) -$

$\binom{S}{5} iw_k^3 A_{n-5,k}^{(5)}(x) - \binom{S}{6} w_k^2 A_{n-6,k}^{(6)}(x) + \binom{S}{7} iw_k A_{n-7,k}^{(7)}(x) + \binom{S}{8} A_{n-8,k}^{(8)}(x).$

But since $A_{0k}(x), A_{1k}(x), A_{2k}(x), A_{3k}(x), A_{4k}(x), A_{5k}(x)$ and $A_{6k}(x)$ are constants, their derivatives vanish. Then we can write:

$A_{1sk} = A_{1k}(x), A_{2sk} = A_{2k}(x), A_{3sk} = A_{3k}(x), A_{4sk} = A_{4k}(x), A_{5sk} = A_{5k}(x), A_{6sk} = A_{6k}(x),$

$A_{7sk} = A_{7k}(x), A_{8sk} = A_{8k}(x) - \binom{S}{1} iw_k^7 A'_{7k}(x),$

$A_{9sk} = A_{9k}(x) - \binom{S}{1} iw_k^7 A'_{8k}(x) - \binom{S}{2} w_k^6 A''_{7k}(x),$

$A_{10sk} = A_{10k}(x) - \binom{S}{1} iw_k^7 A'_{9k}(x) - \binom{S}{2} w_k^6 A''_{8k}(x) + \binom{S}{3} iw_k^5 A'''_{7k}(x),$

$A_{11sk} = A_{11k}(x) - \binom{S}{1} iw_k^7 A'_{10k}(x) - \binom{S}{2} w_k^6 A''_{9k}(x) + \binom{S}{3} iw_k^5 A'''_{8k}(x) + \binom{S}{4} w_k^4 A_{7k}^{(4)}(x),$

$A_{8sk} = A_{8k}(x) - \binom{S}{1} iw_k^7 A'_{7k}(x) - \binom{S}{2} w_k^6 A''_{6k}(x) + \binom{S}{3} iw_k^5 A'''_{5k}(x) + \binom{S}{4} w_k^4 A_{4k}^{(4)}(x) - \binom{S}{5} iw_k^3 A_{3k}^{(5)}(x) - \binom{S}{6} w_k^2 A_{1k}^{(6)}(x) + \binom{S}{7} iw_k A_{1k}^{(7)}(x) + \binom{S}{8} A_{0k}^{(8)}(x),$

$A_{12sk} = A_{12k}(x) - \binom{S}{1} iw_k^7 A'_{11k}(x) - \binom{S}{2} w_k^6 A''_{10k}(x) + \binom{S}{3} iw_k^5 A'''_{9k}(x) + \binom{S}{4} w_k^4 A_{8k}^{(4)}(x) - \binom{S}{5} iw_k^3 A_{7k}^{(5)}(x),$

$A_{13sk} = A_{13k}(x) - \binom{S}{1} iw_k^7 A'_{12k}(x) - \binom{S}{2} w_k^6 A''_{11k}(x) + \binom{S}{3} iw_k^5 A'''_{10k}(x) + \binom{S}{4} w_k^4 A_{9k}^{(4)}(x) - \binom{S}{5} iw_k^3 A_{8k}^{(5)}(x) - \binom{S}{6} w_k^2 A_{7k}^{(6)}(x),$

$A_{14sk} = A_{14k}(x) - \binom{S}{1} iw_k^7 A'_{13k}(x) - \binom{S}{2} w_k^6 A''_{12k}(x) + \binom{S}{3} iw_k^5 A'''_{11k}(x) + \binom{S}{4} w_k^4 A_{10k}^{(4)}(x) - \binom{S}{5} iw_k^3 A_{9k}^{(5)}(x) - \binom{S}{6} w_k^2 A_{8k}^{(6)}(x) + \binom{S}{7} iw_k A_{7k}^{(7)}(x),$

$A_{15sk} = A_{15k}(x) - \binom{S}{1} iw_k^7 A'_{14k}(x) - \binom{S}{2} w_k^6 A''_{13k}(x) + \binom{S}{3} iw_k^5 A'''_{12k}(x) + \binom{S}{4} w_k^4 A_{11k}^{(4)}(x) - \binom{S}{5} iw_k^3 A_{10k}^{(5)}(x) - \binom{S}{6} w_k^2 A_{9k}^{(6)}(x) + \binom{S}{7} iw_k A_{8k}^{(7)}(x) + \binom{S}{8} A_{7k}^{(8)}(x),$

$\vdots$

and for $n > 15$, we have

$A_{nsk} = A_{nk}(x) - \binom{S}{1} iw_k^7 A'_{n-1,k}(x) - \binom{S}{2} w_k^6 A''_{n-2,k}(x) + \binom{S}{3} iw_k^5 A'''_{n-3,k}(x) + \binom{S}{4} w_k^4 A_{n-4,k}^{(4)}(x) - \binom{S}{5} iw_k^3 A_{n-5,k}^{(5)}(x) - \binom{S}{6} w_k^2 A_{n-6,k}^{(6)}(x) + \binom{S}{7} iw_k A_{n-7,k}^{(7)}(x) + \binom{S}{8} A_{n-8,k}^{(8)}(x).$

Hence, we can write the eight linearly independent solution and their derivatives of the differential equation of the form

$$y_k^{(s)}(x,\lambda) = (i\lambda w_k)^s e^{i\lambda w_k x}\left[A_{0k}(x) + \frac{A_{1sk}}{\lambda} + \frac{A_{2sk}}{\lambda^2} + \frac{A_{3sk}}{\lambda^3} + \frac{A_{4sk}}{\lambda^4} + \frac{A_{5sk}}{\lambda^5} + \frac{A_{6sk}}{\lambda^6} + \cdots + \frac{A_{nsk}}{\lambda^n} + O\left(\frac{1}{\lambda^{n+1}}\right)\right].$$

3. Asymptotic behavior of eigenvalues

In this section, we try to find eigenvalue of the problem (1)-(2).

Theorem 2: Consider the boundary value problem (1)- (2), where $q(x)$ is smooth function, such that satisfies the conditions $q'(a) = 0$, $q'(0) = 0, \int_0^a q(x)dx = 0$, and $q(a) \neq 0$, then for $\lambda \in T_0$, where $T_0 = \left\{\lambda : \arg\lambda \in \left[0, \frac{\pi}{8}\right]\right\}$the asymptotic formulas for eigenvalues of the problem for sufficiently large $|m|$, has the following forms

$$\lambda_{0,m} = \left(\frac{2m\pi}{(-1-(\sqrt{2}+1)i)a} + \frac{i}{(-1-(\sqrt{2}+1)i)a} + O\left(\frac{1}{m^8}\right)\right)^8,$$

for $m = N, N+1, N+2, \ldots$ where N is a large integer.

Proof:

If the first eight terms in (3) are chosen so we obtain

$$y_k^{(s)}(x,\lambda) = (i\lambda w'_k)^s e^{i\lambda w_k x}\left[A_{0sk}(x) + \frac{A_{1sk}(x)}{\lambda} + \frac{A_{2sk}(x)}{\lambda^2} + \frac{A_{3sk}(x)}{\lambda^3} + \frac{A_{4sk}(x)}{\lambda^4} + \frac{A_{5sk}(x)}{\lambda^5} + \frac{A_{6sk}(x)}{\lambda^6} + \frac{A_{7sk}(x)}{\lambda^7} + O\left(\frac{1}{\lambda^8}\right)\right],$$

for, $s = 0,1,2,3,4,5,6,7, k = 0,1,2,3,4,5,6,7$. We have:

$A_{0sk} = A_{0k}(x),$

$A_{1sk} = A_{1k}(x), A_{2sk} = A_{2k}(x), A_{3sk} = A_{3k}(x), A_{4sk} = A_{4k}(x), A_{5sk} = A_{5k}(x), A_{6sk} = A_{6k}(x),$

$A_{7sk} = A_{7k}(x).$

And

$A_{0,k}(x) = 1, A_{1,k}(x) = 0, A_{2,k}(x) = 0, A_{3,k}(x) = 0, A_{4,k}(x) = 0, A_{5,k}(x) = 0, A_{6,k}(x) = 0,$

$A_{7,k}(x) = -\frac{iw_k}{8}\int_0^x q(t)A_{0,k}(t)dt.$

And to find the boundary conditions $U_j(y_k)$, for $k, j = 0,1,2,3,4,5$, we have

$U_0(y) = y(0) = 0, U_1(y) = y'(0) = 0, U_2(y) = y''(0) = 0, U_3(y) = y'''(0) = 0,$

$U_j(y) = \sum_{k=1}^{8} (iw_j\lambda)^{8-k} y^{(k-1)}(a,\lambda) = 0, j = 4,5,6,7$

where, $w_k = \sqrt[8]{1} = e^{\frac{2\pi k}{8}i} = e^{\frac{\pi k}{4}i}, k = 0,1,2,3,4,5,6,7.$

Now, $w_0 = 1, w_1 = \frac{1+i}{\sqrt{2}}, w_2 = i, w_3 = \frac{i-1}{\sqrt{2}}, w_4 = -1, w_5 = \frac{-1-i}{\sqrt{2}}, w_6 = -i,\ w_7 = \frac{1-i}{\sqrt{2}},$

and w'_j are the w_j which numbering so that satisfy the inequality:

$\mathrm{Re}(i\lambda w'_0) \leq \mathrm{Re}(i\lambda w'_1) \leq \mathrm{Re}(i\lambda w'_2) \leq \mathrm{Re}(i\lambda w'_3) \leq \mathrm{Re}(i\lambda w'_4) \leq \mathrm{Re}(i\lambda w'_5) \leq \mathrm{Re}(i\lambda w'_6) \leq \mathrm{Re}(i\lambda w'_7).$

We can easily find out the form of each boundary condition up to order eight

$U_0(y_k) = \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]$, ….(12)

$U_1(y_k) = i\lambda w_k' \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]$, … …(13)

$U_2(y_k) = -\lambda^2 w_k'^2 \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]$, … …(14)

$U_3(y_k) = -i\lambda^3 w_k'^3 \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]$, ….(15)

$U_j(y_k) = -i\lambda^7 e^{i\lambda w_k a} \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]\left[(w_k')^7 + w_j(w_k')^6 + (w_j)^2(w_k')^5 + (w_j)^3(w_k')^4 + (w_j)^4(w_k')^3 + (w_j)^5(w_k')^2 + (w_j)^6(w_k') + (w_j)^7\right]$, for $k = 0{:}7, j = 4,5,6,7$. …(16)

If $\lambda \in T_0$, then $w_0' = i, w_1' = \frac{1+i}{\sqrt{2}}, w_2' = \frac{i-1}{\sqrt{2}}, w_3' = 1, w_4' = -1, w_5' = \frac{1-i}{\sqrt{2}}, w_6' = \frac{-1-i}{\sqrt{2}}$ *and* $w_7' = -i.$

$U_0(y_k) = A$ *where* $A = \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]$,

$U_1(y_k) = i\lambda w_k' \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]$,

$U_1(y_0) = -\lambda A, U_1(y_1) = \frac{i-1}{\sqrt{2}}\lambda A, U_1(y_2) = \frac{-i-1}{\sqrt{2}}\lambda A, U_1(y_3) = i\lambda A, U_1(y_4) = -i\lambda A,$

$U_1(y_5) = \frac{i+1}{\sqrt{2}}\lambda A, U_1(y_6) = \frac{1-i}{\sqrt{2}}\lambda A, U_1(y_7) = \lambda A,$

$U_2(y_k) = -\lambda^2 w_k'^2 \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]$,

$U_2(y_0) = \lambda^2 A, U_2(y_1) = -\lambda^2\left(\frac{1+1}{\sqrt{2}}\right)^2 A, U_2(y_2) = -\lambda^2\left(\frac{i-1}{\sqrt{2}}\right)^2 A, U_2(y_3) = -\lambda^2 A,$

$U_2(y_4) = -\lambda^2 A, U_2(y_5) = -\lambda^2\left(\frac{1-i}{\sqrt{2}}\right)^2 A, U_2(y_6) = -\lambda^2\left(\frac{-1-i}{\sqrt{2}}\right)^2 A, U_2(y_7) = \lambda^2 A,$

$U_3(y_k) = -i\lambda^3 w_k'^3 \left[1 + O\left(\frac{1}{\lambda^8}\right)\right]$,

$U_3(y_0) = -\lambda^3 A, U_3(y_1) = -i\lambda^3\left(\frac{1+i}{\sqrt{2}}\right)^3 A, U_3(y_2) = -i\lambda^3\left(\frac{i-1}{\sqrt{2}}\right)^3 A, U_3(y_3) = -i\lambda^3 A,$

$U_3(y_4) = i\lambda^3 A, U_3(y_5) = -i\lambda^3\left(\frac{1-i}{\sqrt{2}}\right)^3 A, U_3(y_6) = -i\lambda^3\left(\frac{-1-i}{\sqrt{2}}\right)^3 A, U_3(y_7) = \lambda^3 A,$

$U_4(y_0) = 0, U_4(y_1) = 0, U_4(y_2) = 0, U_4(y_3) = 0, U_4(y_4) = -i\lambda^7 e^{i\omega_4' a}\left[-8 + O\left(\frac{1}{\lambda^8}\right)\right]$,

$U_4(y_5) = 0, U_4(y_6) = 0, U_4(y_7) = 0, U_5(y_0) = 0, U_5(y_1) = 0, U_5(y_2) = 0, U_5(y_3) = 0,$

$U_5(y_4) = 0,$

$U_5(y_5) = -i\lambda^7 e^{i\omega_5' a}\left[\frac{\sqrt{2}}{16}(64 + 64i) + O\left(\frac{1}{\lambda^8}\right)\right], U_5(y_6) = 0, U_5(y_7) = 0,$

$U_6(y_0) = 0, U_6(y_1) = 0, U_6(y_2) = 0, U_6(y_3) = 0, U_6(y_4) = 0, U_6(y_5) = 0,$

$U_6(y_6) = -i\lambda^7 e^{i\omega_6' a}\left[-\sqrt{2}(4 - 4i) + O\left(\frac{1}{\lambda^8}\right)\right], U_6(y_7) = 0,$

$U_7(y_0) = 0, U_7(y_1) = 0, U_7(y_2) = 0, U_7(y_3) = 0, U_7(y_4) = 0, U_7(y_5) = 0,$

$U_7(y_6) = 0\,, U_7(y_7) = -i\lambda^7 e^{i\omega_7' a}\left[8i + O\left(\frac{1}{\lambda^8}\right)\right].$

We want to find $\Delta(\lambda)$ in T_0.

$$\Delta(\lambda) = \begin{vmatrix} U_0(y_0) & U_0(y_1) & U_0(y_2) & U_0(y_3) & U_0(y_4) & U_0(y_5) & U_0(y_6) & U_0(y_7) \\ U_1(y_0) & U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) & U_1(y_5) & U_1(y_6) & U_1(y_7) \\ U_2(y_0) & U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) & U_2(y_5) & U_2(y_6) & U_2(y_7) \\ U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) & U_3(y_5) & U_3(y_6) & U_3(y_7) \\ U_4(y_0) & U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) & U_4(y_5) & U_4(y_6) & U_4(y_7) \\ U_5(y_0) & U_5(y_1) & U_5(y_2) & U_5(y_3) & U_5(y_4) & U_5(y_5) & U_5(y_6) & U_5(y_7) \\ U_6(y_0) & U_6(y_1) & U_6(y_2) & U_6(y_3) & U_6(y_4) & U_6(y_5) & U_6(y_6) & U_6(y_7) \\ U_7(y_0) & U_7(y_1) & U_7(y_2) & U_7(y_3) & U_7(y_4) & U_7(y_5) & U_7(y_6) & U_7(y_7) \end{vmatrix}$$

Clearly we known from [13], the eigenvalues of the given problem are the zeros of $\Delta(\lambda)$. If $\Delta(\lambda) = 0$ for sufficiently large$|\lambda|$, then

$e^{i\lambda(\omega_4' + \omega_5' + \omega_6' + \omega_7')a} - 1 = -1 + O\left(\frac{1}{\lambda^8}\right)$, …..(17)

or

$e^{i\lambda(-1-(\sqrt{2}+1)i)a} - 1 = -1 + O\left(\frac{1}{\lambda^8}\right)$. … ….(18)

Then according to [2] by using Rouche's theorem we can solve it and we get:

$i\lambda\left(-1 - (\sqrt{2} + 1)i\right)a = 2m\pi i - 1 + O\left(\frac{1}{m^8}\right)$, … …(19)

$\lambda = \frac{2m\pi}{(-1-(\sqrt{2}+1)i)a} + \frac{i}{(-1-(\sqrt{2}+1)i)a} + O\left(\frac{1}{m^8}\right)$, for $m = N, N + 1, N + 2, \dots$

where N is a large integer.

And we know that the eigenvalues of the problem are $\lambda_{0,m} = \lambda^8$,

that is

$$\lambda_{0,m} = \left(\frac{2m\pi}{(-1-(\sqrt{2}+1)i)a} + \frac{i}{(-1-(\sqrt{2}+1)i)a} + O\left(\frac{1}{m^8}\right)\right)^8,$$

for $m = N, N + 1, N + 2, \dots$

where N is a large integer.

4. Asymptotic formula for the eigenfunctions

In this section, we try to find Eigenfunctions of the boundary value problem in the sector $T_0 = \left\{\lambda{:} \arg\lambda \in \left[0, \frac{\pi}{8}\right]\right\}$.

Theorem 3: Asymptotic formulas of the eigenfunction of the boundary value problem (1) - (2) corresponding to $\lambda_{0,m}$ has the following forms:

$y_{0,m}(x,\lambda) = 2e^{\frac{\sqrt{2}}{2}\lambda x(-1-i)} + 2e^{\frac{\sqrt{2}}{2}\lambda x(-1+i)} + 2ie^{-\lambda x} + 2ie^{i\lambda x} - 2\sqrt{2}ie^{\lambda xi} - 2\sqrt{2}e^{\frac{\sqrt{2}}{2}\lambda x(-1-i)} + 0\left(\frac{1}{\lambda^7}\right)$,

$\lambda \in T_0$, for $m = N, N + 1, N + 2, \dots$, *and* N is a large integer.

Proof: If the first eight terms in (3) are chosen then:

$y_k^{(s)}(x,\lambda) = (i\lambda w_k')^s e^{i\lambda w_k x}\left[A_{0sk}(x) + \frac{A_{1sk}(x)}{\lambda} + \frac{A_{2sk}(x)}{\lambda^2} + \frac{A_{3sk}(x)}{\lambda^3} + \frac{A_{4sk}(x)}{\lambda^4} + \frac{A_{5sk}(x)}{\lambda^5} + \frac{A_{6sk}(x)}{\lambda^6} + O\left(\frac{1}{\lambda^7}\right)\right]$,
for, $s = 0,1,2,3,4,5,6,7, k = 0,1,2,3,4,5,6,7$. We have:
$A_{1sk} = 1, A_{1sk} = 0, A_{2sk} = 0, A_{3sk} = 0, A_{4sk} = 0, A_{5sk} = 0, A_{6sk} = 0.$
And to finding the boundary conditions $U_j(y_k)$ for $k = 0,1,2,3,4,5,6,7 j = 1,2,3,4,5,6,7$ up to order $O\left(\frac{1}{\lambda^7}\right)$ and If $\lambda \in T_0$, then
$w_0' = i, w_1' = \frac{1+i}{\sqrt{2}}, w_2' = \frac{i-1}{\sqrt{2}}, w_3' = 1,$
$w_4' = -1, w_5' = \frac{1-i}{\sqrt{2}}, w_6' = \frac{-1-i}{\sqrt{2}}$ *and* $w_7' = -i.$
$U_1(y_k) = i\lambda w_k'\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_1(y_0) = -\lambda\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_1(y_1) = \frac{i-1}{\sqrt{2}}\lambda\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_1(y_2) = \frac{-i-1}{\sqrt{2}}\lambda\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_1(y_3) = i\lambda\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_1(y_4) = -i\lambda\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_1(y_5) = \frac{i+1}{\sqrt{2}}\lambda\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_1(y_6) = \frac{1-i}{\sqrt{2}}\lambda\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_1(y_7) = \lambda\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_2(y_k) = -\lambda^2 w_k'^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_2(y_0) = \lambda^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_2(y_1) = -\lambda^2\left(\frac{1+1}{\sqrt{2}}\right)^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_2(y_2) = -\lambda^2\left(\frac{i-1}{\sqrt{2}}\right)^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right]$
$U_2(y_3) = -\lambda^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_2(y_4) = -\lambda^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_2(y_5) = -\lambda^2\left(\frac{1-i}{\sqrt{2}}\right)^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_2(y_6) = -\lambda^2\left(\frac{-1-i}{\sqrt{2}}\right)^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_2(y_7) = \lambda^2\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_3(y_k) = -i\lambda^3 w_k'^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_3(y_0) = -\lambda^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_3(y_1) = -i\lambda^3\left(\frac{1+i}{\sqrt{2}}\right)^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_3(y_2) = -i\lambda^3\left(\frac{i-1}{\sqrt{2}}\right)^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_3(y_3) = -i\lambda^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_3(y_4) = i\lambda^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_3(y_5) = -i\lambda^3\left(\frac{1-i}{\sqrt{2}}\right)^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_3(y_6) = -i\lambda^3\left(\frac{-1-i}{\sqrt{2}}\right)^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], U_3(y_7) = \lambda^3\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_4(y_0) = 0, U_4(y_1) = 0, U_4(y_2) = 0, U_4(y_3) = 0, U_4(y_4) = -i\lambda^7 e^{i\omega_4' a}\left[-8 + O\left(\frac{1}{\lambda^7}\right)\right],$
$U_4(y_5) = 0, U_4(y_6) = 0, U_4(y_7) = 0,$
$U_5(y_0) = 0, U_5(y_1) = 0, U_5(y_2) = 0, U_5(y_3) = 0, U_5(y_4) = 0,$
$U_5(y_5) = -i\lambda^7 e^{i\omega_5' a}\left[\frac{\sqrt{2}}{16}(64 + 64i) + O\left(\frac{1}{\lambda^7}\right)\right], U_5(y_6) = 0, U_5(y_7) = 0,$
$U_6(y_0) = 0, U_6(y_1) = 0, U_6(y_2) = 0, U_6(y_3) = 0, U_6(y_4) = 0, U_6(y_5) = 0,$
$U_6(y_6) = -i\lambda^7 e^{i\omega_6' a}\left[-\sqrt{2}(4 - 4i) + O\left(\frac{1}{\lambda^7}\right)\right], U_6(y_7) = 0$
$U_7(y_0) = 0, U_7(y_1) = 0, U_7(y_2) = 0, U_7(y_3) = 0, U_7(y_4) = 0, U_7(y_5) = 0,$
$U_7(y_6) = 0, U_7(y_7) = -i\lambda^7 e^{i\omega_7' a}\left[8i + O\left(\frac{1}{\lambda^7}\right)\right],$
$y_0(x,\lambda) = e^{-\lambda x}\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], y_1(x,\lambda) = e^{i\lambda\left(\frac{1+i}{\sqrt{2}}\right)x}\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], y_2(x,\lambda) = e^{i\lambda\left(\frac{i-1}{\sqrt{2}}\right)x}\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$y_3(x,\lambda) = e^{i\lambda x}\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], y_4(x,\lambda) = e^{-i\lambda x}\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], y_5(x,\lambda) = e^{i\lambda\left(\frac{1-i}{\sqrt{2}}\right)x}\left[1 + O\left(\frac{1}{\lambda^7}\right)\right],$
$y_6(x,\lambda) = e^{i\lambda\left(\frac{-1-i}{\sqrt{2}}\right)x}\left[1 + O\left(\frac{1}{\lambda^7}\right)\right], y_7(x,\lambda) = e^{\lambda x}\left[1 + O\left(\frac{1}{\lambda^7}\right)\right].$
As we see in [14], we can write the eigenfunctions as follows
$y_{0,m}(x,\lambda) = \frac{-1}{64\lambda^{34} i^3\left(\frac{\sqrt{2}}{16}(64+64i)\right)(4\sqrt{2}-4\sqrt{2}i)} e^{(i-1)\lambda a} \cdot e^{i\lambda\left(\frac{i-1}{\sqrt{2}}\right)a} \cdot e^{i\lambda\left(\frac{i+1}{\sqrt{2}}\right)a}$

$$\cdot \begin{vmatrix} y_0(x,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & y_4(x,\lambda) & y_5(x,\lambda) & y_6(x,\lambda) & y_7(x,\lambda) \\ U_1(y_0) & U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) & U_1(y_5) & U_1(y_6) & U_1(y_7) \\ U_2(y_0) & U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) & U_2(y_5) & U_2(y_6) & U_2(y_7) \\ U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) & U_3(y_5) & U_3(y_6) & U_3(y_7) \\ U_4(y_0) & U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) & U_4(y_5) & U_4(y_6) & U_4(y_7) \\ U_5(y_0) & U_5(y_1) & U_5(y_2) & U_5(y_3) & U_5(y_4) & U_5(y_5) & U_5(y_6) & U_5(y_7) \\ U_6(y_0) & U_6(y_1) & U_6(y_2) & U_6(y_3) & U_6(y_4) & U_6(y_5) & U_6(y_6) & U_6(y_7) \\ U_7(y_0) & U_7(y_1) & U_7(y_2) & U_7(y_3) & U_7(y_4) & U_7(y_5) & U_7(y_6) & U_7(y_7) \end{vmatrix} \ldots\ldots (20)$$

$$= \begin{vmatrix} e^{-\lambda x} & e^{i\lambda\left(\frac{1+i}{\sqrt{2}}\right)x} & e^{i\lambda\left(\frac{-1+i}{\sqrt{2}}\right)x} & e^{i\lambda x} & e^{-i\lambda x} & e^{i\lambda\left(\frac{1-i}{\sqrt{2}}\right)x} & e^{i\lambda\left(\frac{-1+i}{\sqrt{2}}\right)x} & e^{\lambda x} \\ -1 & \mathrm{i}\left(\frac{1+i}{\sqrt{2}}\right) & i\left(\frac{-1+i}{\sqrt{2}}\right) & \mathrm{i} & -\mathrm{i} & i\left(\frac{1-i}{\sqrt{2}}\right) & \mathrm{i}\left(\frac{-1-i}{\sqrt{2}}\right) & 1 \\ 1 & -\left(\frac{1+i}{\sqrt{2}}\right)^2 & -\left(\frac{-1+i}{\sqrt{2}}\right)^2 & -1 & -1 & -\left(\frac{1-i}{\sqrt{2}}\right)^2 & -\left(\frac{-1-i}{\sqrt{2}}\right)^2 & 1 \\ -1 & -i\left(\frac{1+i}{\sqrt{2}}\right)^3 & -i\left(\frac{-1+i}{\sqrt{2}}\right)^3 & -\mathrm{i} & \mathrm{i} & -i\left(\frac{1-i}{\sqrt{2}}\right)^3 & -i\left(\frac{-1-i}{\sqrt{2}}\right)^3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} + O\left(\frac{1}{\lambda^7}\right). \dots\dots\dots (21)$$

Thus, we obtain

$$y_{0,m}(x,\lambda) = 2e^{\frac{\sqrt{2}}{2}\lambda x(-1-i)} + 2e^{\frac{\sqrt{2}}{2}\lambda x(-1+i)} + 2ie^{-\lambda x} + 2ie^{i\lambda x} - 2\sqrt{2}ie^{\lambda x i} - 2\sqrt{2}e^{\frac{\sqrt{2}}{2}\lambda x(-1-i)} + 0\left(\frac{1}{\lambda^7}\right). \dots\dots (22)$$

5. Conclusion

In our research, we obtained asymptotic expressions for the eight-order differential equations. Also, we demonstrated new asymptotic formulas for the eigenvalues and eigenfunctions to the boundary value problem (1) and (2).

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TJPS

إستنتاج القيم الذاتية والدوال الذاتية بشكل مقارب لمسائل القيمة الحدودية من الرتبة الثامنة

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الملخص

في البحث المقدم، ندرس مسألة القيمة الذاتية الناتجة عن المعادلات التفاضلية من الرتبة الثامنة بشروط حدودية مناسبة ، والتي تحتوي على معلمة طيفية. تم حساب تعبيرات مقاربة دقيقة جديدة للحلول الثمانية المستقلة خطيًا. بعد ذلك ، تم الحصول على صيغ مقاربة جديدة للقيم الذاتية والدوال الذاتية لمسألة القيمة الحدودية