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Alternating Direction Implicit Method for Solving Heat Diffusion Problems

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ABSTRACT

The objective of this paper to find the solution of fractional partial differential equations using numerical methods to solve heat diffusion problems and this method is the implicit alternating direction method ADI with discussion of its algorithm and by this method it is possible to solve heat diffusion problems in a shorter time as well as obtain more accurate results compared to other numerical methods.

Introduction

ADI numerical method is one of many other numerical methods that applying to solve many kinds of problems like boundary value problems, elasticity problems, fluid mechanics,…, this method is on of recently methods because it is depend on two other methods (Implicit method and Explicit method), so this method is so far to reach the solutions and it is also more accuracy than other numerical methods. .Boling , Xueke and Fenghui (2011) are used numerical solutions to solve fractional partial differential equations [1], Horak (2005), used Parallel numerical solution of 2-D heat Equations [2], In [2012], De-Turck D.,"Solving the heat equation [3].Jacob N. (2015) used theta method comparative study of 2D asymmetric Diffusion problem with convection on the wall [4]. Ashaju ,Abimbola, and Samson Bright (2015) are using Alternating-Direction Implicit Finite-Difference method to calculate the heat transfer in a metal bar [5]. Z.F. Tian and T.B.GE (2007) are solving two-dimensional unsteady convection-diffusion problems by ADI numerical method, [6]. In (2014) Aderito, Arauj and Cidalia are introduced a alternating direction implicit method for a second-order, [7]. Also M Dehghan In (2002), he introduced a new ADI technique for two-dimensional parabolic equation with an integral condition, [8].

1. Theoretical part

1.1. ADI Method: The work of this method is summarized by integrating the explicit ,implicit method and alternating methods in its work the alternating direction implicit (ADI) method is an efficient method for solving differential equations by numerical solution and it was proposed by the worlds of Peaceman and Rachford in the united states in 1955 with an easy concept where the solution is done in two stages towards the x axis in the second stages, the points are found in the direction of the y axis, as the same time, by applying them to one and two-dimensional heat equation.

1.2. ADI Algorithm: The general form of ADI method is:

$$u_{i,j+1}^{r+1} - u_{i,j}^{r} = \beta\left[\left(g_0 u_{i+1,j}^{r} + g_1 u_{i,j}^{r} + g_2 u_{i-1,j}^{r} + \cdots + g_{i+1} u_{0,j}^{r}\right) + \left(g_0 u_{i,j+1}^{r+1} + g_1 u_{i,j}^{r+1} + g_2 u_{i,j-1}^{r+1} + \cdots + g_{j+1} u_{i,0}^{r+1}\right)\right]$$

Now, we introduce the summery of basic idea for the ADI numerical method in four steps:

Step 1: Applying Implicit Method in X-axes direction and Explicit Method in Y-axes direction.

Step 2: Solving The specific equation (heat, elasticity, flow, ….) depending on time.

Step 3: now, Applying Implicit Method in Y-axes direction and Explicit Method in X-axes direction.

Step 4: Applying Thomas's Algorithm to solve the system of equations.

2. Practical Part

In this section , we apply the ADI numerical to solve one problem for diffusion heat equation and we explain how the temperature degrees are distribute on the plate in every time period by some shape, as bellow:

If we have a rectangular metal plate which is 3 inch wide and 9 inch high are applicable to the x, y axes, at the point of origin according to the data shown in below:

let m=0 , i=1,2,3 , j=0 to find m=1

$\beta = 0.1\,, g_0 = 1, g_1 = -1.5, g_2 = 0.375, g_3 = 0.125, g_4 = 0.09375$

$T_{i,j+1} = T_{i,j} + \beta \sum_{k=0}^{i+1} g_k T_{i+k-1,j}$

$T_{1,1} = T_{1,0} + \beta[g_0 T_{2,0} + g_1 T_{1,0} + g_2 T_{0,0}]$

$T_{1,1} = 0 + 0.1[0 + 0 + 0.375(10)]$

$T_{1,1} = 0 + 0.1(3.75) = 0.375$

$T_{2,1} = T_{2,0} + \beta[g_0 T_{3,0} + g_1 T_{2,0} + g_2 T_{1,0} + g_3 T_{0,0}]$

$T_{2,1} = 0 + 0.1[0 + 0 + 0 + 0.125(10)]$

$T_{2,1} = 0 + 0.1(1.25) = 0.125$

$T_{3,1} = T_{3,0} + \beta[g_0 T_{4,0} + g_1 T_{3,0} + g_2 T_{2,0} + g_3 T_{1,0} + g_4 T_{0,0}]$

$T_{3,1} = 0 + 0.1[5 + 0 + 0 + 0 + 0.09375(10)]$

$T_{3,1} = 0.5 + 0.1(0.9375) = 0.59375$

Left Boundary 10° C				Right Boundary 5° C
	0.375	0.125	0.59375	

0°C

Let j=1 , i=1,2,3 to find m=2

$T_{1,2} = T_{1,1} + \beta[g_0 T_{2,1} + g_1 T_{1,1} + g_2 T_{0,1}]$

$T_{1,2} = 0.375 + 0.1[0.125 + (-1.5)0.375 + 0.375(10)]$

$T_{1,2} = 0.375 + 0.3875 - 0.05625 = 0.70375$

$T_{2,2} = T_{2,1} + \beta[g_0 T_{3,1} + g_1 T_{2,1} + g_2 T_{1,1} + g_3 T_{0,1}]$

$T_{2,2} = 0.125 + 0.1[0.59375 + (-1.5)0.125 + 0.375(0.375) + 0.125(10)]$

$T_{2,2} = 0.125 + 0.059375 - 0.01875 + 0.0140625 + 0.125 = 0.304687$

$T_{3,2} = T_{3,1} + \beta[g_0 T_{4,1} + g_1 T_{3,1} + g_2 T_{2,1} + g_3 T_{1,1} + g_4 T_{0,1}]$

$T_{3,2} = 0.59375 + 0.09375 + 0.5 + 0.0015625 + 0.007421879 - 0.05625 = 1.140234379$

Left Boundary 10° C				Right Boundary 5° C
	0.70375	0.304688	1.1402344	
	0.375	0.125	0.59375	

0° C

Let j=2 ,i=1,2,3 to find m=3

$T_{1,3} = T_{1,2} + \beta[g_0 T_{2,2} + g_1 T_{1,2} + g_2 T_{0,2}]$

$T_{1,3} = 0.70375 + 0.1[0.3047 + (-1.5)0.70375 + 0.375(10)]$

$T_{1,3} = 0.70375 + 0.375 + 0.03047 - 0.10605 = 1.00317$

$T_{2,3} = T_{2,2} + \beta[g_0 T_{3,2} + g_1 T_{2,2} + g_2 T_{1,2} + g_3 T_{0,2}]$

$T_{2,3} = 0.3047 + 0.1[1.1402 + (-1.5)0.3047 + 0.375(0.70375) + 0.125(10)]$

$T_{2,3} = 0.3047 + 0.125 + 0.111402 - 0.045705 + 0.026391 = 0.75930325$

$T_{3,3} = T_{3,2} + \beta[g_0 T_{4,2} + g_1 T_{3,2} + g_2 T_{2,2} + g_3 T_{1,2} + g_4 T_{0,2}]$

$T_{3,3} = 1.1402 + 0.1[5 + (-1.5)1.1402 + 0.375(0.3047) + 0.125(0.70375) + 0.09375(10)]$

$T_{3,3} = 1.1402 + 0.5 - 0.17103 + 0.00879688 + 0.01143 + 0.09375 = 1.58601688$

Left Boundary 10° C				Right Boundary 5° C
	1.00317	0.524525	1.586017	
	0.70375	0.304688	1.1402344	
	0.375	0.125	0.59375	

0° C

By Implicit Method

If m=0 to find m=1

$T_{i,j+1} = T_{i,j} + \beta \sum_{k=0}^{i+1} g_k T_{i+k-1,j+1}$

$T_{1,1} = T_{1,0} + \beta[g_0 T_{2,1} + g_1 T_{1,1} + g_2 T_{0,1}]$

$T_{1,1} = 0 + 0.1[T_{2,1} + (-1.5)T_{1,1} + 0.375(10)]$

$T_{1,1} = 0.1T_{2,1} - 0.15T_{1,1} + 0.375$

$(1.15)T_{1,1} = 0.375 + (0.1)T_{2,1} \quad \cdots (1)$

We have $T_{1,1} = \frac{0.375+0.1T_{2,1}}{1.15}$

$T_{2,1} = T_{2,0} + \beta[g_0 T_{3,1} + g_1 T_{2,1} + g_2 T_{1,1} + g_3 T_{0,1}]$

$T_{2,1} = 0 + 0.1[T_{3,1} + (-1.5)T_{2,1} + 0.375T_{1,1} + 0.125(10)]$

$T_{2,1} = 0.1T_{3,1} - 0.15T_{2,1} + 0.0375T_{1,1} + 0.125$

$(1.15)T_{2,1} = 0.125 + 0.0375T_{1,1} + 0.1T_{3,1} \quad \cdots (2)$

$T_{3,1} = T_{3,0} + \beta[g_0 T_{4,1} + g_1 T_{3,1} + g_2 T_{2,1} + g_3 T_{1,1} + g_4 T_{0,1}]$

$T_{3,1} = 0 + 0.1[5 + (-1.5)T_{3,1} + 0.375T_{2,1} + 0.125T_{1,1} + 0.09375$

$T_{3,1} = 0.5 - 0.15T_{3,1} + 0.0375T_{2,1} + 0.0125T_{1,1} + 0.09375$

And by compensation, we get

$(1.15)T_{3,1} = 0.59375 + 0.0125\left(\frac{0.375+0.1T_{2,1}}{1.15}\right) + 0.0375T_{2,1}$

$(1.3225)T_{3,1} = 1.15(0.59375) + 0.0125(0.375) + 0.00125T_{2,1} + 0.043125T_{2,1}$

$(1.3225)T_{3,1} = 0.6828125 + 0.0046875 + 0.00125T_{2,1} + 0.043125T_{2,1}$

$(1.3225)T_{3,1} = 0.6875 + 0.044375T_{2,1}$

We have

$T_{3,1} = \frac{0.6875+0.044375T_{2,1}}{1.3225}$

And by compensation, we get

$(1.15)T_{2,1} = 0.125 + 0.0375\left(\frac{0.375+0.1T_{2,1}}{1.15}\right) + 0.1\left(\frac{0.6875+0.0044375T_{2,1}}{1.3225}\right)$

$(1.15)T_{2,1} = 0.125 + 0.0375(0.3261) + 0.0375(0.086957)T_{2,1} + 0.05198 + 0.003858696T_{2,1}$

$(1.15)T_{2,1} = 0.125 + 0.01222875 + 0.003260888T_{2,1} + 0.05198 + 0.003858696T_{2,1}$

$(1.15)T_{2,1} = 0.18920875 + 0.007119584T_{2,1}$

$T_{2,1} = \frac{0.18920875}{1.14288042} = 0.165554284$

And compensation$T_{2,1}$ in $T_{1,1} \mathit{and}\ T_{3,1}$ *we get*

$T_{3,1} = 0.573990342\,, T_{1,1} = 0.340482981$

If m=1 to find m=2

$T_{1,2} = T_{1,1} + \beta[g_0 T_{2,2} + g_1 T_{1,2} + g_2 T_{0,2}]$

$T_{1,2} = 0.340482981 + 0.1[T_{2,2} + (-1.5)T_{1,2} + 0.375(10)]$

$T_{1,2} = 0.340482981 + 0.1T_{2,2} - 0.15T_{1,2} + 0.375$

$(1.15)T_{1,2} = 0.715482981 + (0.1)T_{2,2} \quad \cdots (1)$

We have $T_{1,2} = \frac{0.715482981+0.1T_{2,2}}{1.15}$

$T_{2,2} = T_{2,1} + \beta[g_0 T_{3,2} + g_1 T_{2,2} + g_2 T_{1,2} + g_3 T_{0,2}]$

$T_{2,2} = 0.165554284 + 0.1[T_{3,2} + (-1.5)T_{2,2} + 0.375T_{1,2} + 0.125(10)]$

$T_{2,2} = 0.165554284 + 0.1T_{3,2} - 0.15T_{2,2} + 0.0375T_{1,2} + 0.125$

$(1.15)T_{2,2} = 0.290554284 + 0.0375T_{1,2} + 0.1T_{3,2} \quad \cdots (2)$

$T_{3,2} = T_{3,1} + \beta[g_0 T_{4,2} + g_1 T_{3,2} + g_2 T_{2,2} + g_3 T_{1,2} + g_4 T_{0,2}]$

$T_{3,2} = 0.573990342 + 0.1[5 + (-1.5)T_{3,2} + 0.375T_{2,2} + 0.125T_{1,2} + 0.09375$

$T_{3,2} = 0.5 - 0.15T_{3,2} + 0.0375T_{2,2} + 0.0125T_{1,2} + 0.09375$

$(1.15)T_{3,2} = 1.167740342 + 0.0375T_{2,2} + 0.0125T_{1,2} \quad \cdots (3)$

And by compensation, we get

$(1.15)T_{3,2} = 1.167740342 + 0.0125\left(\frac{0.715482981+0.1T_{2,2}}{1.15}\right) + 0.0375T_{2,2}$

$(1.3225)T_{3,2} = 1.15(1.167740342) + 0.0125(0.715482981) + 0.00125T_{2,2} + 0.043125T_{2,2}$

$(1.3225)T_{3,2} = 1.342901393 + 0.008943537263 + 0.00125T_{2,2} + 0.043125T_{2,2}$

$(1.3225)T_{3,2} = 1.35184493 + 0.044375T_{2,2}$

We have

$T_{3,2} = \frac{1.35184493+0.044375T_{2,2}}{1.3225}$

And by compensation, we get

$(1.15)T_{2,2} = 0.290554284 + 0.0375\left(\frac{0.715482981+0.1T_{2,2}}{1.15}\right) + 0.1\left(\frac{1.35184493+0.0044375T_{2,2}}{1.3225}\right)$

$(1.15)T_{2,2} = 0.290554284 + 0.0375(0.622159113) + 0.0375(0.086956521)T_{2,2} + 0.05198 + 0.0003355387524T_{2,2}$

$(1.15)T_{2,2} = 0.290554284 + 0.023330966 + 0.0032608695657_{2,2} + 1.102218898 + 0.0003355387524T_{2,2}$

$(1.15)T_{2,2} = 1.1416104148 + 0.0003596408317T_{2,2}$

$T_{2,2} = \frac{1.1416104148}{1.149640359} = 0.993015254$

And compensation $T_{2,2}$ in $T_{1,2}$ *and* $T_{3,2}$ *we get*

$T_{1,2} = 0.708508266$, $T_{3,2} = 1.055508493$

If m=2 to find m=3

$T_{1,3} = T_{1,2} + \beta[g_0 T_{2,3} + g_1 T_{1,3} + g_2 T_{0,3}]$

$T_{1,3} = 0.708508266 + 0.1[T_{2,3} + (-1.5)T_{1,3} + 0.375(10)]$

$T_{1,3} = 0.708508266 + 0.1T_{2,3} - 0.15T_{1,3} + 0.375$

$(1.15)T_{1,3} = 1.083508266 + (0.1)T_{2,3} \quad \cdots (1)$

We have $T_{1,3} = \frac{1.083508266+0.1T_{2,3}}{1.15}$

$T_{2,3} = T_{2,2} + \beta[g_0 T_{3,3} + g_1 T_{2,3} + g_2 T_{1,3} + g_3 T_{0,3}]$

$T_{2,3} = 0.993015254 + 0.1[T_{3,3} + (-1.5)T_{2,3} + 0.375T_{1,3} + 0.125(10)]$

$T_{2,3} = 0.993015254 + 0.1T_{3,3} - 0.15T_{2,3} + 0.0375T_{1,3} + 0.125$

$(1.15)T_{2,3} = 1.118015254 + 0.0375T_{1,3} + 0.1T_{3,3} \quad \cdots (2)$

$T_{3,3} = T_{3,2} + \beta[g_0 T_{4,3} + g_1 T_{3,3} + g_2 T_{2,3} + g_3 T_{1,3} + g_4 T_{0,3}]$

$T_{3,3} = 1.055508493 + 0.1[5 + (-1.5)T_{3,3} + 0.375T_{2,3} + 0.125T_{1,3} + 0.09375$

$T_{3,3} = 1.055508493 + 0.5 - 0.15T_{3,3} + 0.0375T_{2,3} + 0.0125T_{1,3} + 0.09375$

$(1.15)T_{3,3} = 1.649258493 + 0.0375T_{2,3} + 0.0125T_{1,3} \quad \cdots (3)$

And by compensation, we get

$(1.15)T_{3,3} = 1.649258493 + 0.0125\left(\frac{1.083508266+0.1T_{2,3}}{1.15}\right) + 0.0375T_{2,3}$

$(1.3225)T_{3,3} = 1.15(1.167740342) + 0.0125(1.083508266) + 0.00125T_{2,3} + 0.043125T_{2,3}$

$(1.3225)T_{3,3} = 1.342901393 + 0.013543853 + 0.00125T_{2,3} + 0.043125T_{2,3}$

$(1.3225)T_{3,3} = 1.356445246 + 0.044375T_{2,3}$

We have

$T_{3,3} = \frac{1.356445246+0.044375T_{2,3}}{1.3225}$

And by compensation, we get

$(1.15)T_{2,3} = 1.118015254 + 0.0375\left(\frac{1.083508266+0.1T_{2,3}}{1.15}\right) + 0.1\left(\frac{1.356445246+0.0044375T_{2,3}}{1.3225}\right)$

$(1.15)T_{2,3} = 1.118015254 + 0.0375(0.9421811) + 0.0375(0.086956521)T_{2,3} + 0.1025667483 + 0.0003355387524T_{2,3}$

$(1.15)T_{2,3} = 1.118015254 + 0.035331791 + 0.003260869565T_{2,3} + 0.1025667483 + 0.0003355387524T_{2,3}$

$(1.15)T_{2,3} = 1.255913793 + 0.0003596408317T_{2,3}$

$T_{2,3} = \frac{1.255913793}{1.149640359} = 1.092440591$

And compensation $T_{2,3}$ in $T_{1,3}$ *and* $T_{3,3}$ *we get*

$T_{1,3} = 1.037175935$,

$T_{3,3} = 1.481034534$

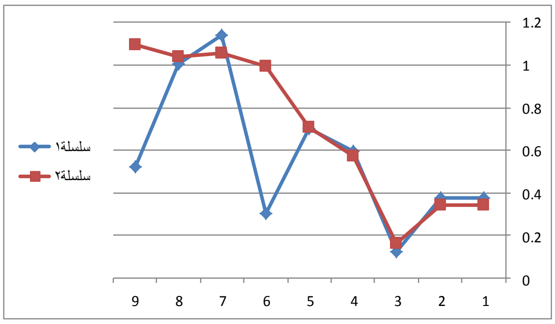

Fig. 1.1: comparison of drawing between the explicit method and the implicit method

The ADI method is absolutely depend on the two methods (as in the above shape) so this method is more accuracy than other methods.

3. Conclusion

In this paper, The Alternating Direction Implicit (ADI) method has been constructed for the purpose of application to some selected problems, and with a view to discussing the accuracy of this technique. Then, many problems have been solved, it is possible to solve these heat diffusion problems in a shorter time and from the figure(1.1), we can see the numerical results of the new method are more efficient and desirable for use in the solution of the heat diffusion problems from than other methods.

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طريقة الاتجاه المتناوب الضمنية في حل مسائل انتشار الحرارة

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الملخص

الهدف من هذا البحث هو لإيجاد حل للمعادلات التفاضلية الجزئية الكسورية باستخدام الطرق العددية في حل مسائل انتشار الحرارة وهذه الطريقة هي طريقة الاتجاه المتناوب الضمنية (ADI) مع مناقشة خوارزميتها. وبواسطة هذه الطريقة يمكن حل مسائل انتشار الحرارة بوقت اقصر وكذلك الحصول على نتائج ادق مقارنة بالطرق العددية الأخرى.