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Fekete -Szego Problem for Certain Subclass of q-Difference Operator Using Quasi-Subordination

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ABSTRACT

In this paper , a new certain class of q-Difference Operator with quasi-subordination is defined and Fekete-Szego problems for functions belongs to the class are derived . the results are presented here equip extensions of these given in some previous works

1. Introduction

Let $\mho(p)$ denote the class of functions in open unit disk

$\mathcal{D} = \{ z \in \mathbb{C} : |z| < 1 \}$.

The multivalent function $\mathcal{F}$ in $\mathcal{D}$ of the form

$\mathcal{F}(z) = z^p + \sum_{t=1}^{\infty} \alpha_{t+p}\, z^{p+t}$, $p \in \mathbb{N} = \{ 1, 2, 3 \dots \}$, $z \in \mathcal{D}$. (1)

Now , if p=1 then $\mho(1) = \mho$, denote by $\mathcal{F} * \hbar$ Hadamard product of the functions $\mathcal{F}$ and $\hbar$ are analytic in $\mathcal{D}$. $\mathcal{F}$ form (1) and

$\hbar(z) = z^p + \sum_{t=1}^{\infty} \beta_{t+p}\, z^{t+p}$

then

$(\hbar * \mathcal{F})(z) = z^p + \sum_{t=1}^{\infty} \alpha_{t+p}\, \beta_{t+p}\, z^{t+p}$

in [3] , [4] Jackson defined and for($0 < q < 1$) , the operator $\mathfrak{D}_q$ is called q-derivative of the function $\mathcal{F}$ by

$$\mathfrak{D}_q \mathcal{F}(z) = \begin{cases} \frac{\mathcal{F}(z) - \mathcal{F}(qz)}{(1-q)z} & \text{if } z \neq 0 , \\ \mathcal{F}'(0) & \text{if } z = 0 \end{cases} \qquad (2)$$

By the equation (2) it follows that if $\mathcal{F} \in \mho(p)$ form the equation (1), then

$\mathfrak{D}_q \mathcal{F}(z) = [p]_q z^{p-1} + \sum_{t=1}^{\infty} [t+p]_q\, \alpha_{t+p}\, z^{t+p-1}$

Where $[t]_q = \frac{1-q^t}{(1-q)}$, $\lim_{q \to 1} [t]_q = t$.

By using q-derivative Jackson well-defined is the operator $\delta_{p,q}^{s} : \mho(p) \to \mho(p)$, s∈N .

If $\mathcal{F} \in \mho(p)$ by the equation (1) it follows that

$\delta_{p,q}^{s} \mathcal{F}(z) = (\mathcal{F} * \mathrm{H}_{p,q}^{s})(z)$, $p \in \mathrm{N}_0, z \in \mathcal{D}$

Where

$\mathrm{H}_{p,q}^{s}(z) = z^p + \sum_{t=1}^{\infty} \left([t+p]_q \right)^s z^{t+p}$

Moreover

$\delta_{p,q}^{s} \mathcal{F}(z) = z^p + \sum_{t=1}^{\infty} \left([t+p]_q \right)^s \alpha_{t+p}\, z^{t+p}$

$\lim_{q \to 1^-} \delta_{p,q}^{s} \mathcal{F}(z) = z^p + \sum_{t=1}^{\infty} (t+p)^s\, \alpha_{t+p}\, z^{t+p}$

For $\lambda \geq 0$, with the assist of the operator $\delta_{p,q}^{s}$ well-define q-derivative the new operator $\mathcal{M}_{\lambda,p,q}^{s,m} : \mho(p) \to \mho(p)$ by

$\mathcal{M}_{\lambda,p,q}^{s,m}\, \mathcal{F}(z) = (1-\lambda)\, \mathcal{M}_{\lambda,p,q}^{s,m-1} \mathcal{F}(z) + \lambda \frac{z}{p} (\mathcal{M}_{\lambda,p,q}^{s,m}\, \mathcal{F}(z))'$

$$\mathcal{M}_{\lambda,p,q,}^{s,0} \mathcal{F}(z) = \delta_{p,q}^{s} \mathcal{F}(z) ,$$

From the definition we can see that if $\mathcal{F} \in \mho(p)$ of the form the equation (1) , then

$\mathcal{M}_{\lambda,p,q}^{s,m}\, \mathcal{F}(z) = z^p + \sum_{t=1}^{\infty} \left([t+p]_q \right)^s \left(\frac{p+\lambda t}{p} \right)^m \alpha_{t+p}\, z^{t+p}$

For real complex numbers a , b , c , the Gaussian-hypergeometric function is formed by

${}_2\mathbb{F}_1(a, b, c; z) = \sum_{t=1}^{\infty} \frac{(a)_t (b)_t}{(c)_t} \frac{z^{t+p}}{t!}$,

the function θ_p(a ,b , c ; z) defined by

$\theta_p(a, b, c; z) = z^p + \sum_{t=1}^{\infty} \frac{(a)_t (b)_t}{(c)_t} \frac{z^{p+t}}{t!}$

The well known Gaussian-hypergeometric function conformable to the function

θ_p(a ,b , c ; z) , using the convolution for $\mathcal{M}_{\lambda,p,q}^{s,m}\ \mathcal{F}(z)$, define the operator $\Psi_{\lambda,p,q}^{s,m}$ by

$\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z) = \theta_p(a\ ,b\ , c\ ; z) * \mathcal{M}_{\lambda,p,q}^{s,m}\ \mathcal{F}(z)$

$$\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z) = z^p + \sum_{t=1}^{\infty}\left([t+p]_q\right)^s \left(\frac{p+\lambda t}{p}\right)^m \frac{(a)_t (b)_t}{(c)_t}\ \alpha_{t+p}\ \frac{z^{t+p}}{t!} \qquad (3)$$

Let

$$H_t = \left([t+p]_q\right)^s \left(\frac{p+\lambda t}{p}\right)^m \frac{(a)_t (b)_t}{t!\,(c)_t}\ .$$

$\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z) = z^p + \sum_{t=1}^{\infty} \Omega_t\ \alpha_{t+p}\ z^{t+p}$

For analytic functions $\mathcal{F}$, $\hbar$, the function $\mathcal{F}$ is subordinate to $\hbar$ in $\mathcal{D}$ (see[5]) .

Can be written

$\mathcal{F}(z) \prec \hbar\ (z)$. (4)

If there exists an analytic function ϖ , with $|\varpi(z)| < 1$ and $\varpi(0) = 0$ such that

$\mathcal{F}(z) = \hbar\ (\varpi\ (z)\ )$. (5)

If $\hbar$ is the univalent function in $\mathcal{D}$, then $\mathcal{F}(z) \prec \hbar\ (z)$ is equipollent to $\mathcal{F}(0) = \hbar(0)$, $\mathcal{F}(\mathcal{D}) = \hbar\ (\mathcal{D})$.

Minda and Ma [6] introduced and studied the classes (Ω) and $\mathcal{S}^*$ (Ω) as below

$(\Omega\ ) = \{\ \mathcal{F}(z) \in \mho : 1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} \prec \Omega\ (z)\}$ (6)

$\mathcal{S}^*\ (\Omega\ ) = \{\ \mathcal{F}(z) \in \mho : \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec \Omega\ (z)\}$, (7)

Where the analytic function is Ω (z) with part real positive in $\mathcal{D}$, Ω ($\mathcal{D}$) is like with respect to real axis and the starlike with respect to Ω '(0) > 0 and Ω (0) = 1 . the class $\mathcal{C}(\Omega$) and $\mathcal{S}^*(\Omega)$ contain several wellknown subclasses of convex and starlike function is specific case .

In [5], $\mathcal{F}$ is the function quasi-subordinate to $\hbar$ in $\mathcal{D}$, written as next

$\mathcal{F}(z) \prec_q \hbar\ (z)$ (8)

If there exist ϖ and $\mathcal{B}$ analytic functions , with $|\mathcal{B}(z)| \le 1$, and $|\varpi(z)| < 1$ and $\varpi(0) = 0$ then

$\mathcal{F}(z) = \mathcal{B}(z)\hbar(\ \varpi(z))$. (9)

"When $\mathcal{B}(z)$=1 , then $\mathcal{F}(z) = \hbar(\ \varpi(z))$, that $\mathcal{F}(z) \prec \hbar\ (z)$ in $\mathcal{D}$. that if $\varpi(z) = z$, then $\mathcal{F}(z) = \mathcal{B}(z)\hbar\ (z)$, it is called that $\mathcal{F}$ is majorized by $\hbar$ and written $\mathcal{F}(z) \ll \hbar\ (z)$ in $\mathcal{D}$. it is obvious that quasi-subordination is a generalization of subordination as well as majorization" . see [10] .

Darus and Mohd [1] introduced the classes $\mathcal{C}_q(\Omega$) and $\mathcal{S}_q^*(\Omega$) as below

$\mathcal{C}_q(\Omega\ ) = \{\ \mathcal{F}(z) \in \mho : \frac{z\ \mathcal{F}''(z)}{\mathcal{F}'(z)} \prec_q \Omega\ (z) - 1\ \}$ (10)

$\mathcal{S}_q^*(\Omega\ ) = \{\ \mathcal{F}(z) \in \mho : \frac{z\ \mathcal{F}'(z)}{\mathcal{F}(z)} - 1 \prec_q \Omega\ (z) - 1\ \}$. (11)

Definition 1.1 let the class $\chi_{s,q}^p$ ($\mathcal{v}$, b ; Ω) consists of function $\mathcal{F}(z) \in \mho$(p) satisfy

The quasi-subordination

$$\frac{1}{b}\left(\frac{z\left(\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z)\right)'}{(1-\mathcal{v})\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z) + \mathcal{v}z\left(\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z)\right)'} - p\right) \prec_q \Omega\ (z) - 1$$

(12)

($z \in$; p $\in$ N , p > s ; s $\in N_0 = \mathbb{N} \cup \{0\}$; b $\in \mathcal{C}^*$; $0 \le \mathcal{v} \le 1$).

Clearly , we have the next relationship : $\chi_{0,q}^1$ (0 , 1 ; Ω) = $\mathcal{S}_q^*(\Omega$) ; $\chi_{0,q}^1$ (1 , 1 ; Ω) = $\mathcal{C}_q(\Omega$)

It is well known this n-th coefficient of the univalent function $\mathcal{F}(z) \in \mho$ bounded by n (see[3]) . bounds for coefficient giving datum about various geometric properties of the function .there are many authors having inspected the bounds for coefficient Fekete -Szego for various class [7 , 9] . in particular , authors start to study the problem of Fekete -Szego for various class use quasi-subordination [7 , 11] .

Lemma 1.2 if $\varpi \in \Pi$, Π is the class of analytic functions $\varpi(z$) , then

$$|\varpi_2 - j\varpi_1^2| \le \begin{cases} -j, \text{ if } j \le -1 \\ 1, \text{ if } -1 \le j \le 1, \\ j, \text{ if } j \ge -1. \end{cases}$$

2 . Main results.

let $\mathcal{F}(z) = z + a_{1+p}\ z^{1+p} + a_{2+p}\ z^{2+p} + \dots$, $\Omega(z) = 1 + B_1\ z + B_2 z^2 + \dots$, $\emptyset(\ z) = C_0 + C_1\ z + C_2\ z^2 + \cdots$, $\varpi(z) = \varpi_1 z + \varpi_2\ z^2 + \dots$, $B_1 > 0$.

Theorem2.1 if $\mathcal{F}(z$) $\in \mho$(p) belongs to $\chi_{s,q}^p$ ($\mathcal{v}$, b ; Ω) , then a

$$|a_{p+1}| \le \frac{|b|\ |C_0|B_1\sqrt{2B_1}}{\sqrt{\ |\ b\ C_0\ B_1^2\left[(\mathcal{v}+2)H_1^2\ (\mathcal{v}-1) + 2H_2\ (\mathcal{v}+2)\right] - 2(B_2 - B_1)(\mathcal{v}+1)^2 H_1^2\ |}}$$

(13)

$$|a_{p+2}| \le \frac{|b|\ |C_0|B_1}{(\mathcal{v}+2)H_2} + \frac{|b|\ |C_1|B_1}{(\mathcal{v}+2)H_2} + \left(\frac{|b|\ |C_0|B_1}{(\mathcal{v}+1)H_1}\right)^2 . \quad (14)$$

and , For each real number m ,

$$\left|a_{p+2} - m\ a_{p+1}{}^2\right| \le \frac{|b|\ |C_1|\ |C_0|B_1\sqrt{2B_1}\ \ (|b|\ |C_0|B_1)^2}{\left((\mathcal{v}+2)\ H_2\right)^2 \left((\mathcal{v}+1)H_1\right)^2}$$

(15)

Proof.

If $\mathcal{F}(z) \in \chi_{s,q}^p$ (, b ; Ω) , then there exist analytic function $\varpi(z)$ and $\emptyset(\ z)$ with $|\varpi(z)\ | \le 1$, $\emptyset(\ z) < 1$ and $\emptyset(\ 0) = 0$ such that

$$1/_b\left(\frac{z\left(\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z)\right)'}{(1-\mathcal{v})\ \Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z) + \mathcal{v}z\left(\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z)\right)'} - p\right) =$$

$\emptyset(\ z)(\omega(\varpi(z) - 1)$. (16)

Since

$$\frac{z\left(\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z)\right)'}{(1-\mathcal{v})\ \Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z) + \mathcal{v}\ z\left(\Psi_{\lambda,p,q}^{s,m}\ \mathcal{F}(z)\right)'} - p =$$

$\sqrt{|\ b\ C_0\ B_1^2[H_1^2(\mathcal{v}-1)(\mathcal{v}+2) + 2H_2\ (\mathcal{v}+2)] - 2(B_2 - B_1)\ (\mathcal{v}+1)^2 H_1^2\ |}$

$\alpha_{1+p}\ z^{1+p}$

$+ [\ 2\ (\mathcal{v}+2)\omega_2\ \alpha_{2+p} - 2(\mathcal{v}+2)\omega_2\ \alpha_{1+p}{}^2\ ]\ z^2 + \dots$,

$\emptyset(\ z)(\omega(\varpi(z) - 1) = B_1 C_0\ \varpi_1 z + [B_1 C_1\ \varpi_1 + C_0\ (B_1\varpi_2 + B_2\ \varpi_1^2\ )]\ z^2 + \dots$,

From the equation (16) it follows that

a_{p+1}

=

$$\frac{\varpi_1 b\, C_0\, B_1\sqrt{2B_1}}{\sqrt{|\,b\, C_0\, B_1^2[(\upsilon+2)\mathrm{H}_1^2\,(\upsilon-1)+2\mathrm{H}_2\,(\upsilon+2)]-2(B_2-B_1)\,(\upsilon+1)^2\mathrm{H}_1^2\,|}}$$ (17)

$$a_{p+2}=\left(\frac{2}{(\upsilon+2)\mathrm{H}_2}+\left(\frac{1}{(\upsilon+1)\mathrm{H}_1}\right)^2\right)(B_1C_1\ \varpi_1+B_1C_0\,\varpi_2+C_0(b\,B_1^2\,C_0+B_2)\varpi_1^2\,].$$

Further ,

$$a_{p+2}-m a_{p+1}=\frac{b}{\left((\upsilon+2)\mathrm{H}_2\right)^2\left((\upsilon+1)\mathrm{H}_1\right)^2}\ [C_1\ \varpi_1+C_0(\varpi_2-\mathrm{j}\varpi_1)],$$ (18)

Since $\emptyset(z)$ is bounded and analytic in $\mathcal{D}$, in [8] we have

$|C_n|\le 1-|C_0|^2\le 1$.

Using by these fact and the wellknown this inequality $|\varpi_1|\le 1$ in (17) and (18) , we get

$$|a_{p+1}|\le\frac{|b|\,|C_0|B_1\sqrt{2B_1}}{\sqrt{|\,b\,C_0\,B_1^2[(\upsilon-1)(\upsilon+2)\mathrm{H}_1^2\ +2(\upsilon+2)\mathrm{H}_2\,]-2(B_2-B_1)(\upsilon+1)^2\mathrm{H}_1^2\,|}}$$

And

$$|a_{p+2}-m\,a_{p+1}{}^2\,|\le\frac{|b|\,|C_1|\ |C_0|B_1\sqrt{2B_1}\ (|b|\,|C_0|B_1)^2}{\left((\upsilon+2)\mathrm{H}_2\right)^2\left((\upsilon+1)\mathrm{H}_1\right)^2}$$

(19)

Applying Lemma1.2 and the tringle inequality to (19) , obtain (15) . the result is severe for the function

$$1/_b\left(\frac{z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}{(1-\upsilon)\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)+\upsilon z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}-p\right)=(1+z)(\,\omega\,(z^2)\,-1\,)$$

Or

$$1/_b\left(\frac{z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}{(1-\upsilon)\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)+\upsilon z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}-p\right)=(1+z)(\,\omega\,(z\ )-1\,).$$

For $m=0$, in (15) , have (14) . the proof of theorem2.1 is complete .

Corollary2.2 if $\mathcal{F}(z)\in\mho$ belongs to $\mathcal{S}_q^*(\Omega)$, then

$$|a_2|\le\frac{|b|\,|C_0|B_1\sqrt{B_1}}{\sqrt{|\,b\,C_0\,B_1^2-(B_2-B_1)|}},$$

$$|a_3|\le\frac{|b|\,|C_0|B_1}{2}+\frac{|b|\,|C_1|B_1}{2}+(|b|\,|C_0|B_1)^2.$$

$$|a_{p+2}-m\,a_{p+1}{}^2\,|\le\frac{|b|\,|C_1|m\ \ |C_0|B_1\sqrt{B_1}\ (|b|\,|C_0|B_1)^2}{2}.$$

Corollary2.3 if $\mathcal{F}(z)\in\mho$ belongs to $\mathcal{C}_q(\Omega)$, then

$$|a_2|\le\frac{|b|\,|C_0|B_1\sqrt{B_1}}{\sqrt{|\,3b\,C_0\,B_1^2-4(B_2-B_1)|}}$$

$$|a_3|\le\frac{|b|\,|C_0|B_1}{3}+\frac{|b|\,|C_1|B_1}{3}+\left(\frac{|b|\,|C_0|B_1}{2}\right)^2$$

$$|a_{2+p}-m\,a_{1+p}{}^2\,|\le\frac{m|b|\,|C_1|\ \ |C_0|B_1\sqrt{B_1}\ (|b|\,|C_0|B_1)^2}{3}.$$

Theorem2.4 $\mathcal{F}(z)\in\mho(\mathrm{p})$ satisfies

$$1/_b\left(\frac{z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}{(1-\upsilon)\,\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)+\upsilon z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}-p\right)\ll\Omega\,(z)-1\ ,$$

Then the following inequalities are hold

$$|a_{p+1}|\le\frac{|b|\,|C_0|B_1\sqrt{2B_1}}{\sqrt{|\,b\,C_0\,B_1^2[(\upsilon-1)\mathrm{H}_1^2\,(\upsilon+2)+2\mathrm{H}_2\,(\upsilon+2)]-2(B_2-B_1)(\upsilon+1)^2\mathrm{H}_1^2\,|}}$$

$$|a_{p+2}|\le\frac{|b|\,|C_0|B_1}{(\upsilon+2)\mathrm{H}_2}+\frac{|b|\,|C_1|B_1}{(\upsilon+2)\mathrm{H}_2}+\left(\frac{|b|\,|C_0|B_1}{(\upsilon+1)\mathrm{H}_1}\right)^2.$$

and , For each real number m ,

$$|a_{p+2}-m\,a_{p+1}{}^2\,|\le\frac{|b|\,|C_1|\ \ |C_0|B_1\sqrt{2B_1}\ (|b|\,|C_0|B_1)^2}{\left((\upsilon+2)\mathrm{H}_2\right)^2\left((\upsilon+1)\mathrm{H}_1\right)^2}$$

Proof., the result follows by taking $\varpi(z)=z$ in the proof of Theorem2.1 .

Theorem2.5 $\mathcal{F}(z)\in\mho(\mathrm{p})$ belongs to $\chi^p_{s,q}(\upsilon,\mathrm{b}\,;\Omega)$, then for each real number m , $\mathrm{b}>0$, and τ_1,τ_2,τ_3 , are the real numbers .

$$|a_{p+2}-m\,a_{p+1}{}^2\,|\le\begin{cases}\mathrm{b}\upsilon(1-\mathrm{j}) & ,m\le\tau_1\,,\\ 2\mathrm{b}\upsilon\,, & \tau_1\le m\le\tau_2\\ \mathrm{b}\upsilon(1+\mathrm{j}) & ,\ m\ge\tau_2.\end{cases}$$

(20)

Further , if $\tau_1\le m\le\tau_3$, then

$$|a_{p+2}-m\,a_{p+1}{}^2\,|+\mathrm{T}_1|a_{p+1}\ |^2\le 2\mathrm{b}\upsilon\,.\quad(21)$$

If $\tau_3\le m\le\tau_2$, then

$$|a_{p+2}-m\,a_{p+1}{}^2\,|+\mathrm{T}_2|a_{p+1}\ |^2\le 2\mathrm{b}\upsilon\,.\qquad(22)$$

For each real number m and $\mathrm{b}>0$,

$$|a_{p+2}-m\,a_{p+1}{}^2\,|\le\begin{cases}-\mathrm{b}\upsilon(1-\mathrm{j}) & ,m\le\tau_2\,,\\ -2\mathrm{b}\upsilon\,, & \tau_2\le m\le\tau_1\\ -\mathrm{b}\upsilon(1+\mathrm{j}) & ,\ m\ge\tau_1.\end{cases}$$

(23)

Further , if $\tau_2\le m\le\tau_3$, then

$$|a_{p+2}-m\,a_{p+1}{}^2\,|+\mathrm{T}_2|a_{p+1}\ |^2\le -2\mathrm{b}\upsilon\,.\qquad(24)$$

If $\tau_3\le m\le\tau_1$, then

$$|a_{p+2}-m\,a_{p+1}{}^2\,|+\mathrm{T}_1|a_{p+1}\ |^2\le -2\mathrm{b}\upsilon\,.\qquad(25)$$

Where

$$\upsilon=\frac{2\,B_1}{\mathrm{H}_2\ (\upsilon+2)}+\left(\frac{B_1}{\mathrm{H}_1\ (\upsilon+1)}\right)^2,$$

$$\tau_1=\frac{b\,|C_0|B_1^2+B_1-B_2}{\left(\mathrm{H}_1\ (\upsilon+1)\right)^2\left(\mathrm{H}_2\ (\upsilon+2)\right)^2\ b\,|C_0|B_1^2},$$

$$\tau_2=\frac{b\,|C_0|B_1^2+B_2+B_1}{\left(\mathrm{H}_1\ (\upsilon+1)\right)^2\left(\mathrm{H}_2\ (\upsilon+2)\right)^2\ b\,|C_0|B_1^2},$$

$$\tau_3=\frac{b\,|C_0|B_1^2+B_2}{\left(\mathrm{H}_1\ (\upsilon+1)\right)^2\left(\mathrm{H}_2\ (\upsilon+2)\right)^2\ b\,|C_0|B_1^2},$$

Proof. we assume that $\mathrm{b}>0$. from (15) , have

$|\alpha_{2+p}-m\,\alpha_{1+p}{}^2\,|\le\mathrm{b}\upsilon[\max\{1,|\mathrm{j}|\}$.

If $m\le\tau_1$, then $\mathrm{j}\le-1$. so , by applying Lemma1.2 , we have the first inequality in (20) .

If $m\le\tau_2$, then $\mathrm{j}\ge 1$. applying Lemma1.2 , we get the last inequality in (20) .

When $\tau_1\le m\le\tau_2$, then $|\mathrm{j}|\le 1$. so applying Lemma1.2 , we obtain the middle inequality in (20) .

If $m\le\tau_1$ *or* $m\le\tau_2$, the result is severe for the function

$$1/_b\left(\frac{z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}{(1-\upsilon)\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)+\upsilon z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}-p\right)=\emptyset(z)(\Omega\,(z^2)\,-1)\ .$$

If $\tau_1\le m\le\tau_2$, the result is severe for the function

$$1/_b\left(\frac{z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}{(1-\upsilon)\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)+\upsilon z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}-p\right)=\emptyset(z)(\Omega\,(z^2)\,-1)\ .$$

If $m=\tau_1$, the result is severe for the function

$$\frac{1}{\upsilon}\left(\frac{\mathcal{D}^{l}\Psi^{n,m}_{\lambda,p,q}\mathcal{F}(z)}{\mathcal{D}^{i}\Psi^{n,m}_{\lambda,p,q}\mathcal{F}(z)}-p^{l-1}\right)=\emptyset(z)(\Omega\left(\frac{z\,(z+\upsilon)}{1+\upsilon z}\right)-1)\ ,0\le\upsilon\le 1\,.$$

If $m=\tau_2$, the result is sharp for the function

$$1/_b\left(\frac{z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}{(1-\upsilon)\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)+\upsilon z\left(\Psi^{s,m}_{\lambda,p,q}\mathcal{F}(z)\right)'}-p\right)=$$

$$\emptyset(z)(\Omega\left(-\frac{z\,(z+\upsilon)}{1+\upsilon z}\right)-1)\ .$$

Moreover, (21) and (22) are established by an application of Lemma1.2.

Applying Lemma1.2, we can prove (23) and (25) for b > 0 . the proof of theorem2.5 is complete .

Corollary2.7 $\mathcal{F}(z) \in \mho$ belong to $\mathcal{S}_q^*(\Omega)$, then for each number m

$$|a_3 - m\, a_2{}^2| \le \begin{cases} \dfrac{|C_0|B_1\sqrt{B_1}\ [1-2m-(|b|\,|C_0|B_1)^2]}{\sqrt{|b\, C_0\, B_1^2-(B_2-B_1)|}} & , m \le \tau_1 , \\ \dfrac{|C_0|B_1\sqrt{B_1}}{\sqrt{|C_0\, B_1^2-(B_2-B_1)|}}, & \tau_1 \le m \le \tau_2 \\ \dfrac{|C_0|B_1\sqrt{B_1}\ [1+2m-(|C_0|B_1)^2]}{\sqrt{|C_0\, B_1^2-(B_2-B_1)|}} & , m \ge \tau_2. \end{cases}$$

If $\tau_1 \le m \le \tau_3$, then

$$|a_3 - m\, a_2{}^2| + \frac{(2m-1)C_0\, B_1^2-(B_2-B_1)}{2\, C_0^2 B_1^2}|\alpha_2|^2 \le \frac{|C_0|B_1\sqrt{B_1}}{\sqrt{|C_0\, B_1^2-(B_2-B_1)|}},$$

If $\tau_3 \le m \le \tau_2$, then

$$|a_3 - m\, a_2{}^2| + \frac{(2m+1)C_0\, B_1^2-(B_2-B_1)}{2\, C_0^2 B_1^2}|\alpha_2|^2 \le \frac{|C_0|B_1\sqrt{B_1}}{\sqrt{|C_0\, B_1^2-(B_2-B_1)|}}.$$

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مشكلة فيكت سيكو لفئة فرعية معينة من المؤثر q-Difference باستخدام شبه التابعية

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الملخص

في هذا البحث ، تم تحديد فئة جديدة معينة من المؤثر q-Difference مع شبه التابعية و مشاكل فيكت سيكو للدوال التي تنتمي إلى الفئة. النتائج المعروضة هنا لتجهيز هذه الملحقات في بعض الأعمال السابقة.