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Characterizations of Weakly Approximately Primary Submodules in Some Types of Modules

Khaled Y. Jhad , Bothaynah N. Shahab

Department of Mathematic Ibn- AL-Haitham College of Education , University of Baghdad, Baghdad , Iraq

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Corresponding Author:

Name: Khaled Y. Jhad

E-mail:

khaledyounis1989@gmail.com

dr.buthyna@yahoo.com

Tel:

ABSTRACT

Our aim in this note is to introduce several characterizations of weakly approximately primary submodules in class of multiplication modules. Furthermore, we characterized weakly approximately primary submodules by theirs resudule in the class of multiplication modules with the help of some types of modules as non-singular, projective, regular and faithful modules. Also, we characterized weakly approximately primary ideal with some kind of weakly approximately primary submodules in the same previous classes of modules with help of finitely generated modules.

1. Introduction

Let T be a commutative ring with identity and S be a unital left T-module. Weakly approximately primary submodule was introdued recently by [1], “where aproper submodule E of a T-module S is called weakly approximately primary if wherever 0≠ ah∈E, for a∈T, h∈S, implies that h∈E+Soc(S) or a∈$\sqrt{[E+Soc(S):_T S]}$, and an ideal A of a ring T is a weakly approximately primary if A is a weakly approximately primary T-submodule of a T-module T”[1], “where Soc(S) is the socal of modules, defined to be the interscetion of all essentail submodule of S”[2],

“where a non-zero submodule K of an T-module S is an essentail if K∩L≠(0) for all non-zero submodule L of S”[5]. “An T-module S is a multiplication if every submodule K of S is of the form K = IS for some ideal I of T, equivalent to S is multiplication if K = $[K:_T S]$S where $[K:_T S]$ = {r∈T: rS⊆K}”[3]. “If K, L are any submodules of multiplication module S, then K = IS, L = JS for some ideals I, J of T such that KL = IJS and KL = IL. Inparticular KS = Iww = K, also for any x∈S, K =Ix”[4]. “If S is a multiplication T-module, then for every elements m, m' ∈S, by $m\,m'$ means the product of two submodules T_m , $T_{m'}$, that is $m\,m'$ = $T_m . T_{m'}$ is a submodule of S”[5]. “In multiplication module S, the S-rad(E) of submodule E denoted by S-rad(E) = $\sqrt{E}$ = {m∈S: m^n⊆E for some positive integer n}”[6]. “A T-module S is regular if for each x∈S there exists f∈Hom_T (S,T) such that x = f(x)x”[6]. “Recall that an T-module S is faithful if ann_T(S) = (0)”[7], “and a T-module S is non-singular if Z(S) = {x∈S: xI = (0) for some essential ideal I of T} = (0)”[2]. “If E is a submodule of a T-module, and I is an ideal of T, then $[E:_S T]$ = E , $[I:_T T]$ = I”[9].

2. Characterizations of weakly approximately primary submodules in class of multiplication modules.

In this section we introduce many characterization of weakly approooximately primary submodule in class of multiplication modules.

First we need to recall the following lemma which appear in [1, prop.2.6]

Lemma (2.1)

“A proper submodule E of a T-module S is a weakly approximately primary if and only if wherever (0)≠ IF⊆E, where I is an ideal of T and F is a submodule of S, implies that F⊆E+Soc(S) or I⊆$\sqrt{[E+Soc(S):_T S]}$".

Proposition (2.2)

Let S be a multiplication T-module, and E is a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ HD⊆E, where H, D are submodules of S, implies that D⊆E+Soc(S) or H^n⊆E+Soc(S) for some positive integer n.

Proof

(⟹) Let HD⊆E, where H, D are submodules of S. Since S is multiplication, then H = IS, D = JS for some ideals I, J of T, it follows that (0)≠ I(JS)⊆E. But E is a weakly approximately primary, then by lemma (2.1) we have JS⊆E+Soc(S) or I^nS⊆E+Soc(S) for some positive integer n, that is D⊆E+Soc(S) or H^n⊆E+Soc(S).

(⟸) Let (0)≠ IK⊆E for I is an ideal of T and K is a submodule of S. But S is a multiplication, then K = JS for some ideal J of T. Thus (0)≠ I(JS)⊆E, it follows that (0)≠ LK⊆E where L = IS, so by hypothesis we have K⊆E+Soc(S) or L^n⊆E+Soc(S), that is I^nS⊆E +Soc(S). Thus K⊆E+Soc(S) or I^n⊆$\sqrt{[E+Soc(S):_T S]}$. Hence by lemma (2.1) E is a weakly approximately primary submodule of S.■

As direct application of proposition (2.2) we get the following corollaries.

Corollary (2.3)

Let S be a multiplication T-module, and E be a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ mm'⊆E, for m, m'∈S, implies that m'⊆E+Soc(S) or m^n⊆E+Soc(S) for some positive integer n.

Corollary (2.4)

Let S be a multiplication T-module, and E be a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ mH⊆E, for m∈S, H is a submodule of S, implies that H⊆E+Soc(S) or m^n⊆E+Soc(S) for some positive integer n.

Corollary (2.5)

Let S be a multiplication T-module, and E be a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ Km⊆E, for K is a submodule of S, m∈S, implies that m ⊆E+Soc(S) or K^n⊆E+Soc(S) for some positive integer n.

It is well known that cyclic T-module is multiplication[3], we get the following corollaries.

Corollary (2.6)

Let S be a cyclic T-module, and E is a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ HD⊆E, where H, D are submodules of S, implies that D⊆E+Soc(S) or H^n⊆E+Soc(S) for some positive integer n.

Corollary (2.7)

Let S be a cyclic T-module, and E be a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ mm'⊆E, for m, m'∈S, implies that m'⊆E+Soc(S) or m^n⊆E+Soc(S) for some positive integer n.

Corollary (2.8)

Let S be a cyclic T-module, and E be a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ mH⊆E, for m∈S, H is a submodule of S, implies that H⊆E+Soc(S) or m^n⊆E+Soc(S) for some positive integer n.

Corollary (2.9)

Let S be a cyclic T-module, and E be a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ Km⊆E, for K is a submodule of S, m∈S, implies that m ⊆E+Soc(S) or K^n⊆E+Soc(S) for some positive integer n.

The following are another characterizations of a weakly approximately primary submodules in class of multiplication modules.

Corollary (2.10)

Let S be a multiplication T-module, and E be a proper submodule of S. Then E is a weakly approximately primary if and only if m'⊆E+Soc(S) or m ⊆S-rad(E +Soc(S)) .

Corollary (2.11)

Let S be a multiplication T-module, and E be a proper submodule of S. Then E is a weakly approximately primary if and only if wherever (0)≠ mH⊆E, for m∈S, H is a submodule of S, implies that H⊆E+Soc(S) or m ⊆ S-rad(E+Soc(S)).

3. Charactrized weakly approximately primary submodules by theirs resudules.

In this part of this paper we introdued many characterizations of weakly approximately primary submodules by their resudules.

First we need to discuse the following fact in this remark.

Remark (3.1)

The resudules of weakly approximately primary submodule of T-module S need not to be a weakly approximately primary ideals of T, so, the following example explained that.

Example (3.2)

The submodule <$\bar{10}$> of the Z-module Z_{60} is weakly approximately primary, since Soc(Z_{60}) = <$\bar{2}$> [1, Example (2.3)]. That is wherever (0)≠ r$\bar{h}$∈<$\bar{10}$> for r∈Z, $\bar{h}$∈ Z_{60} implies that $\bar{h}$∈<$\bar{10}$>+ Soc(Z_{60}) = <$\bar{10}$>+<$\bar{2}$> = <$\bar{2}$> or r∈$\sqrt{[<\bar{10}>+Soc(Z_{60}):_Z Z_{60}]}$ = $\sqrt{[<\bar{2}>:_Z Z_{60}]}$ = 2Z. Thus, if 0≠ 2. $\bar{5}$∈<$\bar{10}$> for 2∈Z, $\bar{5}$∈ Z_{60} implies that 2∈$\sqrt{[<\bar{10}>+Soc(Z_{60}):_Z Z_{60}]}$ = Z. But the resudule $[<\bar{10}>:_Z Z_{60}]$ = 10Z is not weakly approximately primary ideal of Z since 0≠ 2.5∈10Z for 2,5∈Z but 5∉10Z+ Soc(Z) = 10Z+(0) = 10Z and 2∉$\sqrt{[<\bar{10}>+Soc(Z):_Z Z]}$ =10 Z. ■

Befor we introduced the first characterization we need to recall the following lemma

Lemm(3.3) [6, prop.3.25]

"If S is a regular T-module, then Soc(S) = Soc(T)S".

Proposition (3.4)

Let S be a multiplication regular T -module, and E is a proper submodule of S. Then E is a weakly approximately primary if and only if $[E:_T S]$ is a weakly approximately primary ideal of T.

Proof

(⟹) Let (0)≠ IJ⊆$[E:_T S]$ where I, J are ideals of T, implies that (0)≠ IJS⊆E, it follows that (0)≠ KL⊆E, where K = IS, L = JS (since S is a multiplication). Hence by proposition (2.2) L⊆E+Soc(S) or K^n⊆E+Soc(S) for some positive integer n, that is JS⊆ E+Soc(S) or I^nS⊆ E+Soc(S). But S is a regular, then by lemma (3.3) Soc(S) = Soc(T)S. Thus we have JS⊆$[E:_T S]$S + Soc(T)S or I^nS⊆$[E:_T S]S$ + Soc(T)S, it follows that J⊆$[E:_T S]$ + Soc(T) or I^n⊆ $[[E:_T S] + Soc(T) :_T T]$, implies that J⊆$[E:_T S]$ + Soc(T) or I⊆ $\sqrt{[[E:_T S] + Soc(T) :_T T]}$. Hence by lemma(2.1) $[E:_T S]$ is a weakly approximately primary ideal of T.

(⟸) Let (0)≠ IL⊆E, for I is an ideal of Tand L is a submodule of S, since S is a multiplication then L = JS for some ideal J of T. Thus,(0)≠ IJS⊆E, that is (0)≠ IJ⊆$[E:_T S]$. But $[E:_S S]$ is a weakly approximately primary ideal of T, then by lemma (2.1) we have J⊆$[E:_T S]$ + Soc(T) or I^n⊆$[E:_T S]$ + Soc(T) for some positive integer n, it follows that JS⊆$[E:_T S]S$ + Soc(T)S or I^nS⊆$[E:_T S]S$ + Soc(T)S. But S is a regular, then by lemma (3.3) Soc(T)S = Soc(S), so, L⊆E + Soc(S) or I^n⊆$[E + Soc(S) :_T S]$. Hence by lemma(2.1) E is a weakly approximately primary submodule of S. ■

The following corollary is a direct consequence of proposition (3.4)

Corollary (3.5)

Let S be a cyclic regular T-module, and E is a proper submodule of S. Then E is a weakly approximately primary if and only if $[E:_T S]$ is a weakly approximately primary ideal of T.

Befor we introduced the next characterization we need to recall the following lemmas

Lemma (3.6) [1, coro.(2.7)]

"A proper submodule E of an T-module S is a weakly approximately primary if and only if wherever (0)≠ aF⊆E for a∈S, F is a submodule of S, implies that F⊆ E + Soc(S) or a∈$\sqrt{[E + Soc(S) :_T S]}$".

Lemma (3.7) [6, coro.(2.1.4)(1)]

"If S is a faithful multiplication T-module then Soc(S) = Soc(T)S".

Proposition (3.8)

A proper submodule E of a faithful multiplication T-module S is a weakly approximately primary if and only if $[E:_T S]$ is a weakly approximately primary ideal of T.

Proof

(⟹) Let (0)≠ aI⊆$[E:_T S]$, for a∈T, I is an ideal of T, implies that (0)≠ aIS⊆E. Since E is a weakly approximately primary, then by lemma (3.6) we have IS⊆E+ Soc(S) or a∈$\sqrt{[E + Soc(S) :_T S]}$, it follows that IS⊆E+ Soc(S) or a^nS⊆E + Soc(S) for some positive integer n. Since S is faithful multiplication, then by lemma (3.7) Soc(S) = Soc(T)S. Thus, we have IS⊆$[E:_T S]S$ + Soc(T)S or a^nS⊆$[E:_T S]S$ + Soc(T)S, it follows that I⊆ $[E:_T S]$ + Soc(T) or a^n⊆$[E:_T S]$ + Soc(T) = $[[E:_T S] + Soc(T) :_T T]$, that is I⊆ $[E:_T S]$ + Soc(T) or a∈ $\sqrt{[[E:_T S] + Soc(T) :_T T]}$. Hence by lemma (3.6) $[E:_T S]$ is a weakly approximately primary ideal of T.

(⟸) Let (0)≠ mH⊆E, for m∈S, H is a submodule of S. Since S is a multiplication then m = Tm = JS, H = IS for some ideals I, J of T, that is (0)≠ JIS⊆E, so (0)≠ JI⊆$[E:_T S]$. Since $[E:_T S]$ is a weakly approximately primary ideal of T, then by lemma(2.1) we have I⊆$[E:_T S]$ + Soc(T) or J^n⊆$[[E:_T S] + Soc(T) :_T T]$ = $[E:_T S]$ + Soc(T), it follows that IS⊆$[E:_T S]S$ + Soc(T)S or J^nS⊆$[E:_T S]S$ + Soc(T)S. But S is faithful multiplication, Soc(T)S = Soc(S), that is H⊆E + Soc(S) or I^nS⊆E + Soc(S), it follows that H⊆ E + Soc(S) or m^n⊆ E + Soc(S). Hence by corollary (2.4) E is a weakly approximately primary submodule of S. ■

The following corollary is direct applecation of proposition (3.8)

Corollary (3.9)

A proper submodule E of cyclic faithful T-module S is a weakly approximately primary if and only if $[E:_T S]$ is a weakly approximately primary ideal of T.

To introduced next characterization we need to recall the following lemma

Lemma (3.10) [2, coro.1.26]

"If S is a non-singular T-module, then Soc(T)S = Soc(S)".

Proposition (3.11)

A proper submodule E of a non-singular multiplication T-module S is a weakly approximately primary if and only if $[E:_T S]$ is a weakly approximately primary ideal of T.

Proof

(⟹) Let (0)≠ rt∈$[E:_T S]$, for r, t∈T, implies that (0)≠ r(tS)⊆E. Since E is a weakly approximately primary submodule, then by lemma (3.6) we have tS⊆E + Soc(S) or r^nS⊆E + Soc(S). But S is a non-singular, then by lemma (3.10) Soc(S) = Soc(T)S. Thus tS⊆$[E:_T S]S$ + Soc(T)S or r^nS⊆$[E:_T S]S$ + Soc(T)S, it follows that t∈$[E:_T S]$ + Soc(T) or r^n∈$[E:_T S]$ + Soc(T) = $[[E:_T S]$+ Soc(T) $:_T T]$. Hence $[E:_T S]$ is a weakly approximately primary ideal of T.

(⟸) Let (0)≠ Km⊆E, for K is a submodule of S, m∈S. Since S is a multiplication then K = JS, m = Tm = IS, for some ideals I, J of T, it follows that (0)≠ JIS⊆E, implies that (0)≠ JI⊆$[E:_T S]$. But $[E:_T S]$ is a weakly approximately primary ideal of T, then by lemma (2.1) we have I⊆$[E:_T S]$ + Soc(T) or J^n⊆$[[E:_T S] + Soc(T) :_T T]$ = $[E:_T S]$ + Soc(T), for some positive integer n. That is IS⊆$[E:_T S]S$ + Soc(T)S or J^nS⊆$[E:_T S]S$ + Soc(T)S. But S is a non-singular, then Soc(T)S = Soc(S). Thus, we have m⊆E + Soc(S) or K^n⊆E + Soc(S). Hence by corollary (2.5) E is a weakly approximately primary submodule of S. ∎

The following corollary is a direct consequence of proposition (3.11).

Corollary (3.12)

A proper submodule E of a cyclic non-singular T-module S is a weakly approximately primary if and only if $[E:_T S]$ is a weakly approximately primary ideal of T.

We recall the following lemmas befor we introduced the next characterization

Lemma (3.13) [6, prop (3.24)]

"If S is a projective T-module, then Soc(S) = Soc(T)S".

Lemma (3.14) [1, coro (2.8)]

"A proper submodule E of a T-module S is a weakly approximately primary if and only if wherever (0)≠ Ix⊆E for I is an ideal of T, x∈S, implies that x∈E+Soc(S) or I⊆$\sqrt{[E + Soc(S) :_T S]}$"

Proposition (3.15)

A proper submodule E of a projective multiplication T-module S is a weakly approximately primary if and only if $[E:_T S]$ is a weakly approximately primary ideal of T.

Proof

(⟹) Let (0)≠ Ir⊆$[E:_T S]$ for I is an ideal of T and r∈T, so (0)≠ I(rS)⊆E. But E is a weakly approximately primary, then by lemma (2.1) we have rS⊆ E + Soc(S) or I^n⊆$[E + Soc(S) :_T S]$ for some positive integer n, that is rS⊆ E + Soc(S) or I^nS⊆E + Soc(S). But S is a projective then by lemma (3.13) Soc(S) = Soc(T)S. Thus rS⊆$[E:_T S]S$ + Soc(T)S or I^nS⊆$[E:_T S]S$ + Soc(T)S, implies that r∈$[E:_T S]$ + Soc(T) or I^n⊆$[E:_T S]$+ Soc(T) = $[[E:_T S]+ Soc(T) \underset{T}{:} T]$

Hence by lemma (3.14) $[E:_T S]$ is a weakly approximately primary ideal of T.

(⟸) Let (0)≠ mm'⊆E for m, m'∈S. Since S is a multiplication, then $m = Tm$ = IS, $m' = Tm'$ = JS for some ideals I, J of T. That is (0)≠ IJS⊆E, so (0)≠ IJ⊆$[E:_T S]$. But $[E:_T S]$ is a weakly approximately primary ideal of T, then by lemma (2.1) we have J⊆$[E:_T S]$ + Soc(T) or I^n⊆$[E:_T S]$ + Soc(T), implies that JS⊆$[E:_T S]S$ + Soc(T)S or I^nS⊆$[E:_T S]S$ + Soc(T)S for some positive integer n. But S is a projective, then by lemma (3.13) Soc(T)S = Soc(S), that is m'⊆E + Soc(S) or m^n⊆E + Soc(S). So by corollary (2.3) E is a weakly approximately primary submodule of S. ∎

The proof of the following corollary is direct from proposition (3.15).

Corollary (3.16)

A proper submodule E of a cyclic projective T-module S is a weakly approximately primary if and only if $[E:_T S]$ is a weakly approximaitly primary ideal of T.

Characterized weakly approximately ideals with some kind of weakly approximately submodules.

In this section we characterized a weakly approximately ideal A with submodule AS. Befor we introduced the first characterization we recall the following lemma.

Lemma (4.1) [8, coro. of Theo (9)]

"Let S be a finitely generated multiplication T-module, and I,J are ideals of T. Then IS⊆JS if and only if I⊆J+ann_T(S)".

Proposition (4.2)

Let S be a finitely generated multiplication regular T-module and A be an ideal of Twith ann_T(S)⊆A. Then A is a weakly approximately primary if and only if AS is a weakly approximately primary submodule of S.

Proof

(⟹) Let (0)≠ IK⊆AS, for I is an ideal of T, K is a submodule of S. Since S is a multiplication, then K = JS for some ideal J of T, so (0)≠ IJS⊆AS. But S is a finitely generated multiplication then by lemma (4.1) we have (0)≠ IJ⊆A + ann_T(S). Since ann_T(S)⊆A, then A + ann_T(S) = A, so (0)≠ IJ⊆A. It is given that A is a weakly approximately primary ideal, implies that by lemma (2.1) J⊆A + Soc(T) or I^n⊆$[A + Soc(T) \underset{T}{:} T]$ = A + Soc(T), for some positive integer n. Hence JS⊆AS + Soc(T)S or I^nS⊆AS + Soc(T)S. But S is a regular, then by lemma (3.3) we have Soc(T)S = Soc(S). Thus K⊆ AS+ Soc(S) or I^nS⊆AS + Soc(S), it follows that K⊆ AS+ Soc(S) or I^n⊆ $[AS+ Soc(S) \underset{T}{:} S]$. Hence by lemma (2.1) AS is a weakly approximately primary submodule of S.

(⟸) Let (0)≠ rt⊆A, for r, t∈T, implies that (0)≠ rtS⊆AS. But AS is a weakly approximately primary submodule of S, then by lemma (3.6) we have tS⊆ AS+ Soc(S) or r^nS⊆ AS+ Soc(S) for some positive integer n. Since S is a regular then by lemma (3.3) Soc(S) = Soc(T)S, so t∈A + Soc(T) or r^n∈A + Soc(T) = $[A+ Soc(T) \underset{T}{:} T]$. Hence A is a weakly approximately primary ideal of T. ∎

It is well-known that cyclic T-module is a finitely generated we get the following corollary

Corollary (4.3)

Let S be a cyclic regular T-module and A be an ideal of Twith ann_T(S)⊆A. Then A is a weakly approximately primary ideal if and only if AS is a weakly approximately primary submodule of S.

Proposition (4.4)

Let S be a finitely generated multiplication prolective T-module and A be an ideal of Twith ann_T(S)⊆A. Then A is a weakly approximately primary ideal if and only if AS is a weakly approximately primary submodule of S.

Proof

(⟹) Let (0)≠ KL⊆AS, where K, L are a submodules of S. Since S is a multiplication, then L= IS, K = JS for some ideals I, J of T, it follows that (0)≠ JIS⊆AS. But S is a finitely generated, then by lemma (4.1) we have (0)≠ JI⊆ A+ ann_T(S). Since ann_T(S)⊆A, implies that A+ ann_T(S) =A, so (0)≠ JI⊆A. But A is a weakly approximately primary ideal of T, then by lemma (2.1) we have I⊆A + Soc(T) or J^n⊆[A + Soc(T) $\underset{T}{:}$ T] = A + Soc(T), for some positive integer n, it follows that IS⊆AS + Soc(T)S or J^nS⊆AS + Soc(T)S. Since S is a projective, then by lemma (3.13) we have Soc(T)S = Soc(S). Thus L⊆ AS+ Soc(S) or K^n⊆ AS+ Soc(S). Hence by proposition (2.2) AS is a weakly approximately primary submodule of S.

(⟸) Let (0)≠ IrS⊆A, for I is an ideal of T, r∈T, implies that (0)≠ IrS⊆AS. Since AS is a weakly approximately primary submodule of S, then by lemma (3.14) we have rS⊆ AS+ Soc(S) or I^nS⊆ AS+ Soc(S), for some positive integer n. Since S is a projective, then by lemma (3.13) we have Soc(T)S = Soc(S). Hence rS⊆AS + Soc(T)S or I^nS⊆AS + Soc(T)S, it follows that r∈A + Soc(T) or I^n∈A + Soc(T) = [A+ Soc(T) $\underset{T}{:}$ T]. So by lemma (3.14) A is a weakly approximately primary ideal of T. ■

Proposition (4.5)

Let S be a finitely generated multiplication non-singular T-module and A be an ideal of Twith ann_T(S)⊆A. Then A is a weakly approximately primary ideal if and only if AS is a weakly approximately primary submodule of S.

Proof

(⟹) Follows by proposition (2.2) lemma (4.1) and lemma (3.10)

(⟸) Follows by lemma (3.6), lemma (4.1) and lemma (3.10). ■

Proposition (4.6)

Let S be a faithful finitely generated multiplication T-module and A be an ideal of T. Then A is a weakly approximately primary ideal if and only if AS is a weakly approximately primary submodule of S.

Proof

(⟹) Let (0)≠ mH⊆AS, for m∈S, H is a submodules of S. Since S is a multiplication, then H= JS, m= Tm = IS, for some ideals I, J of T, so (0)≠ IJS⊆AS. It follows by lemma (4.1), (0)≠ IJ⊆ A + ann_T(S). Since S is a faithful then ann_T(S) = (0), so, (0)≠ IJ⊆A. But A is a weakly approximately primary ideal of T, then by lemma (2.1) we have J⊆A + Soc(T) or I^n⊆[A + Soc(T) $\underset{T}{:}$ T] = A + Soc(T), it follows that JS⊆AS + Soc(T)S or I^nS⊆AS + Soc(T)S. But S is afaithful multiplication, then by lemma (3.7) we have Soc(T)S = Soc(S). Hence H⊆ AS+ Soc(S) or m^n⊆ AS+ Soc(S). Thus by corollary (2.4) AS is a weakly approximately primary submodule of S.

(⟸) Let (0)≠ ab∈A, for a, b∈S, it follows that (0)≠ abS⊆AS. But AS is a weakly approximately primary submodule, then by lemma (3.6) we have bS⊆ AS+ Soc(S) or a^nS⊆ AS+ Soc(S). Since S is faithful multiplication then Soc(S) = Soc(T)S, so we have bS⊆AS + Soc(T)S or a^nS⊆AS + Soc(T)S, it follows that b∈A + Soc(T) or a^n∈A + Soc(T) = [A+ Soc(T) $\underset{T}{:}$ T]. Hence A is a weakly approximately primary ideal of T■

The following corollary is direct application of proposition (4.4), proposition (4.5) and proposition (4.6).

Corollary (4.7)

Let S be a cyclic (prolective, non-singular, faithful) T-module and A be an ideal of Twith ann_T(S)⊆A. Then A is a weakly approximately primary ideal if and only if AS is a weakly approximately primary submodule of S.

"Recall that a T-module S is weak cancellation if AS = BS, implies that A+ann_T(S) = B+ann_T(S) for A,B are ideals of T"[8].

The following fact appear in [8,prop (3.9)] we needed before we introduce the next propositions.

Lemma (4.8)

"Let S be a multiplication T-module. Then S is a finitely generated if and only if S is a cancellation".

Proposition (4.9)

Let S be a finitely generated multiplication regular T-module, and E be aproper submodule of S. Then the following statements are equivalents:

1. E is a weakly approximately primary submodule of S.
2. [E:$_T$ S] is a weakly approximately primary ideal of T.
3. E = AS for some weakly approximately primary ideal A of T with ann_T(S)⊆A.

Proof

(1)⟹(2) Follows from proposition (3.4)

(2)⟹(3) Since [E:$_T$ S] is a weakly approximately primary ideal and ann_T(S) = [0:$_T$ S]⊆[E:$_T$ S] then by proposition (4.2) [E:$_T$ S]S is a weakly approximately primary submodule of S, since S is a multiplication, then E = [E:$_T$ S]S = AS, where A = [E:$_T$ S] is a weakly approximately primary ideal of T.

(3)⟹(2) Since E = AS with A is a weakly approximately primary ideal of T such that ann_T(S)⊆A. From other hand we have S is a multiplication, then E = [E:$_T$ S]S. But S is a finitely generated, it follows by lemma (4.8) that S is a weak cancellation. That is A + ann_T(S) = [E:$_T$ S] + ann_T(S). But ann_T(S)⊆A and ann_T(S)⊆ [E:$_T$ S], it follows that A + ann_T(S) = A and [E:$_T$ S] + ann_T(S) = [E:$_T$ S]. But A is a weakly approximately primary ideal, it follows that [E:$_T$ S] is a weakly approximately primary ideal of T. ■

Form proposition (3.11), proposition (4.5) and lemma (4.8) we get the following result.

Proposition (4.10)

Let S be a finitely generated multiplication non-singular T-module, and E be aproper submodule of S. Then the following statements are equivalents:

1. E is a weakly approximately primary submodule of S.
2. [E:$_T$ S] is a weakly approximately primary ideal of T.
3. E = AS for some weakly approximately primary ideal A of T with ann_T(S)⊆A.

Form proposition (3.15), proposition (4.4) and lemma (4.8) we get the following result.

Proposition (4.11)

Let S be a finitely generated multiplication projective T-module, and E be aproper submodule of S. Then the following statements are equivalents:

1. E is a weakly approximately primary submodule of S.
2. [E:$_T$ S] is a weakly approximately primary ideal of T.
3. E = AS for some weakly approximately primary ideal A of T with ann_T(S)⊆A.

Form proposition (3.8), proposition (4.6) and lemma (4.8) we get the following result.

Proposition (4.12)

Let S be a faithful finitely generated multiplication T-module, and E be aproper submodule of S. Then the following statements are equivalents:

1. E is a weakly approximately primary submodule of S.
2. [E:$_T$ S] is a weakly approximately primary ideal of T.
3. E = AS for some weakly approximately primary ideal T.

We end this section by the following corollary.

Corollary (4.13)

Let S be a cyclic (projective, non-singular and faithful) T-module, and E be aproper submodule of S. Then the following statements are equivalents:

1. E is a weakly approximately primary submodule of S.
2. [E:$_T$ S] is a weakly approximately primary ideal of T.
3. E = AS for some weakly approximately primary ideal A of T with ann_T(S)⊆A.

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تشخيصات المقاسات الجزئية المتقاربة الابتدائية الضعيفة في بعض انواع المقاسات

خالد يونس جهاد ، بثينه نجاد شهاب

قسم الرياضيات ، كلية التربية ابن الهيثم ، جامعة بغداد ، بغداد ، العراق

الملخص

هدفنا في هذا البحث هو تقديم العديد من تشخيصات المقاسات الجزئية المتقاربة الابتدائية الضعيفة في صنف المقاسات الضربية. بالاضافة الى ذلك تشخص المقاسات الجزئية المتقاربة الابتدائية الضعيفة بواسطة بواقيها في صنف المقاسات الضربيه بمساعده بعض انواع المقاسات الاخرى مثل المقاسات الغير احادية والاسقاطية و المنتظمة والمخلصة.

كذلك تشخص المثاليات المتقاربة الابتدائية الضعيفة مع بعض انواع المقاسات الجزئية المتقاربة الابتدائية الضعيفة في نفس صنف المقاسات اعلاه بمساعده المقاسات المنتهيه التولدز