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Optimizing Fractional Differential Equation Solutions with Novel Müntz Space Basis Functions

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ABSTRACT

This paper introduces innovative basis functions derived from Müntz spaces, aimed at addressing the computational challenges of Fractional Differential Equations (FDEs). Our primary focus is the creation of these functions using singular indices linked to the solutions of FDEs. We thoroughly investigate the properties of these fundamental functions to understand their operational potential. These functions are particularly adept at capturing initial singular indices, making them highly suitable for solving FDEs. The proposed numerical method is distinguished by its rapid convergence rates, showcasing its efficiency in computational evaluations. We validate our approach by presenting numerical examples that highlight its accuracy and reliability. These examples confirm the effectiveness and efficiency of the new basis functions from Müntz spaces in accurately solving FDEs. This research advances numerical methods for FDEs and serves as a valuable resource for researchers seeking robust and reliable techniques.

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تحسين حلول المعادلات التفاضلية الكسرية باستخدام دوال أساس جديدة من فضاء مونتز

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الملخص

في هذه الدراسة، نقدم دوال أساس جديدة من فضاء مونتز لحل المعادلات التفاضلية الكسرية (FDEs) عددياً. نركز على إنشاء هذه الدوال باستخدام المؤشرات الفردية المتعلقة بالحلول المجهولة في المعادلات التفاضلية الكسرية. نقوم بفحص خصائص هذه الدوال بعناية لفهم كيفية عملها. تتميز هذه الدوال الأساس بقدرتها الفائقة على التقاط أول مؤشرين فرديين، مما يجعلها فعالة في حل المعادلات التفاضلية الكسرية. المنهج العددي الجديد الذي نقترحه يتميز بمعدلات تقارب عالية، وهو أمر مهم لتقييم كفاءة المناهج العددية. ندعم فعالية منهجنا بأمثلة عددية تظهر دقته وموثوقيته. تثبت هذه الأمثلة أن دوال الأساس الجديدة من فضاء مونتز يمكن أن توفر حلولاً دقيقة وفعالة للمعادلات التفاضلية الكسرية. تساعد هذه الدراسة في تطوير المناهج العددية لحل المعادلات التفاضلية الكسرية وتقدم أداة واعدة للباحثين الذين يبحثون عن تقنيات قوية وموثوقة.

1. Introduction

Fractional Differential Equations (FDEs) play a crucial role in applied mathematics and are widely used in various scientific and engineering fields. The adaptability of fractional calculus is particularly notable in areas like engineering, physics, signal processing, and anomalous diffusion (see, e.g., [1, 2, 3, 4]). Recent studies suggest that fractional derivatives provide more accurate representations of numerous dynamic processes than traditional derivatives. In these models, different fractional operators, such as the Riemann-Liouville integral/derivative and the Caputo derivative, are utilized. These operators introduce nonlocality and weakly singular kernels, leading to non-smooth behavior in FDE solutions near domain boundaries.

The wide use of FDEs has led to a surge of interest in developing numerical methods for solving them in recent decades. There has been significant focus on approximating fractional integrals and derivatives [5, 6, 7, 8]. Despite the challenges of the nonlocal and non-smooth nature of FDEs, recent years have seen many new numerical methods proposed for their solution. However, most of the existing literature focuses on error analysis for smooth solutions (see, e.g., [9, 10, 11, 12, 13, 14, 15]).

This research aims to address the challenges of solving FDEs, especially those with non-smooth characteristics near domain boundaries. Inspired by the nonlocal nature of fractional operators, global methods like spectral methods have become popular for solving fractional problems. However, singular terms in FDE solutions hinder exponential convergence with classical orthogonal polynomials. Recent methods aim to handle singularities in non-smooth FDE solutions [16, 17].

The domain of numerical solutions for FDEs is ever-evolving, and this research aims to make a substantial contribution to its development.

Our primary objective is to present new basis functions originating from Müntz spaces, meticulously designed using the singular indices of the unknown solutions in Fractional Differential Equations (FDEs). We investigate the characteristics of these basis functions, focusing on their capability to capture the first two singular indices. This study introduces a numerical method founded on these innovative functions, showcasing its rapid convergence rates. To validate the accuracy and efficiency of our approach, we include numerical examples.

This research aims to develop a robust and efficient framework for numerically solving Fractional Differential Equations (FDEs), with the potential to make significant contributions to the field of fractional calculus applications. We thoroughly analyze the properties of these basis functions, particularly their ability to capture the initial two singular indices. The primary objective of this new approach is to enhance the accuracy and efficiency of numerical solutions for FDEs. We introduce a numerical method

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based on these innovative basis functions, characterized by its high convergence rates, and validate it through a series of numerical examples. These examples demonstrate the effectiveness and precision of the proposed method, underscoring its practical applications. By integrating theoretical foundations with numerical implementation, this study provides a solid framework for solving a broad spectrum of FDEs, thereby advancing the field of fractional calculus applications.

The structure of the paper is as follows: The next section provides an overview and introduces some preliminary concepts. In Section 3, we introduce a novel basis function based on the singular index of the function and describe the implementation of the numerical methods used to solve linear FDEs. Section 4 presents numerical examples and applications, demonstrating the effectiveness, accuracy, and convergence rates of the proposed methods. The final section includes our conclusions and a discussion of the approaches used in this study.

2. Preliminaries

In this section, we present essential definitions and properties pertaining to fractional integrals and derivatives, along with Müntz-Legendre polynomials, which will be used extensively throughout this paper. [12, 13].

Definition 2.1: For a function $u:[a,b]\rightarrow \mathcal{R}$ and a real number $\alpha>0$, the Riemann-Liouville fractional integrals are defined as:

$${}_aI_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds, \quad (1)$$

and the Caputo fractional derivative of order α is defined as:

$${}_a^cD_t^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds, \quad n-1<\alpha\le n, \quad (2)$$

here, n is the smallest integer greater than or equal to α, Γ(.) is the gamma function.

In the realm of fractional calculus, a noteworthy connection exists between the Riemann-Liouville fractional integral and the Caputo fractional derivative. For a given function$u(t)$, this relationship can be expressed as follows:

$${}_aI_t^{\alpha}\,{}_a^cD_t^{\alpha}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k, \quad {}_a^cD_t^{\alpha}\,{}_aI_t^{\alpha}u(t)=u(t) \quad (3).$$

This relationship underscores the interplay between fractional integral and fractional derivative operators, offering insights into their combined effects on functions. This understanding is crucial for a comprehensive grasp of fractional calculus. Below, it is evident that fractional Integral and derivative operators applied to power functions result in power functions of the same form. Consider the following:

$${}_aI_t^{\alpha}(t-a)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \quad \beta>-1, \quad (4)$$

and

$${}_a^cD_t^{\alpha}(t-a)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \quad \beta>n-1. \quad (5)$$

Definition 2.2: We define a Müntz sequence as a monotonically increasing sequence of distinct real numbers

$$\Lambda=\{\lambda_0,\lambda_1,\lambda_2,\dots\}, \quad \lambda_0<\lambda_1<\lambda_2<\cdots, \quad (6)$$

and we refer to a system of the form $\{t^{\lambda_0},t^{\lambda_1},t^{\lambda_2},\dots\}$ as a Müntz system, with the corresponding Müntz space associated with the parameter Λ.

From [12], the space associated with $\Lambda=\{0,\lambda_1,\lambda_2,\dots\}$ is a dense subset of $C([0,1])$ if and only if $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\cdots=\infty$. The Müntz−Legendre is defined as polynomials that satisfy the condition of being a linear combination of powers of a given base function over a specified interval. More formally, a sequence of polynomials $\{L_k(t)\}$ is considered a Müntz−Legendre sequence on the interval [a, b] if it can be expressed as:

$$L_k(t)=\sum_{i=0}^{k}c_{i,k}t^{\lambda_k}, \quad c_{i,k}=\frac{\prod_{j=0}^{k-1}(\lambda_i+\lambda_j+1)}{\prod_{j=0,j\neq i}^{k}(\lambda_i-\lambda_j)}. \quad (7)$$

The orthogonality of Müntz−Legendre polynomials in $L^2[0,1]$ concerning the Legendre weight is demonstrated (see [12]),

$$\int_0^1 L_k(t)L_i(t)\,dt=\frac{\delta_{ki}}{2\lambda_k+1}. \quad (8)$$

Moreover, there are several recurrence relations, such as:

$$t\left(L'_k(t)-L'_{k-1}(t)\right)=\lambda_k L_k(t)+(1+\lambda_{k-1})L_{k-1}(t), \quad (9)$$

$$L_k(t)=L_{k-1}(t)-(\lambda_k+\lambda_{k-1}+1)t^{\lambda_k}\int_t^1 s^{-\lambda_k-1}L_{k-1}(s)ds. \quad (\mathbf{10})$$

3. Main Results

In this section, we introduce a numerical method to solve the following linear FDEs of order $0<\alpha\le 1$,

$${}_a^cD_t^{\alpha}u(t)+a(t)u(t)=f(t), \quad t\in(0,1], \quad (11)$$

with initial condition $u(0)=0$.

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To construct the new Müntz polynomials, it is essential to understand the regularity behavior of the exact solution of an FDE. To determine this behavior, we need to identify the singular indexes of the exact solution at the initial time. For finding these indexes, we can use the following Algorithm.

Algorithm

Step 1: Take $a = 0$ and $b = 1$,
Step 2: Compute $r_j = a + \frac{b-a}{10} j$ for $j = 0, 1, \dots, 10$,
Step 3: Compute $s_j = \lim_{t \to 0} \frac{f(t)}{t^{r_j}}$ for $j = 0, 1, \dots, 10$,
Step 4: If $s_j = 0$ for $j < k$ and $s_k \neq 0$, put $a = r_{k-1}$ and $b = r_k$,
Step 5: Do Steps 2-4 until $b - a < 0.01$,
Step 6: Put $\beta_1 - \alpha = r_{k-1}$ and $c_1 = r_k$,
Step 7: Do Steps 2-5 for $f(t) := f(t) - c_1 t^{\beta_1 - \alpha}$,
Step 8: Put put $\beta_2 - \alpha = r_{k-1}$ and $c_2 = r_k$.

Now, we assume that the parameters β_1 and β_2 are the smallest singular index of the function $u(t)$ obtained the above Algorithm. Due to the presence of singular indexes, we consider a Müntz sequence as

$\Lambda = \{0, \beta_1, \beta_2, 1 + \beta_1, 1+, \beta_2, 2 + \beta_1, 2 + \beta_2, \dots\},\ \ \beta_1 < \beta_1,$ (12)

It should be noted that such a sequence holds under condition $\sum_{i=0}^{\infty} \frac{1}{\beta_1 + i} + \sum_{i=0}^{\infty} \frac{1}{\beta_2 + i} = \infty$, making it a dense subspace of $C([0,1])$. Based on this sequence, we define the following Müntz -Legendre polynomials

$$L_k(t) = c_{0,k} + t^{\beta_1} \sum_{i=0}^{\left[\frac{k-1}{2}\right]} c_{2i+1,k} t^{i} + t^{\beta_2} \sum_{i=0}^{\left[\frac{k}{2}-1\right]} c_{2i,k} t^{i} = c_{0,k} + t^{\beta_1} P_{1k}(t) + t^{\beta_2} P_{2k}(t). \quad (13)$$

Also, using (3), we have

$$\int_0^1 L_k(t) L_i(t)\, dt = \begin{cases} \frac{\delta_{ki}}{2\beta_1 + k} & k \text{ is odd} \\ \frac{\delta_{ki}}{2\beta_2 + k - 1} & k \text{ is even} \\ 1 & k = i = 0 \end{cases} \quad (14)$$

where δ_{ki} is Kronecker delta function. Now, we approximate the unknown solution of (11) using truncated Müntz -Legendre polynomials of degree n (n is even number), i.e.,

$$u(t) \approx \sum_{k=0}^{n} d_k L_k(t) = \sum_{k=0}^{n} d_k c_{0,k} + t^{\beta_1} \sum_{k=0}^{n} d_k P_{1k}(t) + t^{\beta_2} \sum_{k=0}^{n} d_k P_{2k}(t). \quad (15)$$

Using the following notations

$$\hat{d} = [d_0, d_1, \dots, d_n]^T, \hat{c}_0 = [c_{0,1}, c_{0,2}, \dots, c_{0,n}], \hat{P}_j = [P_{j0}(t), P_{j1}(t), \dots, P_{jn}(t)], j = 1,2,$$

we can write

$$u_n(t) = \left(\hat{c}_0 + t^{\beta_1} \hat{P}_1 + t^{\beta_2} \hat{P}_2\right) \hat{d}. \quad (16)$$

On the other hand, we have $\hat{P}_1^{\,T} = C_1 T(t)$ and $\hat{P}_2^{\,T} = C_2 T(t)$, where

$$C_1 = \begin{pmatrix} c_{1,1} & & & \\ c_{1,2} & & & \\ c_{1,3} & c_{3,3} & & \\ c_{1,4} & c_{3,4} & & \\ \vdots & \vdots & \vdots & \\ c_{1,n} & c_{3,n} & \cdots & c_{n-1,n} \end{pmatrix}_{n \times \frac{n}{2}}, C_2 = \begin{pmatrix} c_{2,2} & & & \\ c_{2,3} & & & \\ c_{2,4} & c_{4,4} & & \\ \vdots & \vdots & \vdots & \\ c_{2,n} & c_{4,n} & \cdots & c_{n,n} \end{pmatrix}_{n \times \frac{n}{2}},$$

$$h(t) = \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{n-1} \end{pmatrix}. \quad (17)$$

From (16), we can obtain

$$u_n(t) = \left(\hat{c}_0 + t^{\beta_1} h^T(t) C_1^T + t^{\beta_2} h^T(t) C_2^T\right) \hat{d} := G(t) \hat{d}. \quad (18)$$

To have high efficiency and accuracy of the considered Müntz sequence Λ for approximating functions, we consider the following function

$u(t) = t^{0.5} + t^{0.6} \sin(t^{0.1})$.

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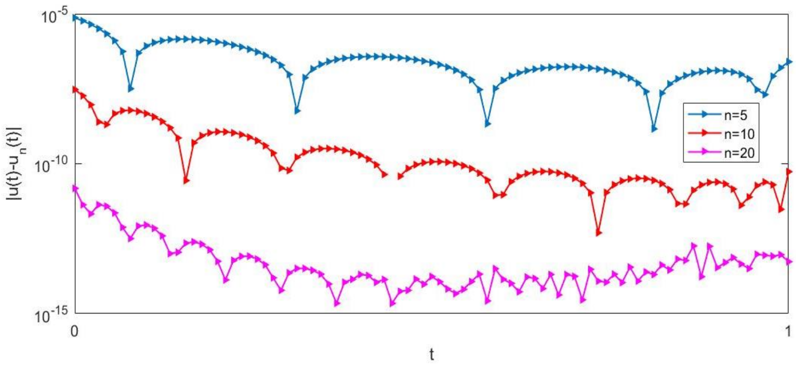

Fig. 1: The errors $|u(t) - u_n(t)|$ using present basis functions.

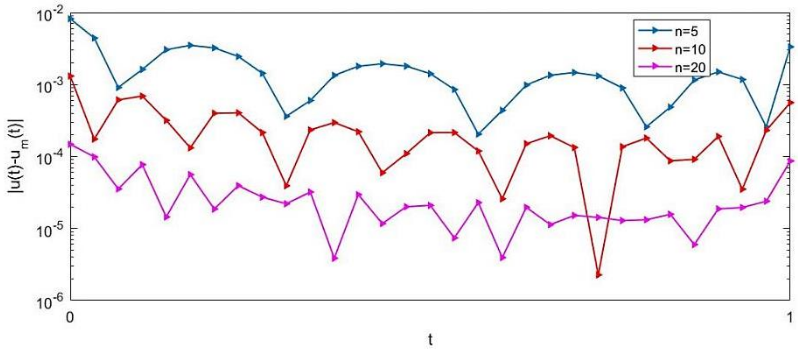

Fig. 2: The errors $|u(t) - u_n(t)|$ using the classical orthogonal polynomials.

Using the presented algorithm, the values of $\beta_1 = 0.499$ and $\beta_2 = 0.699$ have been obtained and we have approximated this function with the introduced bases. The error values for several n values are depicted in Figure 1. We have also approximated this function with classical polynomials and shown its error in Figure 2. It can be clearly seen that the current bases are suitable for approximating non-smooth functions at the endpoints.

For solving FDE defined in (11), we need to evaluate the Caputo fractional derivative of $t^{\beta_1} h^T(t)$ and $t^{\beta_2} h^T(t)$, using (1), we get

$${}_0^C D_t^{\alpha} t^{\beta_1} h^T(t) = t^{\beta_1-\alpha} h^T(t) D_1^T, \qquad (19)$$

where D_1^T is a diagonal matrix with entries $(D_1^T)_{i,i} = \frac{\Gamma(i+\beta_1)}{\Gamma(i+\beta_1-\alpha)}$ for $i = 1,2,\dots,n$. Similarly, we have

$${}_0^C D_t^{\alpha} t^{\beta_2} h^T(t) = t^{\beta_2-\alpha} h^T(t) D_2^T, \ (D_2^T)_{i,i} = \frac{\Gamma(i+\beta_2)}{\Gamma(i+\beta_2-\alpha)}, i = 1,2,\dots,n. \qquad (20)$$

Taking the Caputo fractional derivative of both sides (18) and substituting (19)-(20) in it, we have

$${}_0^C D_t^{\alpha} u_n(t) = \left(\hat{c}_0 + t^{\beta_1-\alpha} h^T(t) D_1^T C_1^T + t^{\beta_2-\alpha} h^T(t) D_2^T C_2^T\right)\hat{d} := H(t)\hat{d}. \qquad (21)$$

From (11), we can define the following residual function

$$R(t,\hat{d}) = (H(t) + a(t)G(t))\hat{d} - f(t). \qquad (22)$$

To evaluate an unknown vector $\hat{d}$, we can use the initial condition $u(0) = G(0)\hat{d} = u_0$ and the following n algebraic equations

$$\int_0^1 R(t,\hat{d})G(t)dt = 0. \qquad (23)$$

Another method to find unknown coefficients $\hat{d}$, we can use the collocation method. For this purpose, we consider the Chebyshev-Gauss-Lobatto points as follows

$$t_k = \frac{1}{2}\left(1 - \cos\left(\frac{k\pi}{n}\right)\right), k = 0,1,\dots,n. \qquad (24)$$

Taking these collocation points in (22), we can obtain the linear system

$$\left(H(t_k) + a(t_k)G(t_k)\right)\hat{d} = f(t_k), k = 1,2,\dots,n, \qquad (25)$$

$$u(0) = G(0)\hat{d} = u_0.$$

The linear system under consideration can be effectively solved by employing various numerical algebraic methods.

Remark 1: It should be noted that the presented method can be easily extended to fractional differential equations of various forms, including both linear and non-linear ones, over arbitrary intervals.

4. Results and Discussion

In this section, we present several examples to demonstrate the effectiveness and applicability of the proposed method. The first example aims to illustrate the theoretical convergence rates

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discussed in this paper. Here, we explore FDEs where the solutions may exhibit singularities at the endpoints.

Example 1: Consider the following linear FDE [10]

${}_0^C D_t^\alpha u(t) = f(t), \;\; u(0) = 0,$

with the exact solution $u(t) = t^{\beta_1} + t^{\beta_2} + t^{\beta_3} + t^{\beta_4}$ and

$f(t) = \sum_{i=1}^{4} \frac{\Gamma(\beta_i+1)}{\Gamma(\beta_i-\alpha+1)} t^{\beta_i-\alpha}.$

The presented method has been employed to solve the equation for various values of α and β_i. For the new method, we compute the matrices C_i and D_i for $i = 1,2$. Then, by solving the linear system (), we obtain the unknown coefficients d_k and the results are depicted in Figures 3-5. In Figures 3-4, the L^2-errors

$\|u - u_n\|_2 = \left(\int_0^1 (u(t) - u_n(t))^2\right)^{\frac{1}{2}},$

are illustrated under different values of n, while the $|u(t) - u_n(t)|$ is portrayed in Figure 5. It is evident from the plots that the proposed method exhibits exceptional efficiency and accuracy. For comparison, one can refer to [10], where they obtained similar results for significantly larger values of n, while we have achieved comparable errors for much smaller values of n.

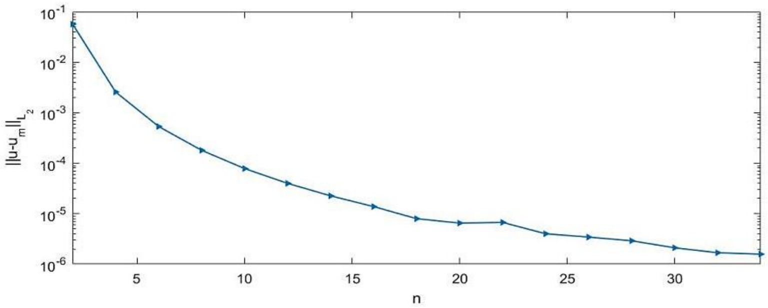

Fig. 3: L_2-error analysis for various case $\alpha = \beta_1 = 0.4, \beta_2 = 0.82, \beta_3 = 0.95, \beta_4 = 1.1$.

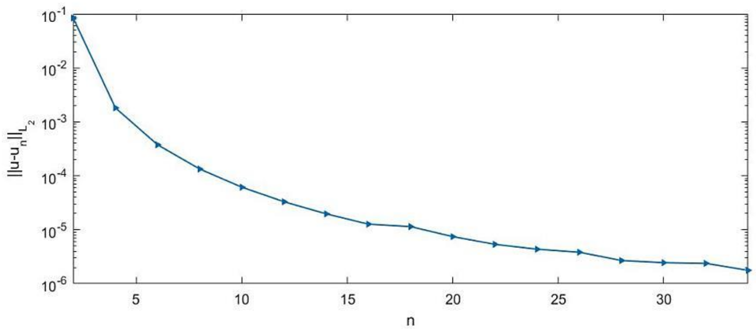

Fig. 4: L_2-error analysis for various case $\alpha = 0.3, \beta_1 = 0.42, \beta_2 = 0.67, \beta_3 = 0.8, \beta_4 = 1.25$.

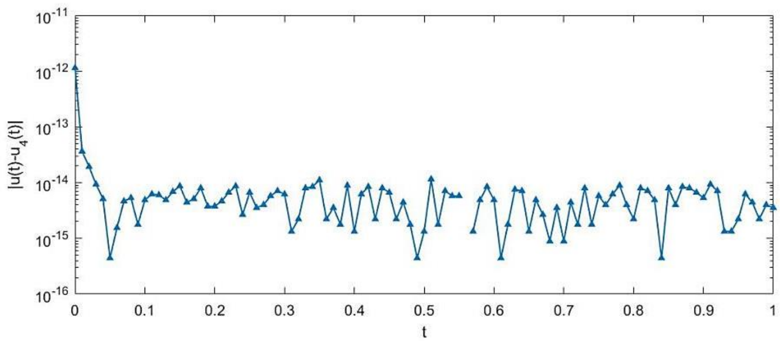

Fig. 5: Graph of $|u(t) - u_4(t)|$ for various the case $\alpha = \beta_1 = 0.1, \beta_2 = 0.2, \beta_3 = 1.1, \beta_4 = 1.2$.

Example 2: Consider the following linear fractional oscillation equation, which can be formulated as [7, 11]:

${}_0^C D_t^\alpha u(t) + u(t) = 0, \;\; u(0) = 1, \; t \in (0, T].$

The exact solution to this problem is $u(t) = E_{\alpha,1}(-t^\alpha)$, where $E_{\alpha,\beta}(.)$ is the Mittag-Leffler function defined as

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$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}$.

The fractional differential equation was solved for various values of the parameter α using the proposed method. By introducing the variable transformation $t := \frac{t}{T}$, the FDE was reformulated, and the proposed method was extended to an arbitrary interval $(0, T)$. In Figure 6, the comparison between the exact solution and the numerically obtained solution is depicted for $n = 20$. It is observed that the numerical solution rapidly converges to the exact solution. Additionally, in Figure 7, the L_2-errors are presented for different values of n and α. The error significantly diminishes with increasing values of n. Furthermore, Table 1 compares the error obtained from our method with several methods presented in [10, 14, 15]. The results indicate that our approach yields substantially more accurate results compared to the methods proposed in previous works. It is crucial to note that our method achieves lower errors, especially for very small values of n.

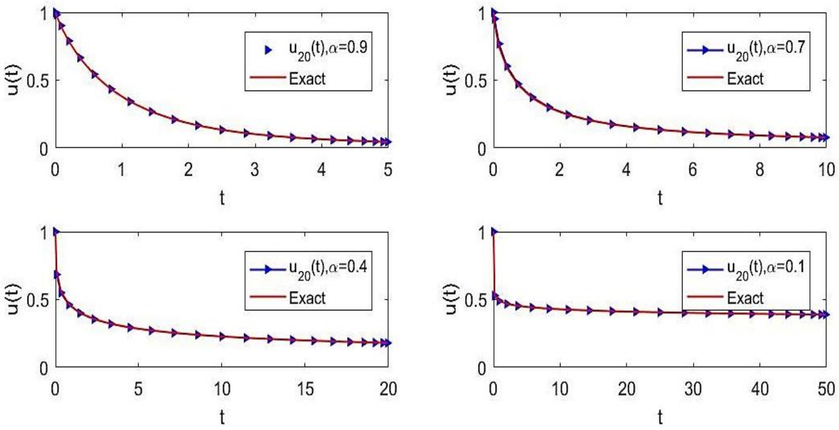

Fig. 6: Comparison between Exact and Numerical solutions for different values of α and T.

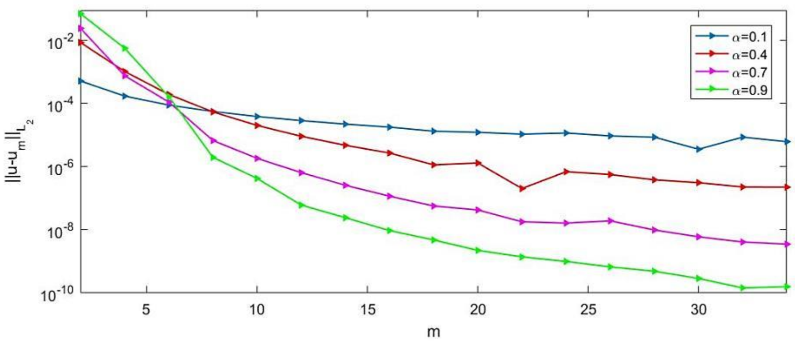

Fig. 7: L_2-errors of the presented method for various values of α and $T = 1$.

Table 1: Comparison errors using the present method and methods in [11,14-15].

$\alpha = 0.3$					$\alpha = 0.6$					
N	In [10]	In [14]	n	Present Method	N	In [10]	In [14]	In [15]	n	Present Method
80	4.67e-05	4.87e-04	5	5.31e-04	80	4.77e-06	1.35e-05	3.36e-05	5	1.41e-04
160	1.45e-05	2.64e-04	10	6.08e-05	160	1.20e-06	4.37e-05	8.49e-06	10	3.14e-06
320	4.40e-06	1.31e-04	20	8.09e-06	320	3.03e-07	1.42e-05	2.13e-06	20	9.12e-08
640	1.31e-06	6.16e-04	34	2.18e-07	640	7.60e-08	4.66e-06	5.35e-07	30	1.51e-08

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Example 3: Consider the following FDE of order $0 < \alpha < 1$, [7]

$${}_0^cD_t^\alpha \sin(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2),\ u(0) = 0, t \in (0,1].$$

This problem is solved using the presented methods with $\beta_1 = 0.9$ and $\beta_2 = 1$. We show the L_2-errors for some values of α in Figure 8. We observe that the exponential convergence rate is obtained. Also in Table 2, we compare the error results derived from the present methods and presented in [7]. The superiority of the present work compared to the previous methods can be seen in this Table.

Table 2: Comparison errors using the present method and method in [8].

	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
n	In [7]	Present Method	In [7]	Present Method	In [7]	Present Method
5	1.17e-01	5.02e-04	2.51e-01	1.21e-03	4.64e-01	1.42e-03
10	4.06e-04	6.37e-07	1.29e-03	5.69e-06	3.14e-03	9.40e-05
15	43.01e-07	2.17e-12	1.08e-07	4.18e-10	3.32e-07	1.88e-08
20	1.02e-11	1.71e-13	8.76e-11	6.82e-13	3.49e-11	1.32e-11

Remark 2: In this section, three numerical examples are considered to evaluate the efficiency and accuracy of the proposed method.

In the first example, the solution of the equation contains four singular terms. The results demonstrated that the method presented in this paper is fully compatible with such singular solutions and can accurately capture at least two of these singular terms. Classical polynomial-based methods exhibit limited accuracy when dealing with such problems.

The numerical results of the second example indicated that the spectral method proposed in this work yields superior results compared to finite difference methods and piecewise polynomial-based methods. With significantly fewer basis functions, a suitable level of accuracy can be achieved.

Furthermore, the third example was designed to compare the proposed method with generalized Jacobi polynomials. Such bases are only capable of capturing a single term from the irregular solution. Overall, it can be concluded that the method presented in this study is considerably more effective than piecewise methods or classical polynomial-based methods, including Jacobi and Chebyshev polynomials.

5. Discussions and Conclusions

In this study, we developed novel basis functions from Müntz spaces to tackle numerical solutions of FDEs.

These functions, developed using singular indices related to the unknown solutions in Fractional Differential Equations (FDEs), excel at capturing the first two singular indices. Our method has shown impressive convergence rates and efficiency, as validated by three numerical examples. In the first example, our method accurately identified at least two out of four singular terms, outperforming traditional polynomial-based methods. The second example demonstrated that our spectral method achieved superior results with fewer basis functions compared to finite difference and piecewise polynomial methods. In the third example, our method proved superior to generalized Jacobi polynomials, which were only able to capture a single term from the irregular solution.

In conclusion, the proposed method delivers a precise and efficient solution for Fractional Differential Equations (FDEs), offering substantial enhancements over traditional techniques in terms of accuracy and computational efficiency. This research makes a significant contribution to the field of numerical solutions for FDEs by introducing a dependable and effective technique, thus serving as a valuable resource for future research and practical applications in solving differential equations with singularities.

Future work can enhance the proposed method for solving FDEs in several directions. One potential area is the extension to higher-dimensional problems, utilizing Müntz space basis functions to expand the method's applicability. Another avenue is the development of adaptive algorithms that dynamically select the most effective basis functions from Müntz spaces, thereby improving both efficiency and accuracy. Integrating this approach with other numerical techniques, such as finite element methods, could result in hybrid methods that combine the strengths of multiple approaches. Additionally, applying the method to real-world problems in fields such as physics, engineering, and finance would demonstrate its practical

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utility and robustness in solving complex differential equations. Conducting thorough error analysis and establishing theoretical error bounds would also provide deeper insights into the method's reliability and performance. These suggestions aim to extend the capabilities and applications of the proposed method, making it a valuable tool for addressing a wide range of FDEs in both theoretical and practical contexts.

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Conflict of Interest

The authors declare no conflicts of interest.

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