n-absorbing I-primary ideals in commutative rings

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ABSTRACT

We define a new generalization of n-absorbing ideals in commutative rings called n-absorbing I-primary ideals. We investigate some characterizations and properties of such new generalization. If P is an n-absorbing I-primary ideal of R and √IP = I√P, then √P is a n-absorbing I-primary ideal of R. And if √P is an (n-1)-absorbing ideal of R such that √(I√P) ⊆ IP, then P is an n-absorbing I-primary ideal of R.

المثلالي الأولي المختزل n من نوع I

المثالي الأولي المختزل n من نوع I في الحلقات التبديلية

المثالي الأولي المختزل n من نوع I

راعيا التعميم جدبا للمثالي الأولي المختزل n من نوع I في الحلقات التبديلية والذي يسمى المثالي الأولي المختزل n من نوع I حيث تم الاستقصاء عن بعض الميزات والخصائص لهذا التعميم الجديد. فإذا كانت P مثالي أولي مختزل لـ R فإن √P هو مثالي أولي مختزل لن (n-1) بهما × I ≡ IP، وليستا P مثالي مختزل من نوع I في R فإن √(I√P) ⊆ IP في R. إذا كانت P مثالي مختزل من نوع (n-1) في R.
Introduction

In our article all rings are commutative ring with non-zero identity. In the recent years many generalizations of prime ideals were defined. Here state some of them. The notion of a weakly prime ideal was introduced by Anderson and Smith, where a proper ideal \( P \) of a commutative ring \( R \) is a weakly prime if \( x, y \in R \), and \( 0 \neq xy \in P \) then \( x \in P \) or \( y \in P \) in \([5]\). Ebrahimi Atani and Farzalipour defined the nation of weakly primary ideals \([8]\). The authors in \([6]\) and \([3]\) introduced the notions \( 2 \) — absorbing and \( n \) — absorbing ideals in commutative rings. A proper ideal \( P \) is said to be \( 2 \) — absorbing (or \( n \) — absorbing) ideal if whenever the product of three (or \( n + 1 \)) elements of \( R \) in \( P \), the product of two (or \( n \)) of these elements is in \( P \).

Throughout the paper the notation \( b_1 \cdots b_n \) means that \( b_i \) is excluded from the product \( b_1 \cdots b_n \). A proper ideal \( P \) of \( R \) is an \( n \) — absorbing \( I \) — primary ideal if for \( b_1, \cdots, b_{n+1} \in R \) such that \( b_1 \cdots b_{n+1} \in P \) — IP , then \( b_1 \cdots b_n \in P \) or \( b_1 \cdots b_i b_{i+1} \cdots b_{n+1} \in \sqrt{P} \) for some \( i \in \{1, 2, \cdots, n\} \) where \( \sqrt{P} \) is the radical of the ideal \( P \).

Assume that \( R \) is an integral domain with quotient field \( F \). The authors in \([7]\) introduced a “proper ideal \( P \) of \( R \) is a strongly primary if, whenever \( cd \in P \) with \( c, d \in F \), we have \( c \in P \) or \( d \in \sqrt{P} \). In \([9]\), a proper ideal \( P \) of \( R \) is a strongly \( I \) — primary ideal if \( cd \in P \) — IP with \( c, d \in F \), then \( c \in P \) or \( d \in \sqrt{P} \). It is said that a proper ideal \( P \) of \( R \) is quotient \( n \) — absorbing \( I \) — primary” if \( b_1 b_2 \cdots b_{n+1} \in P \) with \( b_1, b_2, \cdots, b_{n+1} \in F \), then \( b_1 b_2 \cdots b_n \in P \) or \( b_1 \cdots \hat{b_i} \cdots b_{n+1} \in \sqrt{P} \) for some \( 1 \leq i \leq n \). Set \( P \) is an ideal of a ring \( R \), let \( P \) be an \( n \) — absorbing \( I \) — primary ideal of \( R \) and \( b_1, \cdots, b_{n+1} \in R \). The statement is that \( (b_1, \cdots, b_{n+1}) \) is an \( I \) — \( (n + 1) \) — tuple of \( P \) if \( b_1 \cdots b_{n+1} \in IP \), \( b_1 b_2 \cdots b_n \in P \) and for any \( 1 \leq i \leq n \), \( b_1 \cdots \hat{b_i} \cdots b_{n+1} \in \sqrt{P} \).

2. \( n \) — absorbing \( I \) — primary ideals

In this section, we start with to define the definition of an \( n \) — absorbing \( I \) — primary ideal of a ring \( R \).

Definition: A proper ideal \( P \) of \( R \) is an \( I \) — primary if for \( c, d \in R \) with \( cd \in P \) — IP , then \( c \in P \) or \( d \in \sqrt{P} \).

Definition: A proper ideal \( P \) of \( R \) is an \( n \) — absorbing \( I \) — primary ideal if for \( b_1, \cdots, b_{n+1} \in R \) such that \( b_1 \cdots b_{n+1} \in P \) — IP , then \( b_1 \cdots b_n \in P \) or \( b_1 \cdots b_i b_{i+1} \cdots b_{n+1} \in \sqrt{P} \) for some \( i \in \{1, 2, \cdots, n\} \).

Example 2.1 Consider the ring \( A = k[t_1, t_2, \cdots, t_{n+2}] \), where \( k \) is a field and suppose that \( P = (t_1 t_2 \cdots t_n, t_1^2 t_2 \cdots t_n, t_1 t_2 \cdots t_n) \). Then \( P \) is an \( n \) — absorbing \( I \) — primary ideal but \( P \) is not \( n \) — absorbing.

Proposition 2.2 We set that \( R \) is a ring. Based on this, the following statements can be considered equivalent:

(i) \( P \) is an \( n \) — absorbing \( I \) — primary ideal of \( R \);

(ii) For any elements \( \alpha_1, \cdots, \alpha_n \in R \) with \( \alpha_1 \cdots \alpha_n \) not in \( \sqrt{P} \), \( (P: R \alpha_1 \cdots \alpha_n) \subseteq \bigcup_{i=1}^{n-1} \left( (P: R \alpha_1 \cdots \alpha_{i-1} \cdots \alpha_n) \right) \cup (P: R \alpha_1 \cdots \alpha_{n-1}) \cup (P: R \alpha_1 \cdots \alpha_n) \).

Proof. (i) \( \Rightarrow \) (ii) Set \( \alpha_1, \cdots, \alpha_n \in R \) such that \( \alpha_1 \cdots \alpha_n \not\in \sqrt{P} \). Let \( r \in (P: R \alpha_1 \cdots \alpha_n) \). So \( r \alpha_1 \cdots \alpha_n \in P \). If \( r \alpha_1 \cdots \alpha_n \in IP \), then \( r \in (IP: R \alpha_1 \cdots \alpha_n) \). Let \( r \alpha_1 \cdots \alpha_n \not\in IP \). Since \( \alpha_1 \cdots \alpha_n \not\in \sqrt{P} \), either \( r \alpha_1 \cdots \alpha_{n-1} \not\in P \), that is, \( r \in (P: R \alpha_1 \cdots \alpha_{n-1}) \) or for some \( 1 \leq i \leq n - 1 \) we have
Proposition 2.3 If $V$ be a valuation domain with the quotient field $F$. Then all $n$-absorbing $I$–primary ideal of $V$ is a quotient $n$–absorbing $I$–primary ideal of $R$.

Proof. We can certainly assume that $P$ is $n$–absorbing $I$–primary ideal of $V$ , and $a_1a_2\cdots a_{n+1}\in P$ for some $a_1, a_2, ..., a_{n+1}\in V$ such that $a_1a_2\cdots a_{n}\notin P$. If $a_{n+1}\notin V$, then $a_1a_2\cdots a_{n}\in P$. So $a_1a_2\cdots a_{n+1}a_{n+1}^{-1}=a_1\cdots a_{n}\in P$, which is a contradiction. So $a_{n+1}\in V$. If $a_i\notin V$ for all $1\leq i\leq n$, then there is nothing to prove. If $a_i\notin V$ for some $1\leq i\leq n$, then $a_1\cdots a_i\cdots a_{n+1}\in P\subseteq \sqrt{P}$. Consequently, $P$ is a quotient $n$–absorbing $I$–primary. □

Proposition 2.4 Set $P$ be an $n$–absorbing $I$–primary ideal of $R$ such that $\sqrt{IP}=I\sqrt{P}$, then $\sqrt{IP}$ is a $n$–absorbing $I$–primary ideal of $R$.

Proof. Let us assume $a_1a_2\cdots a_{n+1}\in \sqrt{P} - I\sqrt{P}$ for some $a_1, a_2, ..., a_{n+1}\in R$ such that $a_1\cdots a_i\cdots a_{n+1}\notin \sqrt{P}$ for every $1\leq i\leq n$. Thus, we have $n\in N$ such that $a_1^n a_2^n \cdots a_{n+1}^n \in P$. If $a_1^n a_2^n \cdots a_{n+1}^n \in IP$, then $a_1a_2\cdots a_{n+1}\in I\sqrt{P}$, which is a contradiction. Since $P$ is an $n$–absorbing $I$–primary, our hypothesis implies $a_1^n a_2^n \cdots a_{n}^n \in P$. So $a_1a_2\cdots a_{n}\in \sqrt{P}$ and $\sqrt{P}$ is an $n$–absorbing $I$–primary ideal of $R$. □

Theorem 2.5 Assume that “for any $1\leq i\leq k$, $I_i$ is an $n_i$–absorbing $I$–primary ideal of $R$ such that $\sqrt{P_i}=q_i$ is an $n_i$–absorbing $I$–primary ideal of $R$, respectively. Let $n=n_1+n_2+\cdots+n_k$. The following statements do hold:

(1) $P_1\cap P_2\cap \cdots \cap P_k$ is an $n$–absorbing $I$–primary ideal of $R$.

(2) $P_1P_2\cdots P_k$ is an $n$–absorbing $I$–primary ideal of $R$.

Proof. The proof of the two parts is similar, so we prove just the first. Let $H=P_1\cap P_2\cap \cdots \cap P_k$. Then $\sqrt{H}=P_1\cap P_2\cap \cdots \cap P_k$. Let $a_1a_2\cdots a_{n+1}\in H-IH$ for some $a_1, a_2, ..., a_{n+1}\in R$ and $a_1\cdots a_i\cdots a_{n+1}\notin \sqrt{H}$ for any $1\leq i\leq n$. By, $\sqrt{H}=P_1\cap P_2\cap \cdots \cap P_k$ is an $n$–absorbing $I$–primary, then $a_1a_2\cdots a_{n}\in P_1\cap P_2\cap \cdots \cap P_k$. We prove that $a_1a_2\cdots a_{n}\in H$. For all $1\leq i\leq k$, $P_i$ is an $n_i$–absorbing $I$–primary and $a_1a_2\cdots a_{n}\in P_i-I\overline{P}_i$ , then we have $1\leq \beta_1^i, \beta_2^i, ..., \beta_n^i \leq n$ such that $a_1^i a_2^i \cdots a_{n}^i \in P_i$. If $\beta_1^i = \beta_2^i = \cdots = \beta_n^i = m$ it is for two couples $l, r$ and $m, s$, then $a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_{\overline{n}}^m a_{\overline{n}}^m \cdots a_{\overline{n}}^m a_{\overline{n}}^m a_{\overline{n}}^m a_{\overline{n}}^m \in \sqrt{H}$.

Therefore $a_1^l a_2^l \cdots a_{n}^l a_{n}^l \cdots a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_{\overline{n}}^m a_{\overline{n}}^m \cdots a_{\overline{n}}^m a_{\overline{n}}^m a_{\overline{n}}^m a_{\overline{n}}^m \in \sqrt{H}$, which is a contradiction. So $\beta_1^i$ 's is distinct. Hence $\{a_1^l a_2^l \cdots a_{n}^l a_{n}^l \cdots a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_{\overline{n}}^m a_{\overline{n}}^m \cdots a_{\overline{n}}^m a_{\overline{n}}^m a_{\overline{n}}^m a_{\overline{n}}^m \}$ = $\{a_1, a_2, ..., a_n\}$. If $a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_1^l a_1^r \cdots a_{n}^l a_{n}^r \cdots a_{\overline{n}}^m a_{\overline{n}}^m \cdots a_{\overline{n}}^m a_{\overline{n}}^m a_{\overline{n}}^m a_{\overline{n}}^m \in H$, then $a_1a_2\cdots a_{n}\in H$. □
thus, we are done. Therefore, we may assume that $a_{\beta_1}a_{\beta_2^*}\cdots a_{\beta_{n+1}} \notin P_1$. Since $P_1$ is $I$ – absorbing
$I$ – primary and
\[ a_{\beta_1}a_{\beta_2^*}\cdots a_{\beta_{n+1}} \in P_2 \implies a_{\beta_1}a_{\beta_2^*}\cdots a_{\beta_{n+1}} \in P_2 \cap \cdots \cap P_k. \]
Consequently
\[ a_{\beta_1}a_{\beta_2^*}\cdots a_{\beta_{n+1}} \in \sqrt{H}, \]
which is a contradiction. Similarly, $a_{\beta_1}a_{\beta_2^*}\cdots a_{\beta_{n+1}} \in P_1$ for every
$2 \leq i \leq k$. Then $a_1a_2\cdots a_n \in H$.

**Proposition 2.6** Assume that $P$ is an ideal of a ring $R$ with $\sqrt{IP} \subseteq IP$. If $\sqrt{P}$ is an $(n-1)$ – absorbing ideal of $R$, then $P$ is an $n$ – absorbing $I$ – primary ideal of $R$.

**Proof.** Let $\sqrt{P}$ be an $(n-1)$ – absorbing, and consider $b_1b_2\cdots b_{n+1} \in P \setminus IP$ for some $b_1, b_2, ..., b_{n+1} \in R$ and $b_1b_2\cdots b_n \notin P$. Since
\[ (b_1b_{n+1})(b_2b_{n+1})\cdots (b_nb_{n+1}) = (b_1b_2\cdots b_n)^n b_{n+1} \in P \subseteq \sqrt{P} - I\sqrt{P}. \]
Then for some $1 \leq i \leq n$,
\[ (b_1b_{n+1}) \cdots (b_i\cdots b_n) = (b_1\cdots b_{n+1}) \] and so $b_1\cdots b_i \cdots b_{n+1} \in \sqrt{P}$. Consequently $P$ is an
$n$ – absorbing $I$ – primary ideal of $R$.

We recall that a proper ideal $Q$ of $R$ is an $n$ – absorbing primary if $a_1, a_2, ..., a_{n+1} \in R$
and $a_1a_2\cdots a_{n+1} \in Q$, then $a_1a_2\cdots a_n \in Q$ or the
product of $a_{n+1}$ with $(n-1)$ of $a_1, a_2, ..., a_n$ is in
$\sqrt{Q}$. It is clearly every $n$ – absorbing primary is an $n$ – absorbing $I$ – primary.

**Proposition 2.7** Suppose that $R$ is a ring and $r \notin R$, a nonunit and $m \geq 2$ is not negative integer. Let
$(0: r) \subseteq (a)$, then $(r)$ is an $n$ – absorbing $I$ – primary, for some $I$ with $IP \subseteq I^m$ if and only if $(a)$ is
an $n$ – absorbing primary.

**Proof.** Let $(r)$ be an $n$ – absorbing $I^m$ – primary, and
$a_1a_2\cdots a_{n+1} \in (r)$ for some $a_1, a_2, ..., a_{n+1} \in R$. If
$a_1a_2\cdots a_{n+1} \notin (r)$, then $a_1a_2\cdots a_n \in (r)$ or
$a_1\cdots a_{n+1} \in \sqrt{(r)}$ for some $1 \leq i \leq n$. Based on
this assumption, $a_1a_2\cdots a_{n+1} \in (r^m)$ . Hence
$a_1a_2\cdots a_n(a_{n+1} + r) \in (r)$. If $a_1a_2\cdots a_n(a_{n+1} + r) \notin (r^m)$ ,
then $a_1a_2\cdots a_n(a_{n+1} + r) \in (r)$ or
$a_1\cdots a_{n+1} \in \sqrt{(r)}$ for some $1 \leq i \leq n$. So
$a_1a_2\cdots a_n \in (r)$ or $a_1\cdots a_{n+1} \in \sqrt{(r)}$ for
some $1 \leq i \leq n$. Hence, suppose that
$a_1a_2\cdots a_n(a_{n+1} + r) \in (r^m)$. Thus $a_1a_2\cdots a_{n+1} \in (r^m)$ implies that $a_1a_2\cdots a_n \in (r)$.
Therefore, there exists $s \in R$ such that $a_1a_2\cdots a_n \neq sr^{m-1} \in
(0: r) \subseteq (r)$. Consequently $a_1a_2\cdots a_n \in (r)$.

**Proposition 2.8** Assume $V$ is a valuation domain and
$n \in \mathbb{N}$. Let $P$ be an ideal of $V$ such that $P^{n+1}$ is not principal. Then $P$ is an $n$ – absorbing $I^{n+1}$ – primary if
and only if it is an $n$ – absorbing primary.

**Proof.** ($\Rightarrow$) Let $P$ be an $n$ – absorbing $I^n$ – primary that is not $n$ – absorbing primary. Therefore, there are
$a_1, a_2, ..., a_{n+1} \in R$ such that $a_1\cdots a_{n+1} \in P$, but neither
$a_1\cdots a_n \in P$ nor $a_1\cdots a_{n+1} \in \sqrt{P}$ for any $1 \leq i \leq n$. Hence $(a_i) \nsubseteq P$ for any $1 \leq i \leq n + 1$. And so
$V$ is a valuation domain, thus $P \subseteq (a_i)$ for any $1 \leq i \leq n + 1$, and so $P^{n+1} \subseteq (a_1\cdots a_{n+1})$. Therefore
$P^{n+1}$ is not principal, then $a_1\cdots a_{n+1} \in P - P^{n+1}$. Therefore $P$ is an $n$ – absorbing $I^{n+1}$ – primary implies that either $a_1\cdots a_n \in P$ or $a_1\cdots a_{n+1} \in \sqrt{P}$ for some $1 \leq i \leq n$, which is a contradiction. Hence $P$ is an $n$ – absorbing primary ideal of $R$.

($\Leftarrow$) Is trivial.

**Theorem 2.9** We consider that $f \subseteq P$ are a proper ideal of a ring $R$. 

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1. Let $P$ is an $n$-absorbing $I$-primary ideal of $R$, then $P/J$ is a $n$-absorbing $I$-primary ideal of $R/J$.

2. Let $I \subseteq IP$ and $P/J$ be an $n$-absorbing $I$-primary ideal of $R/J$, then $P$ is an $n$-absorbing $I$-primary ideal of $R$.

3. Let $IP \subseteq J$ and $P$ be an $n$-absorbing $I$-primary ideal of $R$, then $P/J$ is a weakly $n$-absorbing primary ideal of $R/J$.

4. Let $JP \subseteq IP$, $J$ be an $n$-absorbing $I$-primary ideal of $R$ and $P/J$ be a weakly $n$-absorbing primary ideal of $R/J$, then $P$ is an $n$-absorbing $I$-primary ideal of $R$.

**Proof.** (1) Set $b_1, b_2, \ldots, b_{n+1} \in R$ such that $(b_1 + J)(b_2 + J) \cdots (b_{n+1} + J) \in (P/J) - I(P/J) = (P/J) - (I(P) + J)/J$. Then $b_1b_2 \cdots b_{n+1} \in P - IP$ and from being $P$ is an $n$-absorbing $I$-primary, we obtain $b_1 \cdots b_n \in P$ or $b_1 \cdots b_i \cdots b_{n+1} \in \sqrt{P}$ for some $1 \leq i \leq n$. And so $(b_1 + J) \cdots (b_{i-1} + J)(b_{i+1} + J) \cdots (b_{n+1} + J) \in (P/J) - (I(P) + J)/J$. Then $b_1b_2 \cdots b_{n+1} \in P - IP$ and from being $P/J$ is an $n$-absorbing $I$-primary, we obtain $b_1 \cdots b_i \cdots b_{n+1} \in \sqrt{P}$ for some $1 \leq i \leq n$. Therefore $b_1 \cdots b_n \in P$ or $b_1 \cdots b_i \cdots b_{n+1} \in \sqrt{P}$ for some $1 \leq i \leq n$, hence $P$ is an $n$-absorbing $I$-primary ideal of $R$.

(2) Resulted directly from part (1).

(4) Set $b_1 \cdots b_{n+1} \in P - IP$ where $b_1, b_2, \ldots, b_{n+1} \in R$. Note that $b_1 \cdots b_{n+1} \in JP$ because $JP \subseteq IP$. If $b_1 \cdots b_{n+1} \in J$, then either $b_1 \cdots b_n \in J \subseteq P$ or $b_1 \cdots b_i \cdots b_{n+1} \in \sqrt{J} \subseteq \sqrt{P}$ for some $1 \leq i \leq n$, since $J$ is an $n$-absorbing $I$-primary. If $b_1 \cdots b_{n+1} \notin J$, then $(b_1 + J)(b_2 + J) \cdots (b_{n+1} + J) \in (P/J) - (I(P) + J)/J$ and so either $(b_1 + J)(b_2 + J) \cdots (b_n + J) \in P/J$ or $(b_1 + J)(b_2 + J) \cdots (b_i + J)(b_{i+1} + J) \cdots (b_{n+1} + J) \in \sqrt{P}/J = \sqrt{P}/J$ for some $1 \leq i \leq n$. Therefore $b_1 \cdots b_n \in P$ or $b_1 \cdots \hat{b}_i \cdots b_{n+1} \in \sqrt{P}$ for some $1 \leq i \leq n$. Hence $P$ is an $n$-absorbing $I$-primary ideal of $R$.

(3) Resulted directly from part (1).

**Proposition 2.10** Suppose that $P$ is an ideal of a ring $R$ such that $IP$ is an $n$-absorbing primary ideal of $R$.

If $P$ is an $n$-absorbing $I$-primary ideal of $R$, then $P$ is an $n$-absorbing primary ideal of $R$.

**Proof.** Let $a_1, a_2, \ldots, a_{n+1} \in P$ for some elements $a_1, a_2, \ldots, a_{n+1} \in P$ such that $a_1, a_2, \ldots, a_n \notin P$. If $a_1, a_2, \ldots, a_{n+1} \in IP$, then $P = IP$ is an $n$-absorbing primary and $a_1, a_2, \ldots, a_n \notin IP$ implies that $a_1, a_2, \ldots, a_{n+1} \in \sqrt{IP} \subseteq \sqrt{P}$ for some $1 \leq i \leq n$, and so we are done. When $a_1, a_2, \ldots, a_{n+1} \in IP$ clearly the result follows.

**Theorem 2.11** If $P$ is an $n$-absorbing $I$-primary ideal of a ring $R$ and $(a_1, \ldots, a_{n+1})$ is an $I - (n + 1)$-tuple of $P$ for some $a_1, \ldots, a_{n+1} \in R$. Then for every element $a_1, a_2, \ldots, a_m \in \{1, 2, \ldots, n + 1\} $ which $1 \leq m \leq n$,

$$ a_1 \cdot \hat{a}_1 \cdot a_2 \cdot \hat{a}_2 \cdot \ldots \cdot \hat{a}_m \cdot \hat{a}_{n+1} \cdot m \subseteq IP $$

**Proof.** We claim that by using induction on $m$. We take $m = 1$ and assume $a_1 \cdots \hat{a}_1 \cdots a_{n+1}x \notin IP$ for some $x \in P$. Then $a_1 \cdots \hat{a}_1 \cdots a_{n+1}(a_{n+1} \cdot x) \notin IP$. Since $P$ is an $n$-absorbing $I$-primary ideal of $R$ and $a_1 \cdots \hat{a}_1 \cdots a_{n+1} \notin P$, we conclude that $a_1 \cdots \hat{a}_1 \cdots a_{n+1} \cdot (a_{n+1} \cdot x) \in \sqrt{P}$, for some $1 \leq a_2 \leq n + 1$ different from $a_1$. Hence $a_1 \cdots \hat{a}_1 \cdots a_{n+1} \in \sqrt{P},$ a contradiction. Thus $a_1 \cdots \hat{a}_1 \cdots a_{n+1} \cdot P \subseteq IP$. Here assume that $m > 1$ and for every integer less than $m$ the prove does hold. Let $a_1 \cdots \hat{a}_1 \cdots a_{n+1} \cdot x_1, x_2, \ldots, x_m \notin IP$ for some $x_1, x_2, \ldots, x_m \in P$. According to the induction assumption, we conclude that there exists $\zeta \in IP$ such that
\[ a_1 \cdots a_{n-1} a_n + x_1 \in \sqrt{\mathcal{P}} \]

\[ \mathcal{P} = (a_1, a_2, \ldots, a_n) \subseteq \mathcal{M} \]

**Proof.** (i) Since \( \mathcal{P} \) is assumed not to be an \( n \)-absorbing primary ideal of \( R \), so \( \mathcal{P} \) has an \( (n+1) \)-tuple zero \((b_1, \ldots, b_{n+1})\) for some \( b_1, \ldots, b_{n+1} \in R \). Let \( c_1 c_2 \cdots c_{n+1} \notin \mathcal{P} \) for some \( c_1, c_2, \ldots, c_{n+1} \in \mathcal{P} \). Therefore, according to the Theorem 2.11, there is \( \lambda \in \mathcal{P} \) such that \( (b_1 + c_1) \cdots (b_{n+1} + c_{n+1}) = \lambda + c_1 c_2 \cdots c_{n+1} \notin \mathcal{P} \). Hence either \( (b_1 + c_1) \cdots (b_n + c_n) \in \mathcal{P} \) or \( (b_1 + c_1) \cdots (b_i + c_i) \cdots (b_{n+1} + c_{n+1}) \in \sqrt{\mathcal{P}} \) for some \( 1 \leq i \leq n \). Thus either \( b_1 \cdots b_n \in \mathcal{P} \) or \( b_1 \cdots \hat{b}_i \cdots b_{n+1} \in \sqrt{\mathcal{P}} \) for some \( 1 \leq i \leq n \), which is a contradiction. Hence \( \mathcal{P}^{n+1} \subseteq \mathcal{P} \).

(ii) Clearly, \( \sqrt{\mathcal{P}} \subseteq \mathcal{P} \). As \( \mathcal{P}^{n+1} \subseteq \mathcal{P} \), we obtain \( \sqrt{\mathcal{P}} \subseteq \sqrt{\mathcal{P}} \), we are done.

**References**


