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# Nano βθ − Open Sets in Nano Topological Spaces

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## ABSTRACT

This paper's primary objective is to present and study a different class of nano sets known as nano βθ-open sets in nano topological spaces. Additionally, introduced some nano topological concepts by using nano βθ-open set.

# المجموعات المفتوحة النانوية من النمط بيتا ثيتا في الفضاءات التبولوجية النانوية

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**الملخص**

الهدف الأساسي من هذه الدراسة هو تقديم ودراسة فئة مختلفة من مجموعات نانو المعروفة باسم مجموعات النانو المفتوحة من النمط βθ في الفضاءات النانو التبولوجية. بالإضافة إلى ذلك، تم تقديم بعض المفاهيم التبولوجية النانوية باستخدام مجموعة النانو المفتوحة من النمط βθ.

## Introduction

The concepts of β-open sets or semi preopen sets, which were investigated by Abd El Monsef et al. and Andrijevi [1], respectively, are among the most well-known notions and sources of inspiration. Besides. In 2003, Noiri [2] introduced the concept of β-closure of a set to introduce the notions of βθ-open and βθ-closed sets which provide a formulation of the "βθ-closure" of a set in a topological space, using the notions "βθ-open and βθ-closed sets". Using βθ-closed sets, Caldas, Jafari, and Ekici expanded on Noiri's work and established additional concepts. On the other hand, the study of nano topological spaces introduced by Lellis Thivagar and Richard [3], which defined approximations and the boundary region of a subset of the finite universe using an equivalence class as well as defined nano closed sets, nano interior, and nano closure operator. (Nα-open, nano semi open, nano preopen, and nano regular open) sets are the weak variants of nano open sets that he has introduced. Nano open sets refer to the constituent parts of nano topological space. Its root is the Greek word "Nanos," which in modern science refers to a dwarf that is an order of magnitude measuring one billionth. Due to its size and the fact that it has no more than five elements, the topology is known as a nano topology. Additionally, in 2015, Revathy and Ilango [4]

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expanded on the earlier research by proposing the idea of Nβ-open sets. In this work, the researchers introduce and explore a new notion known as Nβθ-open sets, which are both stronger than Nβ-open sets as well as weaker than nano regular β-open sets. further types of Nβθ-open set are investigated in various cases of approximations. In this work, we symbolized nano topological space by (U,τ_R (X)) or U which, unless explicitly stated, does not consider any separation axioms.

We focus on gathering definitions and theorems that are utilized in the sequel in order to make this work as independently as possible. Nearly all of these sections are very well.

**Definition 1.1.** [4] The $nano\ \beta - open$ set is a subset $S$ of $U$ where $S \subset NCl[NInt(NClS)]$.

**Definition 1.2.** [3] The $nano\ regular - open$ (resp., $nano\ pre - open$, $nano\ semi - open$, $nano\ \alpha - open$) set is a subset $S$ of $U$ where $S = NInt(NClS)$ (resp., $S \subset NInt(NClS)$, $S \subset NCl(NIntS)$, $S \subset NInt[NCl(NIntS)]$).

**Definition 1.3.** [5] The $nano\ regular\ semi - open$ set is a subset $S$ of $U$ where $S = NsInt(NsClS)$.

**Definition 1.4.** [6] The $nano\ \theta - open$ (resp., $nano\ semi\ \theta - open$, $nano\ pre\ \theta - open$) set is a subset $S$ of $U$, $\exists$ $nano\ open$ (resp., $nano\ semi\ open$, $nano\ preopen$) set $G$ where $x \in G \subset NClG \subset S$ (resp., $x \in G \subset NSClG \subset S$ , $x \in G \subset NPClG \subset S$).

**Definition 1.5.** [7] The $nano\ \delta - open$ (resp., $nano\ pre\ \delta - open$ ) set is a subset $S$ of $U$, $\exists$ $nano\ open$ ($resp., nano\ preopen$) $set$ $G$ where $x \in G \subset NInt(NClG) \subset S$ (resp., $x \in G \subset NPInt(NPClG \subset S)$.

The family of all $nano\ \beta - open$ (resp., $nano\ regular - open$, $nano\ pre - open$, $nano\ semi - open$, $nano\ regular\ semi - open$, $nano\ \alpha - open$, $nano\ \theta - open$, $nano\ \delta - open$, $nano\ semi\ \theta - open$, $nano\ pre\ \theta - open$, $nano\ pre\ \delta - open$) sets of a nano space $(U, \tau_{\mathcal{R}}(X))$ symbolized by $N\beta O(U,X)$ (resp., $NRO(U,X)$, $NPO(U,X)$, $NSO(U,X)$, $NRSO(U,X)$, $N\alpha O(U,X)$, $N\theta O(U,X)$, $N\delta O(U,X)$, $NS\theta O(U,X)$, $NP\theta O(U,X)$, $NP\delta O(U,X)$).

**Definition 1.6.** [8] The $nano\ regular\ \beta - open$ (resp. $nano\ pre - regular\ p - open$, $nano\ regular\ pre - open$) set is a subset $S$ of $U$ , where $S = N\beta Int(N\beta ClS)$ (resp. $S = NpInt(NpClS)$, $S$ is nano pre-open as well as nano pre-closed). And the family of all $nano\ regular\ \beta - open$ (resp. $nano\ pre - regular\ p - open$, $nano\ regular\ pre - open$) sets of $(U, \tau_{\mathcal{R}}(X))$ symbolized by $NR\beta O(U,X)$ (resp. $NPRPO(U,X)$ , $NRPO(U,X)$).

**Remark1.7** $NInt[NCl(NIntA)] \subset N\beta Int(N\beta clA) \subset Ncl[NInt(NclA)]$

**Theorem 1.8.** Assume that $S \subset U$, then $S \in NR\beta O(U,X)$ if and only if $S \in N\beta O(U,X) \cap N\beta C(U,X)$.

**Proof:** Let $S \in NR\beta O(U,X)$, then $S = N\beta Int(N\beta ClS)$. By Remark 1.7, $NInt[NCl(NIntS)] \subset N\beta Int(N\beta ClS) \subset NCl[NInt(NClS)]$. Hence, $S \in N\beta O(U,X) \cap N\beta C(U,X)$. Conversely, if $S \in N\beta O(U,X) \cap N\beta C(U,X)$ then $S = N\beta Int(N\beta ClS)$ . Therefore, $S$ is $NR\beta O(U,X)$.

**Theorem 1.9.** [9] Assume that $S \subset X$, then $S \in \beta O(X)$ if and only if $\beta ClS \in R\beta O(X)$.

**Theorem1.10.** [10] For each $S \subset X$, $\beta ClS \subset PClS$ and $\beta ClS \subset SClS$.

**Theorem 1.11.** [11] A subset $S \subset X$ is $\delta - open$ (resp. $P\delta$-open) if and only if $S$ is union of regular open (*resp.* pre-regular p-open) sets in $X$.

**Theorem 1.12.** [12] For any space $X$, PRPO($X$) $\subset$ RβO($X$).

**Theorem 1.13.** [4] If $U_r(X) = U$ in $(U, \tau_{\mathcal{R}}(X))$ then $N\beta O(U,X) = P(U)$.

**Theorem 1.14.** [4] If $U_r(X) \neq U$ then any set that intersects with $U_r(X)$ are $N\beta O(U,X)$ in $U$.

**Theorem 1.15.** [13] $S \cap B \in R\beta O(X)$, when $S \in RO(X)$ and $B \in R\beta O(X)$.

**Theorem 1.16.** [9] A space $X$ is extremely disconnected if and only if $PO(X) = \beta O(X)$.

**Theorem 1.17.** [9] For any space $X$, $\theta O(X, PO(X)) = P\theta O(X)$.

**Theorem 1.18.** [9] If $S_i \in R\beta O(X_i)$ for i = 1,2 then $S_1 \times S_2 \in R\beta O(X_1 \times X_2)$.

**Theorem 1.19:** For any nano topological space, $S \in N\delta O(U,X)$ if and only if $\forall\, x \in S$, $\exists\, M \in NRO(U,X)$ where $x \in M \subset S$.

**Proof:** Let $S \in N\delta O(U,X)$, by Definition of $N\delta - open$ set there exists a Nano open set $V$ where $x \in V \subset NInt(NClV) \subset S$, taking $M = NInt(NClV)$, by Definition of *nano regular*, $M$ is *nano regular* and $x \in M \subset S$. *Conversely, for each* $x \in S$, *there exists* $M \in NRO(U,X)$ *where* $x \in M \subset S$. *Therefore,* $\forall x \in S, \exists M \in NRO(U,X)$ *where* $x \in M \subset NInt(NClM) \subset S$. *Hence,* $S \in N\delta O(U,X)$.

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**Theorem 1.20.** A subset S of $(U,\tau_R(X))$ is $N\delta-open$ (resp. $NP\delta$-open) if and only if $S$ is union of nano regular open (resp. pre-regular p-open) sets in $(U,X)$.

**Proof.** Assume that $S\in N\delta O(U,X)$ $\left(resp. S\in NP\delta O(U,X)\right)$. For each $x\in S$, by Theorem 1.19, $\exists\, M_x\in\tau_R(X)$ $\left(\text{resp.} M_x\in NPRPO(U,X)\right)$ where $x\in M_x\subset S$. Therefore, $\bigcup_{x\in S}\{x\}\subset\bigcup_{x\in S}M_x\subset S$ . Hence, $S=\bigcup M$ .

Conversely, suppose that $S$ is union of nano open $(\text{resp. nano pre}-\text{regular P}-\text{open})$ sets $\{M_\lambda,\lambda\in\Delta\}$. Let $x\in S$ then, $x\in M_{\lambda_0}$ for some $\lambda_0\in\Delta$ and $\mathrm{M}_{\lambda_0}$ is nano open $(\text{resp. nano pre}-\text{regular P}-$ open, Therefore $\mathrm{x}\in\mathrm{M}_{\lambda_0}\subset\mathrm{S}$. By Theorem 1.19, $S\in N\delta O(U,X)$ $\left(resp. S\in NP\delta O(U,X)\right)$.

**Theorem 1.21.** [9] *In any space* $X$, $RO\left(X,\beta O(X)\right)=R\beta O(X)$.

**Theorem 1.22.** [9] *Let* $S\in\alpha O(X)$ *and* $M\in R\beta O(X)$, *then* $S\cap M\in R\beta O(S)$.

**Theorem 1.23.** If $(U,\tau_R(X))$ *is extremely disconnected then* $NP\delta O(U,X)=NP\theta O(U,X)$.

***Proof:*** *Let* $S\in NP\delta O(U,X)$, *then by Theorem 1.20,* $S$ *is* a union of $NPRP-open$. Since, $\forall x\in S,\exists V\in NPO(U,X)$ where $x\in V\subset NPInt(NPClV)\subset S$. Since $(U,\tau_R(X))$ is extremely disconnected then $NPClV\in NPO(U,X),\forall V\in NPO(U,X)$. Therefore, $NPInt(NPClV)=NPClV$. Thus, $x\in V\subset NPClV\subset S$. Hence, $S\in NP\theta O(U,X)$.

***Theorem 1.24.*** [4] If $U_r(X)\neq U$ and $L_r(X)\neq\emptyset$ , then any set which intersects $U_r(X)$ are *nano* $\beta-open$ sets in $U$.

## 1. Basic Backgrounds

**Definition 2.1.** [5] Assume that $U\neq\emptyset$ of objects known as the universe and that $\mathcal{R}$ represents an equivalence relation of $U$ called as the indiscernible relation. All the elements in the same equivalence class are assumed to be indiscernibility with each another. The approximation space is stated to consist of the pair $(U,\mathcal{R})$. Let $X\subset U$.

1. The lower approximation of $X$ is symbolized by $L_r(X)$. That is, $L_r(X)=\bigcup_{x\in U}\{\mathcal{R}(x).\mathcal{R}(x)\subset X\}$.
2. The upper approximation of $X$ is symbolized by $U_r(X)$. That is $U_r(X)=\bigcup_{x\in U}\{\mathcal{R}(x).\mathcal{R}(x)\cap X\neq\varphi\}$
3. The boundary region of $X$ is symbolized by $B_r(X)$. That is $B_r(X)=U_r(X)\setminus L_r(X)$.

**Definition 2.2.** [5] Assume that $U$ represents the universe, $\mathcal{R}$ represents an equivalence relation on $U$ and $\tau_{\mathcal{R}}(X)=\{U_r(X),L_r(X),B_r(X),U,\varphi\}$ where $X\subset U$. Then $\tau_{\mathcal{R}}(X)$ satisfies these conditions.

1. $\varphi$ *and* $U$ belong to $\tau_{\mathcal{R}}(X)$,
2. $\bigcup A_\lambda$ is in $\tau_{\mathcal{R}}(X)$ where $A_\lambda$ is any sub-collection of $\tau_{\mathcal{R}}(X)$,
3. $\bigcap A_\lambda$ is in $\tau_{\mathcal{R}}(X)$ where $A_\lambda$ is any finite sub-collection of $\tau_{\mathcal{R}}(X)$.

It implies that $\tau_{\mathcal{R}}(X)$ satisfies the topology's axioms on $U$ and is referred to as nano topology on $U$ with respect to $X$. Additionally, $\left(U,\tau_{\mathcal{R}}(X)\right)$ is known as the nano topological space.

in this work, $NInt(S)$ represents the nano interior of $S$. And $NCl(S)$ denotes the nano closure of $S$.

## 3. Nano βθ-Open Sets

In this section focuses on gathering definition and prove some results about Nano $\beta\theta$-open sets.

**Definition 3.1.** The $Nano\ \beta\theta-open\ set$ is a subset $S$ of $U$, where $\forall\, x\in S$ , $\exists\, G\in N\beta-open$ set where $x\in G\subset N\beta ClG\subset S$ and symbolized by $N\beta\theta O(U,X)$.

**Theorem 3.2.** A subset $S$ of $\left(U,\tau_{\mathcal{R}}(X)\right)$ is $N\beta\theta-open\ set$ if and only if $\forall x\in S$ , $\exists B\in NR\beta O(U,X)$ where $x\in B\subset S$.

**Proof.** Assume that $S\in N\beta\theta O(U,X)$ and $x\in S$ , by Definition of $N\beta\theta-open\ set$, there exists a $N\beta-open\ set\ G$ where $x\in G\subset N\beta ClG\subset S$. Assume $B=N\beta ClG$, by Theorem 1.9, $x\in G\subset N\beta ClG\subset NR\beta O(U,X)\subset S$. Hence, $B$ *is* $NR\beta-open\ set$ and $x\in B\subset S$.

Conversely, assume $x\in S$ there exists a *nano regular* $\beta-open\ set\ G$ where $x\in G\subset S$ , by Theorem 1.8, $G\in N\beta O(U,X)$ and $G\in N\beta C(U,X)$. Since $G$ is nano $\beta$-closed set. Therefore $N\beta ClG=G$. Hence, $x\in G\subset N\beta ClG\subset S$. Therefore, S is $N\beta\theta-open\ set$.

**Theorem 3.3**. A subset $S$ of $\left(U,\tau_{\mathcal{R}}(X)\right)$ is $N\beta\theta-open\ set$ if and only if $S$ is union of *nano regular* $\beta-open\ sets$ in $U$.

**Proof.** Let $S\in N\beta\theta O(U,X)$ , for each $x\in S$ , by Theorem 3.2 , there exists a *nano regular* $\beta-open\ sets\ G_x$ , where $x\in G_x\subset S$ .Therefore, $S=\bigcup_{x\in S}\{x\}\subset\bigcup_{x\in S}G_x\subset S$. Hence, $S=\bigcup_{x\in S}G_x$ .Conversely, let $S=\bigcup_{\lambda\in\Delta}G_\lambda$ , $G_\lambda$ is a nano regular $\beta-$ open sets in $U$ , for each $\lambda\in\Delta$ . let $x\in S$ there exists $\lambda_0\in\Delta$, where $x\in G_{\lambda_0}$ and $G_{\lambda_0}$ is $\mathrm{NR\beta O(U,X)}$. Consequently, $x\in G_{\lambda_0}\subset S$. by Theorem 3.2, $S\in N\beta\theta O(U,X)$.

**Theorem 3.4.** Assume that $S$ represents a subset of $\left(U,\tau_{\mathcal{R}}(X)\right)$, $S\in N\beta O(U,X)$ if and only if $N\beta ClS\in N\beta\theta O(U,X)$.

**Proof.** Consider $S\in N\beta O(U,X)$, then there is $S\subset NCl[NInt(NClS)]$. Hence, $N\beta ClS\subset$

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$N\beta Cl[NCl[NInt(NClS)]]= NCl[NInt(NClS)] \subset NCl[NInt[NCl(N\beta ClS)]]$. Therefore, $N\beta ClS$ is $N\beta - open\ set$ and also $N\beta - closed\ set$. Hence, $N\beta ClS \in N\beta R(U,X)$. Since, $N\beta R(U,X) \subset N\beta\theta O(U,X)$. Thus, $N\beta ClS \in N\beta\theta O(U,X)$. Conversely, if $N\beta ClS \in N\beta\theta O(U,X)$ then $S \in N\beta\theta O(U,X)$ since $S \subset N\beta ClS$. In addition, $N\beta\theta O(U,X) \subset N\beta O(U,X)$. Therefore, $S \in N\beta O(U,X)$.

**Theorem 3.5.** Let $S$ be a subset of $(U,\tau_R(X))$, then $S \in NR\beta(U,X)$ if and only if $S \in N\beta\theta O(U,X) \cap N\beta\theta C(U,X)$.

**Proof:** Let $S \in NR\beta(U,X)$, then $S \in N\beta(U,X)$. Therefore, $S = N\beta Cl(S) = N\beta Cl_\theta(S)$. Hence, $S$ is $N\beta\theta$ −closed. Since, $U \setminus S \in NR\beta(U,X)$, by the argument above, $U \setminus S$ is $N\beta\theta$ −closed and hence, $S$ is $N\beta\theta$ −open.

Therefore, $A \in N\beta\theta O(U,X) \cap N\beta\theta C(U,X)$. Conversely, if $S \in N\beta\theta O(U,X) \cap N\beta\theta C(U,X)$ then $S$ is both $N\beta\theta$ −open and $N\beta\theta$ −closed. Therefore, by Theorem 1.8, $S \in NR\beta(U,X)$.

## 4. Nano $\beta\theta$ − open Sets of $U_r(X)$, $L_r(X)$, and $B_r(X)$ Approximations

In this part, we defined some properties of $N\beta\theta O(U,X)$ by deferent cases of lower, upper approximations and boundary region.

**Proposition 4.1.** In any $(U,\tau_{\mathcal{R}}(X))$ if $U_r(X) = U$ then $N\beta\theta O(U,X) = N\beta O(U,X) = P(U)$.

**Proof.** Assume that $U_r(X) = U$ then by Theorem 1.13, $N\beta O(U,X) = P(U)$. Let $S$ be any subset of $(U,\tau_{\mathcal{R}}(X))$. Then, $\forall\, x \in S$, $\exists\, G \in N\beta O(U,X)$ where, $x \in G \subset NBClG = G \subset S$. Thus, $N\beta\theta O(U,X) = P(U)$.

**Proposition 4.2.** In any $(U,\tau_{\mathcal{R}}(X))$, if $S \subset [U_r(X)]^c$ then $S$ is not $N\beta\theta - open\ set$.

**Proof.** Suppose that $S \subset [U_r(X)]^c$, therefore $S \not\subset U_r(X)$. Hence by Theorem 1.14, $S$ is not $nano\ \beta - open\ set$ in $U$. Which implies that , $S$ is not $nano\ \beta\theta - open\ set$ in $U$.

**Proposition 4.3.** In any $(U,\tau_{\mathcal{R}}(X))$, if $\tau_{\mathcal{R}}(X) = \{\varphi, U\}$ then $N\beta O(U,X) = N\beta\theta O(U,X) = P(U)$.

**Proof.** Suppose that $\tau_{\mathcal{R}}(X) = \{\varphi, U\}$ then $\forall\, S \subset U$, $N\beta Cl(S) = U$, implies that $N\beta Cl[\,N\beta Int(\,N\beta Cl(S))] = U$. It is mean $N\beta O(U,X) = P(U)$. If $N\beta O(U,X) = P(U)$ then each set in $(U,\tau_{\mathcal{R}}(X))$ is clopen set. Therefore, $N\beta ClG = G$. Hence, $\forall\, x \in S, \exists\, G \in N\beta O(U,X)$ where $x \in G \subset N\beta ClG \subset S$. Thus, $N\beta\theta O(U,X) = N\beta O(U,X) = P(U)$.

**Proposition 4.4.** In any $(U,\tau_{\mathcal{R}}(X))$, if $L_r(X) = \varphi$ and $U_r(X) = U$ then $N\beta O(U,X) = N\beta\theta O(U,X) = P(U)$.

**Proof.** Suppose that $L_r(X) = \varphi$ and $U_r(X) = U$, then $B_r(X) = U_r(X) = U$. Therefore, nano topological space is indiscrete nano topology. And, $N\beta O(U,X) = P(U)$. By Proposition 4.3, $N\beta\theta O(U,X) = N\beta O(U,X) = P(U)$.

**Proposition 4.5.** In any $(U,\tau_{\mathcal{R}}(X))$, if $U_r(X) = U$ and $L_r(X) \neq B_r(X) \neq \varphi$ then $N\beta O(U,X) = N\beta\theta O(U,X)$.

**Proof.** Suppose that $U_r(X) = U$, then ***by*** Theorem 1.13, $N\beta O(U,X) = P(U)$. And By proposition 4.3, $N\beta\theta O(U,X) = N\beta O(U,X) = P(U)$.

**Proposition4.6.** In any $(U,\tau_{\mathcal{R}}(X))$, $N\beta O(U,X) = N\beta\theta O(U,X)$ if $L_r(X) = \varphi$ and $U_r(X) = B_r(X) \neq \varphi$.

**Proof.** Let $L_r(X) = \varphi$ and $U_r(X) = B_r(X) \neq \varphi$. By Theorem 1.14, $S$ is $N\beta O(U,X)$ *if and only if* $S \cap U_r(X) \neq \varphi$. And by Proposition of $N\beta - open\ sets$, if $U_r(X) \neq U$, then any subset of $(U_r(X))^c$ is not $N\beta O(U,X)$. Therefore, every subset of $U$ is $N\beta - open\ sets$ except subset of $(U_R(X))^c$. Thus, if $S$ is $N\beta O(U,X)$, then it must be shown that $S$ is $N\beta\theta O(U,X)$. To show that $S$ is $N\beta\theta O(U,X)$, there must be as $\forall x \in S$, $\exists\, G \in N\beta O(U,X)$ where, $x \in G \subset N\beta ClG \subset S$. Now, if $S$ is also $N\beta C(U,X)$, then $G = S$. Therefore, $\forall\, x \in S, \exists\, G = S \in N\beta O(U,X)$ where, $x \in G \subset N\beta ClG = G \subset S$. But, if $S$ is not $N\beta C(U,X)$, then $\forall\, x \in S, \exists\, G = \cup\{x\}$ or $G = \cup\{x\} \cup \{x,a\}$, where $\forall\, x \in U_r(X)$ and $\forall a \in (U_r(X))^c$. First, if $G = \{x\}$, since $\exists\, y \in G^c \subset U_r(X)$ then $G^c$ is also $N\beta O(U,X)$. Therefore, $(G^c)^c = G$ is $N\beta C(U,X)$. Thus, $G$ is also $N\beta C(U,X)$. Hence, $\exists\, G = \{x\} \in N\beta O(U,X)$, $N\beta ClG = \{x\}$. $x \in G \subset N\beta ClG \subset S$. Consequently, $S$ is $N\beta\theta O(U,X)$. Second, If $G = \{x,a\}$. $\{a\} \subset (U_r(X))^c$. Therefore, $U_r(X) \subset U \setminus \{a\} = \{a\}^c$. Thus, there exists at least one element of $G^c$ in $U_r(X)$. Accordingly, $G^c$ is $N\beta O(U,X)$. $(G^c)^c = G$ is $N\beta C(U,X)$. $\forall\, x \in S, \exists\, G \in N\beta O(U,X)$ where, $x \in G \subset N\beta ClG = G \subset S$. Consequently, $S$ is $N\beta\theta O(U,X)$.

**Proposition 4.7.** In any $(U,\tau_{\mathcal{R}}(X))$, if $B_r(X) = \varphi$ and $U_r(X) = L_r(X) \neq \varphi$ then $N\beta O(U,X) \neq N\beta\theta O(U,X)$.

**Proof.** It is clear.

**Corollary 4.8.** If $L_r(X) = \varphi$ or $B_r(X) = \varphi$ and $U_r(X) \neq U$, then $S$ is $N\beta\theta O(U,X)$ *if and only if* $S \cap U_r(X) \neq \varphi$.

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**Proof.** Propositions 4.6 and 4.7 provide evidence for the claim.

**Proposition 4.9.** For any $(U, \tau_{\mathcal{R}}(X))$ , if $X = \{x\} \in U/R$ then $N\beta\theta O(U, X)$ are $\varphi$ and $U$.

**Proof.** Suppose that $X = \{x\} \in U/R$ then $L_r(X) = U_r(X) = \{x\}$ and $B_r(X) = \varphi$. Hence, $\tau_{\mathcal{R}}(X) = \{\varphi, U, U_r(X)\}$. Since $(U_r(X))^c$ is not intersect with $U_r(X)$ thus, by Theorem 1.14, $(U_r(X))^c$ is not $N\beta - open\ set$. So, any $S \in N\beta O(U, X)$, $N\beta Cl = U \not\subset \{x\}$. Therefore, nano $\beta\theta - open\ set$ are $\varphi$ and $U$.

**Proposition *4.10.*** If $U_r(X) \neq U$ and $L_r(X) \neq \emptyset$ then $N\beta\theta O = N\beta O$ and every proper subset of $U$ which intersects $U_r(X)$ is *nano* $\beta\theta - open$ set in $U$.

**Proof.** Directly from Theorem 1.24, proposition 4.5 and Corollary 4.8.

**Proposition 4.11.** In any $(U, \tau_{\mathcal{R}}(X))$, $N\beta\theta O = P(U)$ if $U_r(X) = L_r(X) = U$.

**Proof.** Let $U_r(X) = L_r(X) = U$. Then $\tau_{\mathcal{R}}(X) = \{\varphi, U\}$. Implies that by Proposition 4.3, $N\beta O(U, X) = N\beta\theta O(U, X) = P(U)$.

## 5. Relationship Between Nβθ − open sets and other sets

In this section, we explain the relation between $N\beta\theta - open\ sets$ and some nearly nano open sets.

**Theorem 5.1.** Every $N\theta - Open\ set$ is $N\beta\theta - open$.

**Proof.** Assume that $G \in N\theta O(U, X)$. $\forall\, x \in G$ , $\exists\, H \in \tau_{\mathcal{R}}(X)$ , where $x \in H \subset NClH \subset G$. Since nano open set is also nano $\beta - Open$ set. $H \in N\beta O(U, X)$ and $N\beta ClH \subset NClH$. $x \in H \subset N\beta ClH \subset G$. Hence, G is $N\beta\theta O(U, X)$.

**Theorem 5.2.** Every $N\beta\theta - open\ set$ is $N\beta - open\ set$ .

**Proof.** Assume that $H \in N\beta\theta O(U, X)$ , $\forall\, x \in H$ . $\exists\, G \in N\beta O(U, X)$ where $x \in G \subset N\beta ClG \subset H$. $\cup\{x, x \in H\} \subset \cup G_x \subset H$. So, $H = \cup G_x$ , where $\cup G_x \in N\beta O(U, X)$ And union of $N\beta - open\ sets$ is also $N\beta - open\ set$. Hence, $H \in N\beta O(U, X)$.

**Theorem 5.3.** For any $(U, \tau_{\mathcal{R}}(X))$ , if $X = \{x\} \in U/R$ then $X$ is nano $\beta - open\ set$ but not nano $\beta\theta - openset$.

**Proof.** Suppose that $X = \{x\}$ then $L_r(X) = U_r(X) = \{x\}$ and $B_r(X) = \varphi$ . Therefore, $\tau_{\mathcal{R}}(X) = \{\varphi, U, U_r(X)\}$. By Theorem 1.14, any subset $S$ of $U$ is $N\beta - open\ set$ if $S \cap U_r(X) \neq \emptyset$.Therefore, $U_r(X) = \{x\}$ is $N\beta - open\ set$. On the other hand, if $S \cap U_r(X) \neq \emptyset$ then $S^c \cap U_r(X) = \varphi$. $U_r(X) \not\subset S^c$ implies that $U_r(X) \subset S$. Henceforth $N\beta ClU_r(X) = N\beta Cl\{x\} = U \not\subset \{x\}$. Therefore, $x \in U_r(X) \subset N\beta ClU_r(X) \not\subset \{x\}$. Consequently, $\{x\}$ *is not* $N\beta\theta - open\ set$.

**Theorem 5.4.** Any union of $N\beta\theta - open\ sets$ in a nono topological space is $N\beta\theta - open\ set$.

**Proof.** Let $\{G_\lambda\} \in N\beta\theta O(U, X)$, there has to be shown that $G = \cup G_\lambda$ is also $N\beta\theta -$ open of $U$. *Let* $x \in \cup G_\lambda$ then $x \in G_{\lambda_0}$ , for some $\lambda_0 \in \Delta$ . And $G_{\lambda_0} \in N\beta\theta O(U, X)$, $\exists\, H \in N\beta(U, X)$, where $x \in H \subset N\beta ClH \subset G_{\lambda_0} \subset \cup G_\lambda$ .Consequently, $\cup G_\lambda \in N\beta\theta O(U, X)$.

**Remark 5.1.** If $S$ and $B$ are two $N\beta\theta - open\ sets$, then $S \cap B$ may not be $N\beta\theta - open\ set$ in general as shown in the example below.

**Example 5.1.** Assume that $U = \{p_1, p_2, p_3, p_4\}$ with $U/X$ =$\{\{p_1\}, \{p_4\}, \{p_2, p_3\}\}$, and assume that $X = \{p_1, p_2\}$ then $\tau_{\mathcal{R}}(X) = \{\varphi, U, \{p_1\}, \{p_1, p_2, p_3\}, \{p_2, p_3\}\}$ , and $N\beta O(U, X) = P(U) \setminus \{p_4\}$. Consider $S = \{p_1, p_4\}$ , and $B = \{p_2, p_4\}$ . $S$ is $N\beta\theta O(U, X)$ since there exists $G = \{p_1, p_4\} \in N\beta O(U, X)$ where $x \in G \subset N\beta ClG \subset S$ , and $B$ is $N\beta\theta O(U, X)$ because there exists $H = \{p_2, p_4\} \in N\beta O(U, X)$ where $x \in H \subset N\beta ClH \subset B$. But $S \cap B = \{p_4\}$. Since, $\nexists G \in N\beta O(\mathbb{U}, \mathbb{X})$ , where $x \in G \subset N\beta ClG \subset \{p_4\}$. therefore $\{p_4\}$ is not $N\beta\theta O(U, X)$.

**Theorem 5.5.** For any $(U, \tau_{\mathcal{R}}(X))$, $NS\theta O(U, X) \subset N\beta\theta O(U, X)$.

**Proof.** Let $S \in NS\theta O(U, X)$ and $x \in S$, $\exists G \in NSO(U, X)$ where, $x \in G \subset NSClG \subset S$. By Theorem 1.10, $N\beta ClG \subset NSClG$, Therefore, $x \in G \subset N\beta ClG \subset S$. Since, $NSO(U, X) \subset N\beta O(U, X)$. Consequently $S \in N\beta\theta O(U, X)$.

**Theorem 5.6.** For any $(U, \tau_{\mathcal{R}}(X))$, $NP\theta O(U, X) \subset N\beta\theta O(U, X)$.

**Proof.** Follows directly from Theorem 1.10 and their definitions.

**Theorem 5.7.** For any $(U, \tau_{\mathcal{R}}(X))$, $NP\delta O(U, X) \subset N\beta\theta O(U, X)$.

**Proof**. Assume that $S \in NP\delta O(U, X)$, by Theorem 1.11, $S$ is union of nano pre-regular p-open sets in $(U, X)$. Besides by Theorem 1.12, $NPRPO(U, X) \subset NR\beta O(U, X)$ and by Theorem 3.3, $S \in N\beta\theta O(U, X)$.

The diagram below represents the relationship in nano topological space between the class of Nβθ − open sets and several associated classes of nano open sets.

$NRPO(U, X) \implies NP\theta O(U, X)$

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$$\Downarrow \quad \Downarrow$$

$$NPRPO(U,X) \Rightarrow NP\delta O(U,X) \Rightarrow NPO(U,X)$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$NR\beta O(U,X) \Rightarrow \beta\theta O(U,X) \Rightarrow N\beta O(U,X)$$

$$\Uparrow \quad \Uparrow \quad \Uparrow$$

$$NRSO(U,X) \Rightarrow NS\theta O(U,X) \Rightarrow NSO(U,X)$$

**Diagram 1**

It should be noted that the inverse of implications untrue in general as shown in the following examples

**Example 5.2**. In Example 5.1, let $S = \{p_1, p_3\}$ and $G = \{p_1, p_3\}$, then $N\beta ClG = \{p_1, p_3\}$. Accordingly, $x \in S, \exists\, G$ *is* $N\beta O(U,X)$, where $x \in G \subset N\beta ClG \subset S$. Hence, $S$ *is* $N\beta\theta O(U,X)$. But if $x = p_1 \in S$, consider $H = \{p_1, p_3\}$ then $NClH = U$. Hence, $p_1 \in \{p_1, p_3\} \subset NClH \not\subset S$. Therefore, $S$ is not $N\theta O(U,X)$ when $S$ is $N\beta\theta O(U,X)$.

**Example 5.3.** Assume that U= $\{p_1, p_2, p_3\}$ with $U/R = \{\{p_1\}, \{p_2, p_3\}\}$ , and consider that $X = \{p_1\}$ then $\tau_{\mathcal{R}}(X) = \{\varphi, U, \{p_1\}\}$ and $N\beta O(U,X) = \{\{p_1\}, \{p_1, p_2\}, \{p_1, p_3\}, \varphi, U\}$. $N\beta C(U,X) = \{\{p_2, p_3\}, \{p_3\}, \{p_2\}, U, \varphi\}$. Assume that $S = \{p_1\}$ is $N\beta - open\ set$. Assume that $G = \{p_1\}$ then $N\beta ClG = U$. $p_1 \in G \subset N\beta ClG \not\subset S$. Consequently, $S$ is $N\beta - open\ set$ but it is not $N\beta\theta - open\ set$.

**Example 5.4.** In Example 5.1,

$NPO(U,X) = \{\varphi, U, \{p_1\}, \{p_2\}, \{p_3\}, \{p_1, p_2\}, \{p_1, p_3\}, \{p_1, p_2, p_3\}, \{p_1, p_2, p_4\}, \{p_1, p_3, p_4\}\}$

$NPC(U,X) = \{U, \varphi, \{p_2, p_3, p_4\}, \{p_1, p_3, p_4\}, \{p_1, p_2, p_4\}, \{p_3, p_4\}, \{p_2, p_4\}, \{p_4\}, \{p_3\}, \{p_2\}\}$. Assume that $S = \{p_2, p_3, p_4\}$. Since, $\forall\, x \in S, \exists\, G = \{p_2, p_3, p_4\} \in N\beta O(U,X)$ where $x \in G \subset N\beta ClG \subset S$. Therefore, $S \in N\beta\theta O(U,X)$. But $S \notin NP\delta O(U,X)$, $\nexists\, G \in NPO(U,X)$ where $x \in G \subset NPInt(NPClG) \subset S$. Consequently, $S$ is $N\beta\theta O(U,X)$ but not $NP\delta O(U,X)$ .

**Example 5.5.** In Example 5.1,

$NSO(U,X) = \{\varphi, U, \{p_1\}, \{p_1, p_4\}, \{p_2, p_3\}, \{p_1, p_2, p_3\}, \{p_2, p_3, p_4\}\}$. Assume that $S = \{p_2\}$. $S$ is $N\beta\theta O(U,X)$ , Since, $\forall\, x \in S, \exists\, G = \{p_2\} \in N\beta O(U,X)$ where $x \in G \subset N\beta ClG \subset S$. But $S \notin NS\theta O(U,X)$, since $\nexists\, G \in NSO(U,X)$ where $x \in G \subset NSClG \subset S$. Consequently, $S$ is $N\beta\theta O(U,X)$ but not $NS\theta O(U,X)$ .

**Example 5.6.** Assume that $U = \{p_1, p_2, p_3, p_4\}$ with $U/R = \{\{p_1\}, \{p_2\}\{p_3, p_4\}\}$, and assume that $X = \{p_3\}$, then $\tau_{\mathcal{R}}(X) = \{\varphi, U, \{p_3, p_4\}\}$ and

$$N\beta O(U,X) = \left\{ \begin{matrix} \varphi, U, \{p_3\}, \{p_4\}, \{p_1, p_3\}, \{p_1, p_4\}, \{p_2, p_3\}, \{p_2, p_4\}, \{p_3, p_4\}, \{p_1, p_2, p_3\}, \\ \{p_1, p_2, p_4\}, \{p_1, p_3, p_4\}, \{p_2, p_3, p_4\} \end{matrix} \right\}$$

Now, consider $S = \{p_2, p_3\}$. To show that $S$ is $N\beta\theta O(U,X)$, it needs evidence to prove that $\forall\, x \in S, \exists\, G \in N\beta O(U,X)$ where $x \in G \subset N\beta ClG \subset S$. $p_3 \in S, \exists\, G = \{p_3\} \in N\beta O(U,X)$. and $N\beta ClG = \{p_3\} \subset S$. Therefore, $p_3 \in G \subset N\beta ClG \subset S$. $p_4 \in S$, $\exists\, G = \{p_4\} \in N\beta O(U,X)$ and $N\beta ClG = \{p_4\} \subset S$. Therefore, $p_4 \in G \subset N\beta ClG \subset S$. Hence, $S$ is $N\beta\theta O(U,X)$. But, to show $S$ is $NR\beta O(U,X)$, it must be proven that $S = N\beta Int(N\beta ClS)$. $N\beta ClS = U$ which implies that $N\beta Int(N\beta ClS) = U$. Therefore, $S \neq N\beta Int(N\beta ClS)$. Hence, $S$ is $N\beta\theta O(U,X)$ but not $NR\beta O(U,X)$.

**Theorem 5.8.** If $S_i \in N\beta\theta O(U_i, X)$ for $i = 1,2$ then $S_1 \times S_2 \in N\beta\theta O(U_1 \times U_2)$.

***Proof:*** Let $S_i \in N\beta\theta O(U_i, X)$ for $i = 1,2$, by Theorem 3.3, $S_i = \bigcup_{\ell_i \in \Delta_i} V_{\ell_i}$ and $V_{\ell_i} \in NR\beta O(U_i)$. By Theorem1.18, $V_{\ell_1} \times V_{\ell_2} \in NR\beta O(U_1 \times U_2)$, $\forall\, \ell_1 \in \Delta_1$ and $\ell_2 \in \Delta_2$. Since, $S_1 \times S_2 = \bigcup_{\ell_1 \in \Delta_1} V_{\ell_1} \times \bigcup_{\ell_2 \in \Delta_2} V_{\ell_2} = \bigcup_{\ell_1 \in \Delta_1} \bigcup_{\ell_2 \in \Delta_2} V_{\ell_1} \times V_{\ell_2}$. By Theorem 3.3, $S_1 \times S_2 \in N\beta\theta O(U_1 \times U_2)$.

**Theorem 5.9**. Intersection $N\beta\theta -$ open set and $NR -$ open set is $N\beta\theta -$ open set

**Proof.** Let $S \in NRO(U,X)$ and $V \in N\beta\theta O(U,X)$, so by Theorem 3.3, $V = \bigcup_{\lambda \in \Delta} V_\lambda$ , where $V_\lambda \in NR\beta O(U,X)$, Hence, by Theorem 1.15, $S \cap V_\lambda \in NR\beta O(U,X)$. Since, $NR\beta O(U,X) \subset N\beta\theta O(U,X)$ then $S \cap V_\lambda \in N\beta\theta O(U,X)$.And by Theorem 5.4 , $\bigcup_{\lambda \in \Delta} S \cap V_\lambda \in N\beta\theta O(U,X)$. Hence, $S \cap V \in N\beta\theta O(U,X)$.

## CONCLUSION

In this study, we defined a new type of nano open set in nano topological spaces called ($Nano\ \beta\theta - open\ sets$) and compared it with other types of nano open sets. We have also studied the features of this set, and obtained new results.

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