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The Stability Of Crank – Nicholson and Explicit Methods for Numerical Solution for Sine – Gordon Equation

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ABSTRACT

In This Paper, We Solve The Sine – Gordon Equation by two Numerical Methods: Crank – Nicholson and Explicit and we discuss The Stabilities, and we obtained That The Stability Crank – Nicholson Methods is More Than The Stability Of Explicit Methods.

استقراريه الحل العددي بطريقتي كرانك –نيكلسون Crank-Nicholson والطريقة الصريحة Explicit Method لمعادلة Sine-Gordon

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الملخص

في هذا البحث قمنا بحل معادلة Sine-Gordon باستخدام طريقتين عدديتين. هما طريقة كرانك – نيكلسون , (Crank-Nicholson) وطريقة Explicit Method وناقشنا استقرارية الحلول لكل منهما ووجدنا أن طريقة كرانك – نيكلسون أكثر استقراراً من الطريقة الصريحة.

الكلمات المفتاحية: الاستقرارية، معادلة Sine-Gordon، طريقة Crank-Nicholson، الطريقة الصريحة (Explicit Method)

Introduction

The differential equation is one of the types to describe many of Phenomena in Physics , Chemical , Engineer . and there are Many numerical Methods that are Used to Solving the Problems , Finite difference Methods (FDM) and Finite Elements Methods (FEM) are two Methods are More using , but . There are a lot of risks When applying These Methods [1] . The numerical Solution for Partial differentail equations is absultly find The experimental solution specially When We Using (FDM) , [2] .

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In (1939) Frenkel , Kontorova are in traduced Sine – Gordon equation in Physics. And in (1962) Peering and Skyrme are Studied the numerical result for some Chemical Problems [11] . Forest and Mclaughlin , Ercolani in (1990) are improved the geometry formula of Sine – Gordon equation which can be integrable under periodically boundary Conditions , and they are obtain the good numerical results [5].

In (2003) Ares, Guesta, Sanchez and Toral are study the model of Sine – Gordon equation analytically and numerically[4] .

In (2005)Khomeriki and Leon are described the Bi – Stability behavior Case as appositive non linear equation and they are explain this numerically [7].

2- The Mathematical Model

Now, we derive the mathematical Model for Sine-Gordon equation by Klein – Gordon equation, where the governing equation is in the bellow formula

$$\frac{\partial^2 u}{\partial t^2} - d^2 \frac{\partial^2 u}{\partial x^2} + f(u) = 0 \qquad (1)$$

Where d is the wave speed, $f(u)$ the density of elasticity power.

Note: if $f(u) = g(u)$ then, the equation (1) becomes Klein – Gordon linear equation

i.e. : $$\frac{\partial^2 u}{\partial t^2} - d^2 \frac{\partial^2 u}{\partial x^2} + gu = 0 \qquad (2)$$

if $g = 0$ then Klein – Gordon become traditional Wave equation , and if $f(u) = \sin u$ the equation (2) become Sine – Gordon equation [5] , [8] .

$$\frac{\partial^2 u}{\partial t^2} - d^2 \frac{\partial^2 u}{\partial x^2} = -\sin u \qquad (3)$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\sin u \qquad (4)$$

And with boundary and initial Conditions

$$u(x, 0) = p + e_0 \cos[mx] \quad , \quad u_t(x, 0) = 0$$

$$m \frac{1}{\sqrt{2}} \quad , \quad l = 2\sqrt{2}\, p \quad , \quad 0 \le e_o \le 1000 \quad , \quad -L \le x \le L$$

$u(-L, t) = u(L, t) = b$, where b is constant

The following equation is called [Perturbed Sine Gordon Equation]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\sin u + e \sin[d_t + d_o] \qquad (5)$$

Where e is a capacity and d is frequency , d_o is a Phase [10].

Also, the equation bellow is called Fractional Sine – Gordon equation

$$u_{xx} - {}^{R}D^{a} u + Sin\, u = 0 \qquad (6)$$

${}^{R}D^{a}$ is the Fractional derivative on Riesz space

3- Deriving the general form of Explicit Method

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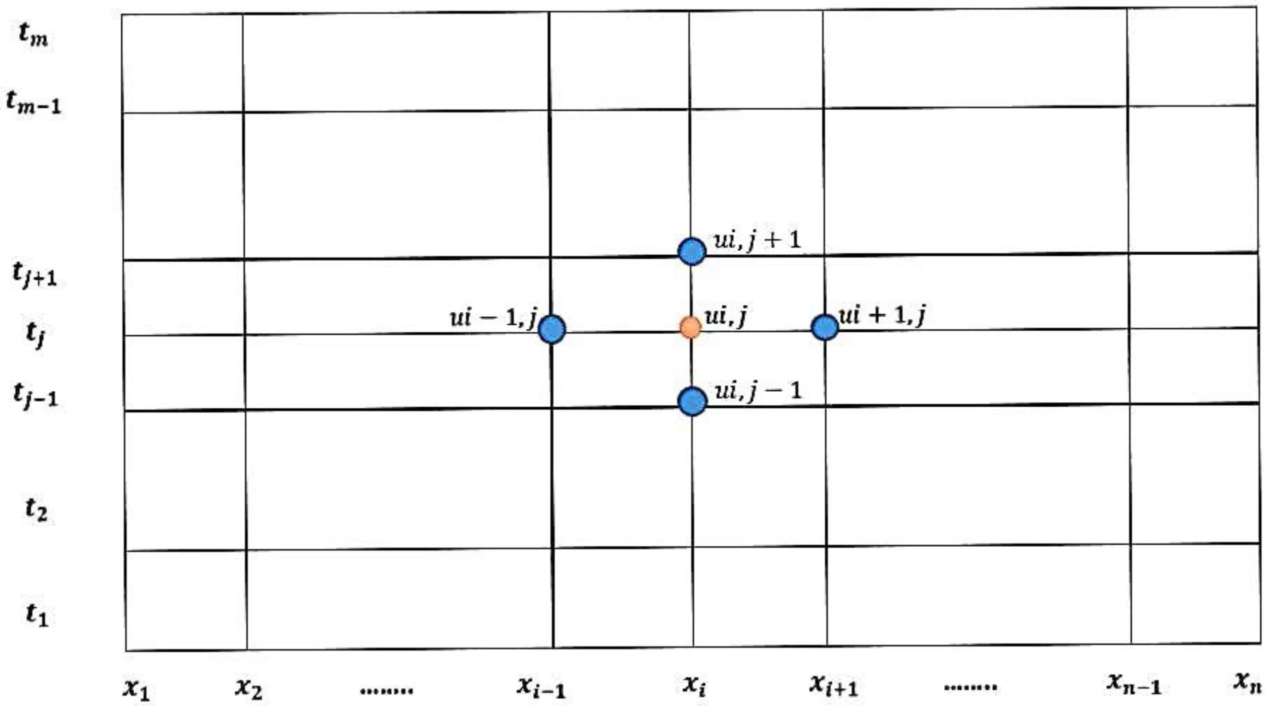
First Step we divide the region. Rectangles: $R = [(x, t) = -L \le x \le L, 0 \le t \le C]$ to $[n-1], [m-1]$ from the Rectangles for Length Side $\Delta x = h$, $\Delta t = k$, as show bellow [9]



The form (1) represent mesh (Explicit method)

From level 1: Calculate the value $t = t_1 = 0$ [3]

$u(x_i, t_1) = p + e_o \cos[mx_i]$, $i = 1, 2, 3, 4, \ldots, n-1$ where

$0 \le e_o \le 1000$, $m = \frac{1}{\sqrt{2}}$, $l = 2\sqrt{2}\, p$

Where the differences are used to approximate u_{tt} (x,t) , $u_{xx}(x,t)$

$$u_{tt}(x,t) = \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{k^2} + O(k^2) \quad (7)$$

$$u_{xx}(x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2) \quad (8)$$

And the nodes are :

$x_{i+1} = x_i + h, x_{i-1} = x_i - h, t_{j+1} = t_j + k, t_{j-1} = t_j - k$

Ignore $O(k^2)$, $O(h^2)$ the we obtain getting:

$$u_{tt}(x,t) = \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{k^2} \quad (9)$$

$$u_{xx}(x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} \quad (10)$$

Also , to find the values 2nd ($t = t_2$) level by Taylor series [9] :

$$u(x,t) = u(x,0) + u_t(x,0)k + \frac{u_{tt}(x,0)k^2}{2!} + O[k^3] \quad (11)$$

Put $x = x_i$

$$u(x_i,t) = u(x_i,0) + u_t(x_i,0)k + \frac{u_{tt}(x_i,0)k^2}{2} + O[k^3] \quad (12)$$

Where

$$u_{tt}(x_i,0) = \frac{u(x_{i+1},0) - 2u(x_i,0) + u(x_{i-1},0)}{h^2} - \sin[u(x_i,0)] \quad (13)$$

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$$u_t\,(x_i\,,0) = 0 \qquad (14)$$

And substitute equations (13,14) in equation (12) we get :

$$u(x_i,k) = u(x_i,0) + \left[\frac{u\,(x_{i+1}\,,0\,)-2u\,(\,x_i\,,0\,)+u\,(\,x_{i-1}\,,0\,)}{h^2} - sin[u\,(x_i\,,0)]\right]\frac{k^2}{2} + O(k^3)$$

$$u(x_i,k) = u(x_i,0) + \frac{r^2}{2}\,[u\,(\,x_{i+1}\,,0\,) - 2u\,(\,x_i\,,0\,) + u\,(\,x_{i-1}\,,0\,)] - \frac{k^2}{2} sin[u\,(x_i\,,0)] \qquad (15)$$

And with $r = \frac{k}{h}$, and substitute ($u_{i\,,1}$) of with $u\,(x_i\,,0)$, and with taking operation simplied to equation (15) we get on level two (FDM)

$$u_{i\,,2} = [1 - r^2]u_{i\,,1} + \frac{r^2}{2}\,[u_{i+1\,,1} + u_{i-1\,,1}] - \frac{k^2}{2}\,\sin[u_{i\,,1}] \qquad (16)$$

Now show that Calculate value $u\,(x\,,\,t)$ as nodes points for class other :

$$u\,(x_i\,,\,t_j)\;: i = 1\,,2\,,3\,,\,\dots,n-1 \quad , \quad j = 2\,,3,4\,,\dots,m$$

As substitute $(u_{i\,,j})$ with out $u(x_i\,,\,t_j)$ *in* equations (10 ,11) , and by substitute equations complete in equation (3) , we get :

$$\frac{u_{i\,,j+1} - 2\,u_{i\,,j} + u_{\,i\,,j-1}}{k^2} - \frac{u_{\,i+1\,,j} - 2\,u_{i\,,j} + u_{i-1\,,j}}{h^2} = \sin[u_{i\,,j}] \qquad (17)$$

And from equation (17) we get :

$$u_{i\,,j+1} = -u_{i\,,j-1} + [2 - 2r^2]\,u_{i\,,j} + r^2[u_{i+1\,,j} + u_{i-1\,,j}] - k^2\,\sin[u_{i\,,j}] \qquad (18)$$

And equation (18) represent (FDM) by used Explicit Method for equation $Sine - Gordon$ and used equation (18) Calculate $(j+1)$, and by value $(j-1\,,j)$, these the method Calculate form Explicit for value undefined $(u_{i\,,j+1})$, and by value defined

$$u_{i\,,j-1}\;,\;u_{i+1\,,j}\;,\;u_{i\,,j}\;,u_{i-1\,,j}$$

That show in form (1)

4- The Stability of Explicit Method

Now . we explain the Stability of Explicit Method with Von- Neumann the solution step are

$$u_{x\,,t} = e^{gt}\,.\,e^{ibx} \;\;,\; x = nh \;\;,\; t = mk \;\;,\; h = \Delta x \;\;,\; k = \Delta t$$

$$g\,,b > 0 \;\;,\; i = \sqrt{-1}$$

$$u_{x\,,t} = e^{gmk}\,.\,e^{ibnh} = [e^{g\Delta t}]^m\,e^{ibn\Delta x} = x^m\,e^{ibn\Delta x}$$

The main step that is substitute the solution of (FDM) at time t with $x^m\,e^{ibnDx}$ [12] and with Von- Neumann can be used $[Linearized\ Stability\ Analysis]$ [6]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - u \qquad (19)$$

And by explicit Method

$$u_{n\,,m+1} = -u_{n\,,m-1} + [2 - 2r^2]u_{n\,,m} + r^2[u_{\,n+1\,,m} + u_{\,n-1\,,m}] - [\Delta t]^2 u_{n\,,m} \quad (20)$$

And by substitute $u_{n\,,m} = x^m e^{ibn\Delta x}$ *in equation* (20) *we geting*

$$x^{m+1}\,.\,e^{ibn\Delta x} = -x^{m-1}\,e^{ibn\Delta x} + [2 - 2r^2]x^m e^{ibn\Delta x} + r^2\left[x^m e^{ib[n+1]\Delta x} + x^m e^{ib[n-1]\Delta x}\right] - [\Delta t]^2\,x^m e^{ibn\Delta x}$$

$$\mathrm{x^m\,x\,.\,e^{ibn\Delta x}} = -\mathrm{x^m\,x^{-1}\,e^{ibn\Delta x}} + [2 - 2\mathrm{r}^2 - (\Delta t)^2]\mathrm{x^m e^{ibn\Delta x}} + \mathrm{r}^2\left[\mathrm{x^m e^{ib\,n\Delta x} e^{ib\Delta x}} + \mathrm{x^m e^{ib\,n\Delta x} e^{-ib\Delta x}}\right]$$

And by divide on $x^m\,e^{ibnDx}$ we getting

$$x = -x^{-1} + [2 - 2r^2 - (\Delta t)^2] + r^2[\,e^{ib\,\Delta x} + e^{-ib\,\Delta x}] \rightarrow$$

$$\frac{x^2+1}{x} = [2 - 2r^2 - (\Delta t)^2] + 2r^2\,\cos[b\,\Delta x] \rightarrow$$

$$\frac{x^2+1}{x} = [2 - 2r^2 - (\Delta t)^2] + 2r^2\,\left[1 - 2Sin^2\left(\frac{b\,\Delta x}{2}\right)\right] \rightarrow$$

$$\frac{x^2+1}{x} = 2\,\left[1 - \frac{(\Delta t)^2}{2} - 2r^2\,Sin^2\,\left(\frac{b\,\Delta x}{2}\right)\right] \rightarrow$$

$$\frac{x^2+1}{x} = 2a$$

$$a = 1 - \frac{(\Delta t)^2}{2} - 2r^2\,Sin^2\,\left(\frac{b\,\Delta x}{2}\right)$$

$$x^2 - 2ax + 1 = 0 \quad \Longrightarrow \quad x = am\,\sqrt{a^2 - 1}$$

The must $|a| \leq 1$, x is factor

$$\left|1 - \frac{(\Delta t)^2}{2} - 2r^2\,Sin^2\left(\frac{b\Delta x}{2}\right)\right| \leq 1 \qquad (21)$$

And from Inequalities (21) , we getting

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$$-1 \le 1 - \frac{(\Delta t)^2}{2} - 2r^2\, Sin^2\left(\frac{b\Delta x}{2}\right) \le 1$$

And by taking the side Inequality (21) , we getting right

$$1 - \frac{(\Delta t)^2}{2} - 2r^2\, Sin^2\left(\frac{b\Delta x}{2}\right) \le 1 \qquad (22)$$

And from Inequalities (22) , we getting

$$r^2 \ge \frac{-(\Delta t)^2}{4\, Sin^2\left(\frac{b\,\Delta x}{2}\right)}$$

Since $r^2 = \frac{(Dt)^2}{(Dx)^2}$, the value r^2 *is positve* , Inequality (22) way to $r^2 > 0$ this true early , for to suites Inequality (21) , we need to :

$$-1 \le 1 - \frac{(\Delta t)^2}{2} - 2r^2\, Sin^2\left(\frac{b\Delta x}{2}\right)$$

$$2 \ge \frac{(\Delta t)^2}{2} + 2r^2\, Sin^2\left(\frac{b\Delta x}{2}\right)$$

$$1 - \frac{(\Delta t)^2}{2} \ge r^2\, Sin^2\left(\frac{b\Delta x}{2}\right)$$

And since $Sin^2\left(\frac{b\Delta x}{2}\right)$ *equal* 1 *for value b* :

$$r^2 \le \frac{4-(\Delta t)^2}{4} \qquad (23)$$

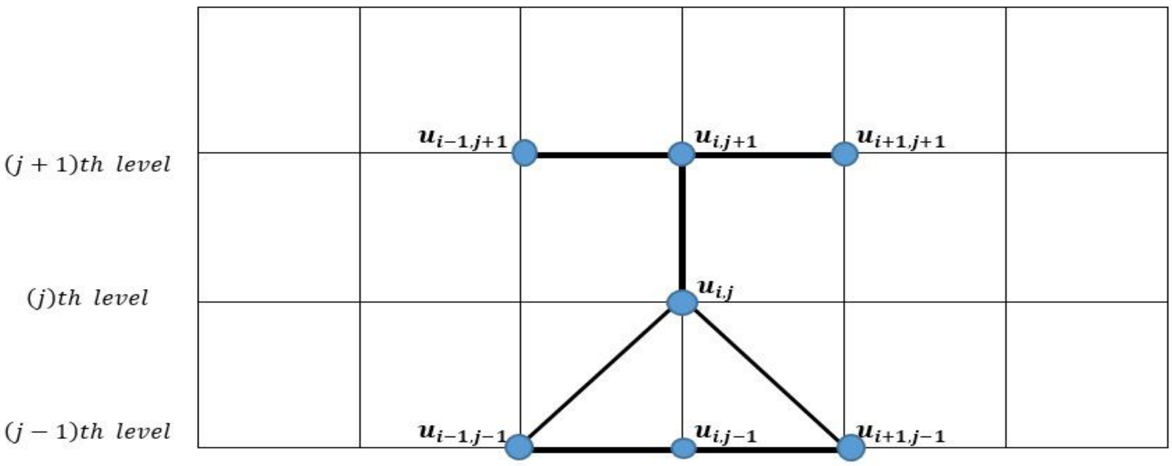
And since $r^2 = \frac{(\Delta t)^2}{(\Delta x)^2}$ And from Inequality (22) , we getting

$$(\Delta t)^2 \le \frac{4\,(\Delta x)^2}{4+(\Delta x)^2} \qquad (24)$$

And represent Inequalities (23,24) on Explicit Method , must the Stability no operation by Sine-Gordon equation .

5- Deriving The General formula of $[\boldsymbol{Crank-Nicholson}]$

In $[Crank - Nicholson]$ method we replace the 2nd partial derivative u_{xx} by the average of central difference at time t and points $(j-1, j+1)$ as shown in the figure (2) bellow [13] :



The form (2) show that solution by using $Crank - Nicholson$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j-1}-2u_{i,j-1}+u_{i-1,j-1}}{h^2}\right) \qquad (25)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2} \qquad (26)$$

And as substitute the equation (25 , 26) in equation (3) we get :

$$\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2} + \frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{2h^2} - \frac{u_{i+1,j-1}-2u_{i,j-1}+u_{i-1,j-1}}{2h^2} = \sin[u_{i,j}] \qquad (27)$$

And from equation (28) we get :

$$-r^2\left[u_{i+1,j+1} + u_{i-1,j+1}\right] + [2+2r^2]\, u_{i,j+1}$$

$$= 4\,u_{i,j} - 2k^2 \sin[u_{i,j}] + r^2\left[u_{i+1,j-1} + u_{i-1,j-1}\right] - [2+2r^2]u_{i,j-1} \qquad (28)$$

Where the equation (28) is the approximate the finite difference of $[Crank - Nicholson]$ method for $[Sine - Gordon\,]$ *method* , and hence we have : three variables $u_{i+1,j+1}$, $u_{i,j+1}$, $u_{i-1,j+1}$ and

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all terms in right hand side of equation (28) as know , so the equation (28) can be lead to the algebraic liner system with three diameters

$$AX = B \qquad (29)$$

A : is a matrix have three diameters , X: Vertical vector , B: Vertical vector

Example: Let we have rectangular plate (2m, 4m) and it is coincide on x and y axis at original point

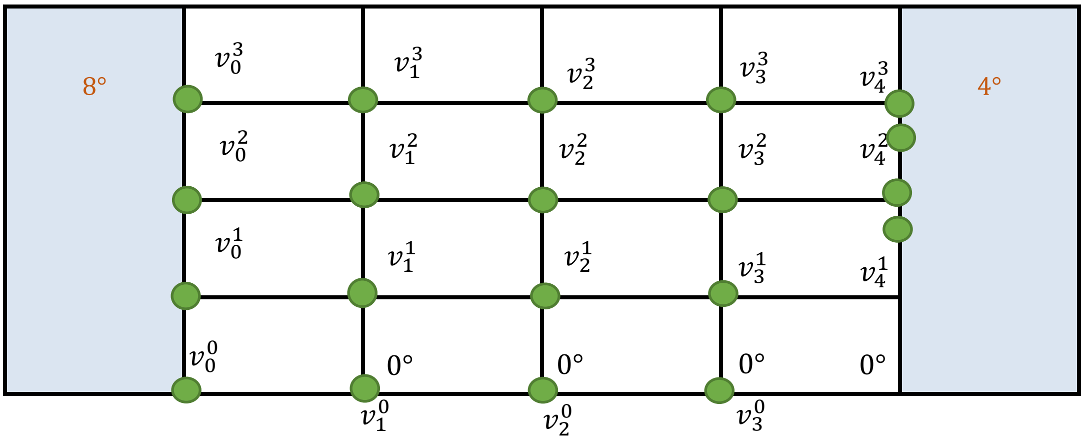
$\Delta t = 4\ sec,\ \Delta x = 5cm\ ,\ a = 1.25$

Initial condition $v(x\ ,t) = 0°\quad 0 < x < 10\quad t = 0$

Boundary condition $v(0\ ,t) = 8°,\ v(10\ ,t) = 4°$

Solution:

1) Make the mesh and substitute the values on shapes



2) The crank – Nicolson general formula

$$-rv_{l-1}^{i+1} + 2(1+m)v_l^{i+1} - rv_{l+1}^{i+1} = rv_{l-1}^{i} + 2(1-m)v_l^{i} + rv_{l+1}^{i}$$

To find m

$$m = a\frac{\Delta t}{\Delta x^2} \rightarrow r = 1.25\frac{5}{5^2} = 0.2$$

Substitute from l = 123 , i= 0 , m= 0.2

$$-0.2v_0^1 + 2.4v_1^1 - 0.2v_2^1 = 0.2v_0^0 + 1.6v_1^0 + 0.2v_2^0$$

$$-0.2v_1^1 + 2.4v_2^1 - 0.2v_3^1 = 0.2v_1^0 + 1.6v_2^0 + 0.2v_3^0$$

$$-0.2v_2^1 + 2.4v_3^1 - 0.2v_4^1 = 0.2v_2^0 + 1.6v_3^0 + 0.2v_4^0$$

Substitute initial and boundary condition , we get

$$-0.2(8) + 2.4v_1^1 - 0.2v_2^1 = 0.2(0) + 1.6(0) + 0.2(0)$$

$$-0.2v_1^1 + 2.4v_2^1 - 0.2v_3^1 = 0.2(0) + 1.6(0) + 0.2(0)$$

$$-0.2v_2^1 + 2.4v_3^1 - 0.2(4) = 0.2(0) + 1.6(0) + 0.2(0)$$

Hence , we have

$2.4v_1^1 - 0.2v_2^1 = 1.6$ ……… (1)

$-0.2v_1^1 + 2.4v_2^1 - 0.2v_3^1 = 0$ ……… (2)

$-0.2v_2^1 + 2.4v_3^1 = 0.8$ ……… (3)

Multiply equation (2) by 12 and solve it with equation (1) we have

$-2.4v_1^1 + 28.8v_2^1 - 2.4v_3^1 = 0$

$2.4v_1^1 - 0.2v_2^1 = 1.6$

$28.6v_2^1 - 2.4v_3^1 = 1.6$ ……… (4)

Solve equation (4) and adding to equation (3)

$28.6v_2^1 - 2.4v_3^1 = 1.6$

$-0.2v_2^1 + 2.4v_3^1 = 0.8$

$28.4v_2^1 = 2.4$

$v_2^1 = 0.0845070423$

Substitute v_2^1 value in equation (1) we obtain v_1^1

$2.4v_1^1 - 0.2(0.0845070423) = 1.6$

$2.4v_1^1 = 1.6169014085$

$v_1^1 = 0.6737089202$

Substitute v_1^1 value in equation (3) we obtain v_3^1

$-0.2(0.0845070423) + 2.4v_3^1 = 0.8$

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$2.4v_3^1 = 0.8169014085$

$v_3^1 = 0.3403755869$

Continue to Substitute from l = 123 , i= 0 , m= 0.2 in general formula

$-0.2v_0^2 + 2.4v_1^2 - 0.2v_2^2 = 0.2v_0^1 + 1.6v_1^1 + 0.2v_2^1$

$-0.2v_1^2 + 2.4v_2^2 - 0.2v_3^2 = 0.2v_1^1 + 1.6v_2^1 + 0.2v_3^1$

$-0.2v_2^2 + 2.4v_3^2 - 0.2v_4^2 = 0.2v_2^1 + 1.6v_3^1 + 0.2v_4^1$

Substitute initial and boundary condition , v_1^1 , v_2^1 , v_3^1

$-0.2(8) + 2.4v_1^2 - 0.2v_2^2 = 0.2(8) + 1.6(0.6737089202) + 0.2(0.0845070423)$

$-0.2v_1^2 + 2.4v_2^2 - 0.2v_3^2 = 0.2(0.6737089202) + 1.6(0.0845070423) +0.2(0.3403755869)$

$-0.2v_2^2 + 2.4v_3^2 - 0.2(4) = 0.2(0.0845070423) + 1.6(0.3403755869) + 0.2(4)$

Hence , we have

$2.4v_1^2 - 0.2v_2^2 = 4.294835681 \dots\dots\dots (1)$

$-0.2v_1^2 + 2.4v_2^2 - 0.2v_3^2 = 0.338028169. \dots\dots\dots (2)$

$-0.2v_2^2 + 2.4v_3^2 = 2.161502347 \dots\dots\dots (3)$

Multiply equation (2) by 12 and solve it with equation (1) we have

$-2.4v_1^2 + 28.8v_2^2 - 2.4v_3^2 = 4.056338028$

$2.4v_1^2 - 0.2v_2^2 = 4.294835681$

$28.6v_2^2 - 2.4v_3^2 = 8.351173709 \dots\dots\dots (4)$

Solve equation (4) and adding to equation (3)

$28.6v_2^2 - 2.4v_3^2 = 8.351173709$

$-0.2v_2^2 + 2.4v_3^2 = 2.161502347$

$28.4v_2^2 = 10.51267606$

$v_2^2 = 0.37016465$

Substitute v_2^2 value in equation (1) we obtain v_1^2

$2.4v_1^2 - 0.2(0.37016465) = 4.294835681$

$2.4v_1^2 = 4.368868611$

$v_1^2 = 1.820361921$

Substitute v_2^2 value in equation (3) we obtain v_3^2

$-0.2(0.1850823249) + 2.4v_3^2 = 2.161502347$

$2.4v_3^2 = 2.235535277$

$v_3^2 = 0.931473032$

Continue to Substitute from l = 123 , i= 2 , m= 0.2 in general formula

$-0.2v_0^3 + 2.4v_1^3 - 0.2v_2^3 = 0.2v_0^2 + 1.6v_1^2 + 0.2v_2^2$

$-0.2v_1^3 + 2.4v_2^3 - 0.2v_3^3 = 0.2v_1^2 + 1.6v_2^2 + 0.2v_3^2$

$-0.2v_2^3 + 2.4v_3^3 - 0.2v_4^3 = 0.2v_2^2 + 1.6v_3^2 + 0.2v_4^2$

Substitute initial and boundary condition , v_1^1 , v_2^1 , v_3^1

$-0.2(8) + 2.4v_1^3 - 0.2v_2^3 = 0.2(8) + 1.6(1.820361921) + 0.2(0.37016465)$

$-0.2v_1^3 + 2.4v_2^3 - 0.2v_3^3 = 0.2(1.820361921) + 1.6(0.37016465) +0.2(0.931473032)$

$-0.2v_2^3 + 2.4v_3^3 - 0.2(4) = 0.2(0.37016465) + 1.6(0.931473032) + 0.2(4)$

Hence , we have

$2.4v_1^3 - 0.2v_2^3 = 6.186612004 \dots\dots\dots (1)$

$-0.2v_1^3 + 2.4v_2^3 - 0.2v_3^3 = 1.14263043 \dots\dots\dots (2)$

$-0.2v_2^3 + 2.4v_3^3 = 3.164389781 \dots\dots\dots (3)$

Multiply equation (2) by 12 and solve it with equation (1) we have

$-2.4v_1^3 + 28.8v_2^3 - 2.4v_3^3 = 13.71156516$

$2.4v_1^3 - 0.2v_2^3 = 6.186612004$

$28.6v_2^3 - 2.4v_3^3 = 19.89817717 \dots\dots\dots (4)$

Solve equation (4) and adding to equation (3)

$28.6v_2^3 - 2.4v_3^3 = 19.89817717$

$-0.2v_2^3 + 2.4v_3^3 = 3.164389781$

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$28.4v_2^3 = 23.06256695$

$v_2^3 = 0.812062217$

Substitute v_2^3 value in equation (1) we obtain v_1^3

$2.4v_1^3 - 0.2(0.812062217) = 6.186612004$

$2.4v_1^3 = 6.349024447$

$v_1^3 = 2.645426853$

Substitute v_2^3 value in equation (3) we obtain v_3^3

$-0.2(0.812062217) + 2.4v_3^3 = 3.164389781$

$2.4v_3^3 = 3.326802225 \rightarrow v_3^3 = 1.386167594$

Crank-Nicolson	*Exact solution*
0.67370892	0.98
0.084507042	0.24
0.340375587	0.398
1.820361921	1.98
0.37016465	0.55
0.931473032	0.6
2.645426853	2.398
0.812062217	1.06
1.386167594	1.4

Table (1) Stability of Crank-Nicolson

7- Conclusion

1- the Stability of Crank – Nicholson is un condition $\forall\, r^2$.
2- the Stability of Explicit Methods is stable if $r^2 \le \frac{4-(Dt)^2}{4}$.
3- The Explicit Method is faster than Crank – Nicholson method to get the result (50%).
4- The Crank – Nicholson is more accuracy the Explicit Method .

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