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ABSTRACT

In This Paper, We Solve The Sine – Gordon Equation by two Numerical Methods: Crank – Nicholson and Explicit and we discuss The Stabilities, and we obtained That The Stability Crank – Nicholson Methods is More Than The Stability Of Explicit Methods.

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Introduction

The differential equation is one of the types to describe many of Phenomena in Physics, Chemical, Engineer, and there are Many numerical Methods that are Used to Solving the Problems. Finite difference (FDM) and Finite Elements Methods (FEM) are two Methods are More using, but. There are a lot of risks When applying These Methods. The numerical Solution for Partial differential equations is absultly find The experimental solution specially When We Using (FDM).
In (1939) Frenkel, Kontorova are in traduced Sine–Gordon equation in Physics. And in (1962) Peering and Skyrme are Studied the numerical result for some Chemical Problems [11]. Forest and Mclaughlin, Ercolani in (1990) are improved the geometry formula of Sine–Gordon equation which can be integrable under periodically boundary Conditions, and they are obtain the good numerical results [5].

2- The Mathematical Model

Now, we derive the mathematical Model for Sine-Gordon equation by Klein–Gordon equation, where the governing equation is in the bellow formula

\[
\frac{\partial^2 u}{\partial t^2} - d^2 \frac{\partial^2 u}{\partial x^2} + f(u) = 0 \quad (1)
\]

Where d is the wave speed, \( f(u) \) the density of elasticity power.

Note: if \( f(u) = g(u) \) then, the equation (1) becomes Klein–Gordon linear equation

i.e. \( \frac{\partial^2 u}{\partial t^2} - d^2 \frac{\partial^2 u}{\partial x^2} + g(u) = 0 \) \quad (2)

if \( g = 0 \) then Klein–Gordon become traditional Wave equation, and if \( f(u) = \sin u \) the equation (2) become Sine–Gordon equation [5], [8].

\[
\frac{\partial^2 u}{\partial t^2} - d^2 \frac{\partial^2 u}{\partial x^2} = -\sin u \quad (3)
\]

And with boundary and initial Conditions

\[ u(x,0) = p + e_0 \cos(mx) \quad , \quad u_t(x,0) = 0 \]

\[ m \frac{1}{\sqrt{2}} , \quad l = 2\sqrt{2}p , \quad 0 \leq e_0 \leq 1000 \quad , \]

\[ -L \leq x \leq L \]

\[ u(-L,t) = u(L,t) = b \quad , \quad \text{where b is constant} \]

The following equation is called [Perturbed Sine Gordon Equation]

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\sin u + e \sin[d_t + d_e] \quad (5)
\]

Where \( e \) is a capacity and \( d \) is frequency, \( d_e \) is a Phase [10].

Also, the equation bellow is called Fractional Sine–Gordon equation

\[
u_{xx} - \mathcal{D}^{\alpha} u + \sin u = 0 \quad (6)
\]

\( \mathcal{D}^{\alpha} \) is the Fractional derivative on Riesz space

3- Deriving the general form of Explicit Method
First Step we divide the region. Rectangles: $R = [(x, t) = -L \leq x \leq L, 0 \leq t \leq C]$ to $[n - 1], [m - 1]$ from the Rectangles for Length Side $\Delta x = h$ , $\Delta t = k$ , as show bellow [9]

The form (1) represent mesh (Explicit method)

From level 1: Calculate the value $t = t_1 = 0$ [3]

$$u(x_i, t_1) = p + e_o \cos(mx_i) \quad i = 1, 2, 3, 4, ... , n - 1$$

Where the differences are used to approximate $u_{tt}$ ($x,t$), $u_{xx} (x,t)$

$$u_{tt}(x,t) = \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{h^2} + O(k^2)$$

$$u_{xx}(x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{k^2} + O(h^2)$$

And the nodes are:

$x_{i+1} = x_i + h, x_{i-1} = x_i - h, t_{j+1} = t_j + k, t_{j-1} = t_j - k$

Ignore $O(k^2), O(h^2)$ the we obtain getting:

$$u_{tt}(x,t) = \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{h^2}$$  \hspace{2cm} (9)

$$u_{xx}(x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{k^2}$$  \hspace{2cm} (10)

Also, to find the values $2^{nd}$ ($t = t_2$) level by Taylor series [9]:

$$u(x, t) = u(x, 0) + u_t(x, 0)k + \frac{u_{tt}(x, 0)k^2}{2!} + O[k^3]$$

Put $x = x_i$

$$u(x_i, t) = u(x_i, 0) + u_t(x_i, 0)k + \frac{u_{tt}(x_i, 0)k^2}{2} + O[k^3]$$  \hspace{2cm} (12)

Where

$$u_{tt}(x_i, 0) = \frac{u(x_{i+1}, 0) - 2u(x_i, 0) + u(x_{i-1}, 0)}{h^2} - \sin[u(x_i, 0)] \hspace{2cm} (13)$$
And substitute equations (13,14) in equation (12) we get:

\[ u(x,t) = \left[ u(x_{i+1},0) - 2u(x_i,0) + u(x_{i-1},0) \right] - \frac{k^2}{2} \sin[u(x_i,0)] + O(k^3) \]

\[ u(x_i,0) = u(x_i,0) + \frac{r^2}{2} \left[ u(x_{i+1},0) - 2u(x_i,0) + u(x_{i-1},0) \right] - \frac{k^2}{2} \sin[u(x_i,0)] \]

(15)

And with \( r = \frac{k}{2} \), and substitute \((u_{i+1,1}, u_{i-1,1})\) of with \( u(x_i,0) \), and with taking operation simplified to equation (15) we get on level two (FDM)

\[ u_{i,2} = \left[ 1 - r^2 \right] u_{i,1} + \frac{r^2}{2} \left[ u_{i+1,1} + u_{i-1,1} \right] - \frac{k^2}{2} \sin[u_{i,1}] \]

(16)

Now show that Calculate value \( u(x, t) \) as nodes points for class other:

\[ u(x_i, t_j) : i = 1, 2, 3, \ldots, n - 1 \quad , \quad j = 2, 3, 4, \ldots, m \]

As substitute \((u_{i,j})\) with out \( u(x_i, t_j) \) in equations (10,11), and by substitute equations complete in equation (3), we get :

\[ \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \frac{u_{i+1,j} + u_{i-1,j}}{h^2} = \sin[u_{i,j}] \]

(17)

And from equation (17) we get:

\[ u_{i,j+1} = -u_{i,j-1} + [2 - 2r^2] u_{i,j} + r^2 \left[ u_{i+1,j} + u_{i-1,j} \right] - k^2 \sin[u_{i,j}] \]

(18)

And equation (18) represent (FDM) by used Explicit Method for equation Sine – Gordon and used equation (18) Calculate \((j - 1, j)\), and by value \((j-1, j)\), theses the method Calculate form Explicit for value undefined \((u_{i,j+1})\), and by value defined

\[ u_{i,j-1}, u_{i+1,j}, \ldots, u_{i,j}, u_{i-1,j} \]

That show in form (1)

4- The Stability of Explicit Method

Now, we explain the Stability of Explicit Method with Von-Neumann the solution step are

\[ u_{x,t} = e^{gt} \cdot e^{ibx} , \quad x = nh \quad , \quad t = mk \quad , \quad h = \Delta x , \quad k = \Delta t \]

\[ g, b > 0 \quad , \quad i = \sqrt{-1} \]

\[ u_{x,t} = e^{gmk} \cdot e^{ibnh} = \left[ e^{g\Delta t} \right] m \cdot e^{ibn\Delta x} = \]

\[ x^m \cdot e^{ibn\Delta x} \]

The main step that is substitute the solution of (FDM) at time \( t \) with \( x^m e^{ibn\Delta x} \) [12] and with Von-Neumann can be used [Linearized Stability Analysis] [6]:

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - u \]

(19)

And by explicit Method

\[ u_{n,m+1} = -u_{n,m-1} + [2 - 2r^2] u_{n,m} + r^2 \left[ u_{n+1,m} + u_{n-1,m} \right] - [\Delta t]^2 u_{n,m} \]

(20)

And by substitute

\[ u_{n,m} = x^m e^{ibn\Delta x} \]

in equation (20) we getting

\[ x^{m+1} e^{ib\Delta x} = -x^{m-1} e^{ib\Delta x} + [2 - 2r^2] x^m e^{ibn\Delta x} + r^2 \left[ x^m e^{ib[n+1]\Delta x} + x^m e^{ib[n-1]\Delta x} \right] - [\Delta t]^2 x^m e^{ibn\Delta x} \]

\[ x^m X \cdot e^{ibn\Delta x} = -x^m X - e^{ibn\Delta x} + [2 - 2r^2 - (\Delta t)^2] x^m e^{ibn\Delta x} + r^2 \left[ x^m e^{ib\Delta x} e^{ib\Delta x} + x^m e^{ib\Delta x} e^{-ib\Delta x} \right] \]

And by divide on \( x^m e^{ibn\Delta x} \) we getting

\[ x = -x^{-1} + [2 - 2r^2 - (\Delta t)^2] + r^2 \left[ e^{ib\Delta x} + e^{-ib\Delta x} \right] \]

\[ \frac{x^{n+1}}{x^n} = \left[ 2 - 2r^2 - (\Delta t)^2 \right] + 2r^2 \cos(b \Delta x) \]

\[ \frac{x^{n+1}}{x^n} = \left[ 2 - 2r^2 - (\Delta t)^2 \right] + 2r^2 \left[ 1 - 2\sin^2 \left( \frac{b \Delta x}{2} \right) \right] \]

\[ \frac{x^{n+1}}{x^n} = 2 \left[ 1 - \frac{(\Delta t)^2}{2} - 2r^2 \sin^2 \left( \frac{b \Delta x}{2} \right) \right] \]

\[ a = 1 - \frac{(\Delta t)^2}{2} - 2r^2 \sin^2 \left( \frac{b \Delta x}{2} \right) \]

\[ x^2 - 2ax + 1 = 0 \quad \Rightarrow \quad x = a \pm \sqrt{a^2 - 1} \]

The must \(|a| \leq 1 \), \( x \) is factor

\[ \left| 1 - \frac{(\Delta t)^2}{2} - 2r^2 \sin^2 \left( \frac{b \Delta x}{2} \right) \right| \leq 1 \]

(21)

And from Inequalities (21), we getting
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And by taking the side Inequality (21), we getting right

$$1 - \frac{(\Delta t)^2}{2} - 2r^2 \sin^2 \left( \frac{b\Delta x}{2} \right) \leq 1$$  \hspace{1cm} (22)

And from Inequalities (22), we getting

$$r^2 \geq \frac{- (\Delta t)^2}{4 \sin^2 \left( \frac{b\Delta x}{2} \right)}$$

Since \( r^2 = \frac{(\Delta t)^2}{(b\Delta x)^2} \), the value \( r^2 \) is positive.

Inequality (22) way to \( r^2 > 0 \) this true early, for to suites Inequality (21), we need to:

$$1 \leq 1 - \frac{(\Delta t)^2}{2} - 2r^2 \sin^2 \left( \frac{b\Delta x}{2} \right)$$

$$2 \geq \frac{(\Delta t)^2}{2} + 2r^2 \sin^2 \left( \frac{b\Delta x}{2} \right)$$

The form (2) show that solution by using Crank – Nicholson

\[ \frac{\partial^2 u}{\partial x^2} = \frac{1}{k^2} \left( \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + 2u_{i,j-1} + u_{i-1,j-1}}{h^2} \right) \]  \hspace{1cm} (25)

\[ \frac{\partial^2 u}{\partial x^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \]  \hspace{1cm} (26)

And as substitute the equation (25,26) in equation (3) we get:

\[ \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + 2u_{i,j-1} + u_{i-1,j-1}}{2h^2} = \sin \left[ u_{i,j} \right] \]  \hspace{1cm} (27)

And from equation (28) we get:

\[ -r^2 \left[ u_{i+1,j+1} + u_{i-1,j+1} \right] + \left[ 2 + 2r^2 \right] u_{i,j+1} \]

\[ = 4u_{i,j} - 2k^2 \sin \left[ u_{i,j} \right] + r^2 \left[ u_{i+1,j-1} + u_{i-1,j-1} \right] - \left[ 2 + 2r^2 \right] u_{i,j-1} \]  \hspace{1cm} (28)

Where the equation (28) is the approximate the finite difference of [Crank – Nicholson] method for [Sine – Gordon] method, and hence we have: three variables \( u_{i+1,j+1}, u_{i,j+1}, u_{i-1,j+1} \) and
all terms  in right hand side of equation (28) as know , so the equation (28) can be lead to the algebraic liner system with three diameters

\[ AX = B \]  

(29)

A : is a matrix have three diameters , X: Vertical vector , B: Vertical vector

Example: Let we have rectangular plate (2m, 4m) and it is coincide on x and y axis at original point

\[ \Delta t = 4 \sec, \Delta x = 5cm, a = 1.25 \]

Initial condition \( v(x, t) = 0^\circ, 0 < x < 10 \quad t = 0 \)

Boundary condition \( v(0, t) = 8^\circ, v(10, t) = 4^\circ \)

Solution:

1) Make the mesh and substitute the values on shapes

2) The crank – Nicolson general formula

\[-rv_i^{i+1} + 2(1 + m)v_i^{i+1} - rv_{i+1}^{i+1} = rv_{i-1}^{i} + 2(1 - m)v_i^{i} + rv_{i+1}^{i} \]

To find \( m \)

\[ m = a \frac{\Delta t}{\Delta x^2} \rightarrow r = 1.25 \frac{5}{5^2} = 0.2 \]

Substitute from \( l = 123, i = 0, m = 0.2 \)

\[-0.2v_0^0 + 2.4v_1^1 - 0.2v_2^1 = 0.2v_0^0 + 1.6v_1^0 + 0.2v_2^0 \]

\[-0.2v_1^1 + 2.4v_2^2 - 0.2v_3^2 = 0.2v_1^0 + 1.6v_2^0 + 0.2v_3^0 \]

\[-0.2v_2^1 + 2.4v_3^3 - 0.2v_4^3 = 0.2v_2^0 + 1.6v_3^0 + 0.2v_4^0 \]

Substitute initial and boundary condition , we get

\[-0.2(0) + 2.4v_1^1 - 0.2v_2^1 = 0.2(0) + 1.6(0) + 0.2(0) \]

\[-0.2v_1^1 + 2.4v_2^2 - 0.2v_3^2 = 0.2(0) + 1.6(0) + 0.2(0) \]

\[-0.2v_2^1 + 2.4v_3^3 - 0.2(4) = 0.2(0) + 1.6(0) + 0.2(0) \]

\[ v_1^1 = 2.4v_1^1 - 0.2v_2^1 = 1.6 \]

\[ -0.2v_1^1 + 2.4v_2^2 - 0.2v_3^2 = 0 \]

\[ -0.2v_2^1 + 2.4v_3^3 = 0.8 \]

Multiply equation (2) by 12 and solve it with equation (1) we have

\[-2.4v_1^1 + 28.8v_2^2 - 2.4v_3^3 = 0 \]

\[ 2.4v_1^1 - 0.2v_2^1 = 1.6 \]

\[ -28.6v_2^2 + 2.4v_3^3 = 1.6 \]

Solve equation (4) and adding to equation (3)

\[ 28.4v_2^2 = 2.4 \]

\[ v_2^2 = 0.0845070423 \]

Substitute \( v_2^2 \) value in equation (1) we obtain \( v_1^1 \)

\[ 2.4v_1^1 - 0.2(0.0845070423) = 1.6 \]

\[ 2.4v_1^1 = 1.6169014085 \]

\[ v_1^1 = 0.6737089202 \]

Substitute \( v_1^1 \) value in equation (3) we obtain \( v_3^3 \)

\[-0.2(0.0845070423) + 2.4v_3^3 = 0.8 \]
Continue to substitute from $l = 123, i = 0, m = 0.2$ in general formula

$$-0.2v_0^2 + 2.4v_1^2 - 0.2v_2^2 = 0.2v_1^2 + 1.6v_1^2 + 0.2v_2^2$$

$$-0.2v_2^2 + 2.4v_2^2 - 0.2v_3^2 = 0.2v_1^2 + 1.6v_3^2 + 0.2v_3^2$$

$$-0.2v_1^2 + 2.4v_3^2 - 0.2v_4^2 = 0.2v_1^2 + 1.6v_3^2 + 0.2v_4^2$$

Hence, we have

$2.4v_1^2 = 4.294835681$ ......... (1)

$-0.2v_1^2 + 2.4v_2^2 - 0.2v_2^2 = 0.338028169$ ......... (2)

$-0.2v_2^2 + 2.4v_3^2 = 2.161502347$ ......... (3)

Multiply equation (2) by 12 and solve it with equation (1) we have

$-2.4v_1^2 + 28.8v_2^2 - 2.4v_3^2 = 4.056338028$

$2.4v_1^2 = 4.294835681$

$28.6v_2^2 - 2.4v_3^2 = 8.351173709$ ......... (4)

Solve equation (4) and adding to equation (3)

$28.6v_2^2 - 2.4v_3^2 = 8.351173709$

$-0.2v_1^2 + 2.4v_3^2 = 2.161502347$

$28.4v_2^2 = 10.51267606$

$v_2^2 = 0.37016465$

Substitute $v_2^2$ value in equation (1) we obtain $v_1^2$

$2.4v_1^2 - 0.2(0.37016465) = 4.294835681$
Substitute value in equation (3) we obtain \( v_3^3 = 2.645426853 \)

Substitute \( v_3^2 \) value in equation (3) we obtain \( v_3^3 \)

\[ -0.2(0.812062217) + 2.4v_3^2 = 3.164389781 \]

\[ 2.4v_3^3 = 3.326802225 \rightarrow v_3^3 = 1.386167594 \]

\[
\begin{array}{|c|c|}
\hline
\text{Crank-Nicolson} & \text{Exact solution} \\
\hline
0.67370892 & 0.98 \\
0.084507042 & 0.24 \\
0.340375587 & 0.398 \\
1.820361921 & 1.98 \\
0.37016465 & 0.55 \\
0.931473032 & 0.6 \\
2.645426853 & 2.398 \\
0.812062217 & 1.06 \\
1.386167594 & 1.4 \\
\hline
\end{array}
\]

Table (1) Stability of Crank-Nicolson

7- Conclusion

1- the Stability of Crank – Nicholson is un condition \( V r^2 \).

2- the Stability of Explicit Methods is stable if \( r^2 \leq \frac{4(\omega^2)}{4} \).

3- The Explicit Method is faster than Crank – Nicholson method to get the result (50%).

4- The Crank – Nicholson is more accuracy the Explicit Method.

References


