IASJ

المجلات الاكاديمية العراقية

Tikrit Journal of Pure Science

ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

Studying the Itõ 's formula for some stochastic differential equation: (Quotient stochastic differential equation)

Abdulghafoor J. Salim , Ali F. Ali

Department of Mathematics, college of computer science and Mathematics, Mosul University, Mosul, Iraq

https://doi.org/10.25130/tjps.v26i3.150

ARTICLE INFO.

Article history:

-Received: 1 / 3 / 2021

-Accepted: 9 / 4 / 2021

-Available online: / / 2021

Keywords: stochastic differential equation, Itõ's formula, quotient stochastic differential equation.

Corresponding Author:

Name: Abdulghafoor J. Salim

E-mail:

'drabdul_salim@uomosul.edu.iq

ali.csp109@student.uomosul.edu.iq

Tel:

ABSTRACT

The aim of this paper is to study Itõ 's formula for some stochastic differential equation such as quotient stochastic differential equation, by using the function F (t, x (t)) which satisfies the product Ito's formula, then we find some calculus relation for the quotient stochastic differential equation and we generalize the method for all m supported by some examples to explain the method.

Introduction

A Stochastic differential equation is one of which more of its terms are stochastic process. [Arnold, 1974][1]. According the feature of randomly (stochastic) the numerical analysis for solving stochastic differential equations are also different in some key array from that of ordering differential equation .

The stochastic process $\{x(t),\ t \in [0,\infty)\}$ of the form: [2]

$dx(t)= \alpha(x(t),t)dt + \beta(x(t),t)\, dw(t)$ …(1)

Where $\alpha(x(t),t)$ is drift term, $\beta(x(t),t)$ is diffusion coefficient and $\{w(t)\}$ is a wiener process.

Or equivalently,

$X(t) = X(0) + \int_0^t \alpha(x(t),t)dt + \int_0^t \beta(x(t),t)dw(t)$ … (2)

$0 \le t \le T$, X (0) is the initial value.[11]

let $t \in [t_0, T]$, $\alpha(.)$, $\beta(.)$ sufficiently smooth in some region and F(x(t),t) is twice differential in that region . Then Itõ 's formula is:[11]

$$F(x(t),t) = F(x(t_0),t) + \int_{t_0}^{t} \frac{\partial F}{\partial t} ds + \int_{t_0}^{t}((\alpha(x(s),s)\frac{\partial F}{\partial x} + \frac{1}{2}\beta^2(x(s),s)\frac{\partial^2 F}{\partial x^2})ds + \int_{t_0}^{t} \beta(x(s),s)\frac{\partial F}{\partial x} dw_s \ \ldots (3)$$

Or equivalently by using Taylor Series :[10]

$$dF(x(t),t) = \left(\frac{\partial F(x,t)}{\partial t} + \alpha(x,t)\frac{\partial F(x,t)}{\partial x} + \frac{1}{2}\beta^2(x,t)\frac{\partial^2 F(x,t)}{\partial x^2}\right)dt + \beta(x,t)\frac{\partial F(x,t)}{\partial x}dw_t \ \ldots (4)$$

The stochastic differential equation models have been used after with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. The concept of the important of these models initialized in 1905, [3]. Various authors have explained and study their contribution in this field. Kloeden and Platen [4] discussed the numerical solution of stochastic differential in detail. Platens [5] study the strong and weak approximation methods for the numerical methods to get the solution of stochastic differential equations. Further work on some stochastic differential equation as they presented two explicit methods for *Itô* SDEs with Poisson-driven jumps. Nayak and Chakraverty [6] worked on numerical solution of fuzzy stochastic differential equation. Christios H.skiadas, [7] Study the exact solution of stochastic differential equation (Gomertz, Generalized logistic and revised exponential. Akinbo B.J. et al [2] study numerical solution of stochastic differential equation .And so on.

In this paper, we investigate the form for some (quotient) stochastic differential equation with constant coefficients with first order, second order and then we generalize it for m-order (quotient) stochastic differential equations. We also give some examples with exact solution to explain the method.

Related works and method

Definition 1 :(random variable) [8]

A random variable is a mapping or a function from the sample space Ω onto the real line R, (i.e. X: Ω →R)

Definition 2: (stochastic process) [9]

A stochastic process is a family of random variables denoted by $\{x(t), t \in T\}$ where t is time parameter and $T \in R$.

Definition 3: (Wiener process) [11]

A wiener process (Brownian motion) over [0, T] denoted by {w (t)} is a continuous-time stochastic process satisfying:

1: W (0) =0

2: For all t, s ≥ 0, W (t)-W(s) is normally distributed with mean zero and variance$|t-s|$.

3: The increment's W (t)-W(s) and W (v)-W (u) are independent.

Definition 4 :(Itõ – integral) [10]

Consider the stochastic integral $I(F)=I(F)(w)=\int_a^b f(s,w)dw(s,w)$ then the Itõ-integral is $\int_a^b f(t)dt=\lim_{n\to\infty}\sum_{k=1}^{\infty} f(t_k)\Delta w_k$ where $\Delta w_k=[w(t_{k+1})-w(t_k)]$ and w(t) is wiener proocess or we can write it by $\int_a^b f(t)dw(t)$ then the mean $E(\int_a^b f(t)dw(t))=0$ $E((\int_a^b f(t)dw(t))^2)=\int_a^b E(f^2(t))dt$ .The Itõ 's formula for $y_t(w)=v(t,x_t)$ with $dx_t= f(t,w)dw_t$ is given by

$$dy_t=(\frac{\partial v}{\partial t}+\frac{1}{2}f^2\frac{\partial v}{\partial x^2})dt+f\frac{\partial v}{\partial x}dw_t .$$

And then a stochastic process {x (t): t ϵ (0,∞) } is said to satisfy Itõ – stochastic differential equation .

$dx(t)= \alpha(x(t),t)dt + \beta(x(t),t)dw(t)$ if for $t \geq 0$ it is a solution of $x(t)=x(0)+\int_0^t \alpha(x(t),t)dt+\int_0^t \beta(x(t),t)dw(t)$. … (5)

Definition 5 :(Itõ 's formula) [11]

Let X (t) be a real-valued stochastic process then

$$x(a)=x(b)+\int_a^b Fdt+\int_a^b Gdw \quad \ldots (6)$$

For some $G \in L^2(0, T)$, $F \in L^1(0, T)$ and 0<a<b<T. Then X (t) is a stochastic differential equation satisfies:

$dX = Fdt+Gdw$; for $0 \leq t \leq T$.

Remark .[14]

L^1 [0, T], L^2[0, T] denotes the space of all real-valued, adaptive processes $\{x_t\}$, $\{y_t\}$ respectively , such that

$$E\left(\int_0^T |x_t|dt\right)<\infty$$

$$E\left(\int_0^T |y_t|dt\right)<\infty$$

If $u: R\times[0,T]\to R$ is continuous and their first and second partial derivative for t exist and are continuous. If we take $Y(t)=u(x(t),t)$,

Then the following equation is called **(Itõ 's formula)**:

$$dY=\frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial x}dX+\frac{1}{2}\frac{\partial u^2}{\partial x^2}G^2dt =(\frac{\partial u}{\partial t}+F\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial u^2}{\partial x^2}G^2)dt+\frac{\partial u}{\partial x}Gdw$$

The Ito product rule: [12]

Let the change in the process x_t between t and t+Δt is given by $\Delta x_t=x_{t+\Delta t}-x_t$ when Δt is very small, then we have

$dx_t=x_{t+dt}-x_t$ or $x_{t+dt}=x_t-dx_t$ … (7)

Suppose x_{1t} and x_{2t} are two stochastic processes, then:

$d(x_{1t}x_{2t})=x_{1t}dx_{2t}+x_{2t}dx_{1t}+dx_{1t}dx_{2t}$

Proof: by using equation (7) , we get

$dx_{1t}x_{2t}=x_{1(t+dt)}x_{2(t+dt)}-x_{1t}x_{2t}$

$dx_{1t}=x_{1(t+dt)}-x_{1t}\to x_{1(t+dt)}=dx_{1t}+x_{1t}$ … (8)

$dx_{2t}=x_{2(t+dt)}-x_{2t}\to x_{2(t+dt)}=dx_{2t}+x_{2t}$ … (9)

Then we get:

$d(x_{1t}x_{2t})=(dx_{1t}+x_{1t})(dx_{2t}+x_{2t})-(x_{2t}x_{2t})$

$=x_{1t}dx_{2t}+dx_{1t}dx_{2t}+x_{1t}dx_{2t}+x_{1t}x_{2t}-x_{1t}x_{2t}$

$=x_{2t}dx_{1t}+x_{1t}dx_{2t}+dx_{1t}dx_{2t}$ … (10)

Theorem :(1): (Itõ 's product rule) [11]

Let $dx_i=\alpha_i(t)dt+\beta_i(t)dW(t)$; $(i=1,2)$

$(0\leq t\leq T)$: $\alpha_i(t)\epsilon L^1(0,T)$, $\beta_i(t)\epsilon L^2(0,T)$,. Then $d(x_1(t)x_2(t))=X_1(t)dX_2(t)+X_2(t)dX_1(t)+\beta_1(t)\beta_2(t)dt$.

Let $\alpha_i(t)=\alpha_i$; $\beta_i(t)=\beta_i$ independent of t, where $i=1,2$

Therefore $d(x_1(t)x_2(t))=X_1(t)dX_2(t)+X_2(t)dX_1(t)+\beta_1\beta_2dt$ … (11)

Theorem :(2) [11]

Let $u(x)=x^m$, m=0,1,2,… then $d(x^m)=mx^{m-1}dx+\frac{1}{2}m(m-1)x^{m-2}G^2dt$

Clearly for m=0, 1 and m=2, follows from the Ito's product.

Now let the formula from -1 then.

$d(x^{m-1})=(m-1)x^{m-2}dx+\frac{1}{2}(m-1)(m-2)x^{m-3}G^2dt$

$=(m-1)x^{m-2}d(fdt+gdw)+\frac{1}{2}(m-1)(m-2)x^{m-3}G^2dt$

$d(x^m)=d(xx^{m-1})=x\,d(x^{m-1})+x^{m-1}dx+(m-1)x^{m-2}G^2dt$

$=x((m-1)x^{m-2}dx+\frac{1}{2}(m-1)(m-2)x^{m-3}G^2dt)+(m-1)x^{m-2}G^2dt+x^{m-1}dx$

$=mx^{m-1}dx+\frac{1}{2}m(m-1)x^{m-2}G^2dt$ … (12)

With these tools in hand, we can find the Itõ 's quotients rule for stochastic differential equation.

The quotient rule:

Proposition (1): If x_{t1} and $xt_2$are two stochastic processes satisfies

$dx(t)_i=\alpha_i(x(t),t)dt+\beta_i(x(t),t)dw(t): i=1,2$

Or equivalently satisfies the solution

$X(t)=X(0)+\int_0^t \alpha(x(t),t)dt+\int_0^t \beta(x(t),t)dw(t)$:

Then $d\left(\frac{x_1(t)}{x_2(t)}\right)=\frac{x_2(t)dx_1(t)-x_1(t)dx_2(t)-dx_1(t)dx_2(t)}{(x_2(t))^2}+\frac{x_1(t)}{(x_2(t))^3}(dx_2(t))^2$

Proof:

Let $x_1(t)=x_1$; $x_2(t)=x_2$ and suppose that h=x_1 and g=x_2 where h and g are time-independent

$d\left(\frac{x_1}{x_2}\right)=d\left(\frac{h}{g}\right)=d(hg^{-1})$, let $f(h,g)=\frac{h}{g}$ then

$f_h=\frac{1}{g}$, $f_g=\frac{-h}{g^2}$, $f_{hh}=0$, $f_{gg}=\frac{2h}{g^3}$, $f_{hg}=f_{gh}=\frac{-1}{g^2}$, applying Itõ multi dimentional formula yields[7] .

$d(\frac{h}{g})=d(f(h,g))=f_h\,dh+f_g dg+\frac{1}{2}f_{hh}(dh)^2+\frac{1}{2}f_{gg}(dg)^2-f_{hg}dhdg$

$=\frac{1}{g}dh-\frac{h}{g^2}dg+\frac{1}{2}(0)(dh)^2+\frac{1}{2}(\frac{2h}{g^3})(dg)^2-\frac{1}{g^2}dhdg$

$=\frac{1}{g}dh-\frac{h}{g^2}dg+(\frac{h}{g^3})(dg)^2-\frac{1}{g^2}dhdg$

$=\frac{hdg-gdh-dhdg}{g^2}+\frac{h}{g^3}(dg)^2$, where $h=x_1$ and $g=x_2$ then we have

$$d\left(\frac{x_1}{x_2}\right)=\frac{x_2dx_1-x_1dx_2-dx_1dx_2}{x_2^2}+\frac{x_1}{x_2^3}(dx_2)^2 \quad \dots (13)$$

So the proof is complete.

Proposition (2): let $x_1(t)$ and $x_2(t)$are two stochastic processes, then

$$d\left(\frac{x_1(t)}{x_2(t)}\right)^2=\frac{(x_2(t))^2(2x_1(t)dx_1(t)+G_1^2dt)-(x_1(t))^2(2x_2(t)dx_2(t)+G_2^2dt)}{(x_2(t))^4}-\frac{(2x_1(t)dx_1(t)+G_1^2dt)(2x_2(t)dx_2(t)+G_2^2dt)}{(x_2(t))^4}+\frac{(x_1(t))^2}{(x_2(t))^6}(2x_2(t)dx_2(t)+G_2^2dt)^2$$

Proof:

Let $(x_1(t))^2=x_1^2$; $(x_2(t))^2=x_2^2$ and suppose that h=x_1^2 and g=$x_2^2$where h and g are time-independent:

$d(\frac{x_1}{x_2})^2=d(\frac{x_1^2}{x_2^2})=d(\frac{h}{g})=d(hg^{-1})$, let $f(h,g)=\frac{h}{g}$ then $f_h=\frac{1}{g}$, $f_{hh}=0$, $f_g=\frac{-h}{g^2}$, $f_{gg}=\frac{2h}{g^3}$, $f_{hg}=f_{gh}=\frac{-1}{g^2}$, applying Itõ multi- dimensional formula yields . $d(\frac{h}{g})=d(f(h,g))$

$=f_h dh+f_g dg+\frac{1}{2}f_{hh}(dh)^2+\frac{1}{2}f_{gg}(dg)^2+f_{hg}dhdg$

$=\frac{1}{g}dh-\frac{h}{g^2}dg+\frac{1}{2}(0)(dh)^2+\frac{1}{2}(\frac{2h}{g^3})(dg)^2-\frac{1}{g^2}dhdg$

$=\frac{1}{g}dh-\frac{h}{g^2}dg+\frac{h}{g^3}(dg)^2-\frac{1}{g^2}dhdg=\frac{gdh-hdg-dhdg}{g^2}+\frac{h}{g^3}(dg)^2$

Where h=x_1^2 and g=x_2^2 , then:

$$d\left(\frac{x_1}{x_2}\right)^2=\frac{x_2^2dx_1^2-x_1^2dx_2^2-dx_1^2dx_2^2}{x_2^4}+\frac{x_1^2}{x_2^6}(dx_2^2)^2 ,$$

From equation (12) we have $dx^m=mx^{m-1}dx+\frac{1}{2}m(m-1)x^{m-2}G^2dt$

$dx^2=2xdx+\frac{1}{2}2(2-1)x^0G^2dt=2xdx+G^2dt$, so that

$$d\left(\frac{x_1}{x_2}\right)^2=\frac{x_2^2(2x_1dx_1+G_1^2dt)-x_1^2(2x_2dx_2+G_2^2dt)}{x_2^4}-\frac{(2x_1dx_1+G_1^2dt)(2x_2dx_2+G_2^2dt)}{x_2^4}+\frac{x_1^2}{x_2^6}(2x_2dx_2+G_2^2dt)^2 \quad \dots (14)$$

Then the proof is complete.

Now, if we have $\left(\frac{x_1}{x_2}\right)^m$ with $x_1(t)$ and $x_2(t)$ are given as before then :

$$d\left(\frac{x_1}{x_2}\right)^m=\frac{x_2^m\left(mx_1^{m-1}dx_1+\frac{1}{2}m(m-1)x_1^{m-2}G_1^2dt\right)}{(x_2^2)^m}-\frac{x_1^m\left(mx_2^{m-1}dx_2+\frac{1}{2}m(m-1)x_2^{m-2}G_2^2dt\right)}{(x_2^2)^m}-\frac{\left(mx_1^{m-1}dx_1+\frac{1}{2}m(m-1)x_1^{m-2}G_1^2dt\right)\left(mx_2^{m-1}dx_2+\frac{1}{2}m(m-1)x_2^{m-2}G_2^2dt\right)}{(x_2^2)^m}+\frac{x_1^m}{(x_2^3)^m}\left(mx_2^{m-1}dx_2+\frac{1}{2}m(m-1)x_2^{m-2}G_2^2dt\right) \quad \dots (15)$$

Proof:

Let h=x_1^m and g=x_2^m where h and g are time independent then

$d(\frac{x_1}{x_2})^m=d(\frac{x_1^m}{x_2^m})=d(\frac{h}{g})=d(hg^{-1})$, let $f(h,g)=\frac{h}{g}$ then $f_h=\frac{1}{g}$, $f_{hh}=0$, $f_g=\frac{-h}{g^2}$, $f_{gg}=\frac{2h}{g^3}$, $f_{hg}=f_{gh}=\frac{-1}{g^2}$, a plying Itõ multi -dimensional formula yields .

$d(\frac{h}{g})=d(f(h,g))=f_h dh+f_g dg+\frac{1}{2}f_{hh}(dh)^2+\frac{1}{2}f_{gg}(dg)^2+f_{hg}dhdg$

$=\frac{1}{g}dh-\frac{h}{g^2}dg+\frac{1}{2}(0)(dh)^2+\frac{1}{2}(\frac{2h}{g^3})(dg)^2-\frac{1}{g^2}dhdg$

$=\frac{1}{g}dh-\frac{h}{g^2}dg+\frac{h}{g^3}(dg)^2-\frac{1}{g^2}dhdg=\frac{gdh-hdg-dhdg}{g^2}+\frac{h}{g^3}(dg)^2$

Where h=x_1^m and g=x_2^m then:

$$d\left(\frac{x_1}{x_2}\right)^m=\frac{x_2^m dx_1^m-x_1^m dx_2^m-dx_1^m dx_2^m}{(x_2^2)^m}+\frac{x_1^m}{(x_2^3)^m}(dx_2^m)^2 ,$$

From equation (11) $dx^m=mx^{m-1}dx+\frac{1}{2}m(m-1)x^{m-2}G^2dt$[11]

$$d\left(\frac{x_1}{x_2}\right)^m=\frac{x_2^m\left(mx_1^{m-1}dx_1+\frac{1}{2}m(m-1)x_1^{m-2}G_1^2dt\right)}{(x_2^2)^m}-\frac{x_1^m\left(mx_2^{m-1}dx_2+\frac{1}{2}m(m-1)x_2^{m-2}G_2^2dt\right)}{(x_2^2)^m}-\frac{\left(mx_1^{m-1}dx_1+\frac{1}{2}m(m-1)x_1^{m-2}G_1^2dt\right)\left(mx_2^{m-1}dx_2+\frac{1}{2}m(m-1)x_2^{m-2}G_2^2dt\right)}{(x_2^2)^m}+\frac{x_1^m}{(x_2^3)^m}\left(mx_2^{m-1}dx_2+\frac{1}{2}m(m-1)x_2^{m-2}G_2^2dt\right)$$

The proof in the similar way likewise for m=1, 2.

The exact solution for the proposed formulas:

In this paragraph we give some examples for some order of quotient stochastic differential equation with two functions f(x (t), t) and g(x (t), t) which satisfies ito formula to, explain the method.

Lemma: (see [13])

Suppose $\{w_t\}$ is a Brownian motion then, by using Ito's Formula, we get:

$dw_t^2=2w_tdw_t+dt$, $(dw_t)^2=dt$, $dtdw_t=0$ and

$dw_t^3=3w_t^2dw_t+3w_tdt \quad \dots (16)$

Example (1):

Let $f(x)=\sin(x)$ and $g(y)=y-1: y\neq 1$, where f (.) and g (.) are two processes satisfies equation (2), then find $\int_a^b d(\frac{f}{g})$ where a=0, $b=\frac{\pi}{2}$, $\beta_1=\beta_2=1$, $\alpha_1=\alpha_2=0$

Solution: by the Ito product rule equation (11) then we have

$$d\left(\frac{f}{g}\right)=d\left(\frac{\sin(x)}{y-1}\right)=\frac{1}{y-1}d(\sin(x))+\sin(x)\,d\frac{1}{y-1}+\beta_1\beta_2dt$$

$\int_a^b d\left(\frac{f}{g}\right) = \int_a^b d\left(\frac{\sin(x)}{y-1}\right) = \int_a^b (\frac{1}{y-1} d(\sin(x)) + \sin(x)\, d\frac{1}{y-1} + \beta_1\beta_2 dt)$

$= \int_a^b \frac{1}{y-1} d(\sin(x)) + \int_a^b \sin(x)\, d\frac{1}{y-1} + \int_a^b \beta_1\beta_2 dt$

$\int_a^b \frac{1}{y-1} d(\sin(x)) = \frac{\sin(b)}{y(b)-1} - \frac{\sin(a)}{y(a)-1} - \int_a^b \sin(x)\, d\frac{1}{y-1} - \int_a^b \beta_1\beta_2 dt$ ... (17)

$\int_a^b \sin(x)\, d\frac{1}{y-1} = \frac{\sin(b)}{y(b)-1} - \frac{\sin(a)}{y(a)-1}$ ... (18)

From (17) and (18) we get

$\int_a^b \frac{1}{y-1} d(\sin(x)) = \left[\frac{\sin(b)}{y(b)-1} - \frac{\sin(a)}{y(a)-1}\right] - \left[\frac{\sin(b)}{y(b)-1} - \frac{\sin(a)}{y(a)-1}\right] - \int_a^b \beta_1\beta_2 dt = -\int_a^b \beta_1\beta_2 dt$ ... (19)

From (18) and (19) we get

$\int_a^b d\left(\frac{\sin(x)}{y-1}\right) = \left[-\int_a^b \beta_1\beta_2 dt\right] + \left[\frac{\sin(b)}{y(b)-1} - \frac{\sin(a)}{y(a)-1}\right] + \left[\int_a^b \beta_1\beta_2 dt\right]$

$= \frac{\sin(b)}{y(b)-1} - \frac{\sin(a)}{y(a)-1}$, since a=0, b=$\frac{\pi}{2}$ then

$\int_a^b d\left(\frac{\sin(x)}{y-1}\right) = \int_0^{\frac{\pi}{2}} d\left(\frac{\sin(x)}{y-1}\right) = \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}-1} - \frac{\sin(0)}{0-1} = \frac{1}{\frac{\pi}{2}-1} - 0 = \frac{1}{\frac{\pi-2}{2}} = \frac{2}{\pi-2}$

Example (2):

Let $f(x) = \sin(x)$ and $g(y) = y-1; (y \neq 1)$, then find $\int_a^b d(\frac{f}{g})^2$ where a=0, b=$\frac{\pi}{2}$,$\beta_1 = \beta_2 = 1$,$\alpha_1 = \alpha_2 = 0$

Solution: by the Ito product rule equation (11) then we have

$\int_a^b d(\frac{f}{g})^2 = \int_a^b d(\frac{\sin(x)}{y-1})^2 = \int_a^b d\left(\frac{\sin^2(x)}{(y-1)^2}\right) = \int_a^b [\frac{1}{(y-1)^2} d\sin^2(x) + \sin^2(x) d\left(\frac{1}{(y-1)^2}\right) + \beta_1\beta_2 dt] = \int_a^b \frac{1}{(y-1)^2} d\sin^2(x) + \int_a^b \sin^2(x) d\left(\frac{1}{(y-1)^2}\right) + \int_a^b \beta_1\beta_2 dt$

$\int_a^b \frac{1}{(y-1)^2} d\sin^2(x) = \frac{\sin^2(x(b))}{(y(b)-1)^2} - \frac{\sin^2(x(a))}{(y(a)-1)^2} - \int_a^b \sin^2(x) d\left(\frac{1}{(y(a)-1)^2}\right) - \int_a^b \beta_1\beta_2 dt$ ... (20)

$\int_a^b \sin^2(x) d\left(\frac{1}{y^2}\right) = \frac{\sin^2(x(b))}{(y(b)-1)^2} - \frac{\sin^2(x(a))}{(y(a)-1)^2}$ ... (21)

From (20) and (21) we get

$\int_a^b \frac{1}{y^2} d\sin^2(x) = \frac{\sin^2(x(b))}{(y(b)-1)^2} - \frac{\sin^2(x(a))}{(y(a)-1)^2} - [\frac{\sin^2(x(b))}{(y(b)-1)^2} - \frac{\sin^2(x(a))}{(y(a)-1)^2}] - \int_a^b \beta_1\beta_2 dt = -\int_a^b \beta_1\beta_2 dt$ ... (22)

From (21) and (22) then we have

$\int_a^b d(\frac{\sin(x)}{y})^2 = \left[-\int_a^b \beta_1\beta_2 dt\right] + \left[\frac{\sin^2(x(b))}{(y(b)-1)^2} - \frac{\sin^2(x(a))}{(y(a)-1)^2}\right] + \left[\int_a^b \beta_1\beta_2 dt\right]$

$= \frac{\sin^2(x(b))}{(y(b)-1)^2} - \frac{\sin^2(x(a))}{(y(a)-1)^2}$, since a=0 , b=$\frac{\pi}{2}$ then

$\int_a^b d(\frac{\sin(x)}{y-1})^2 = \int_0^{\frac{\pi}{2}} d(\frac{\sin(x)}{y})^2 = \frac{\sin^2(\frac{\pi}{2})}{(\frac{\pi}{2}-1)^2} - \frac{\sin^2(0)}{(0-1)^2}$

$= \frac{\frac{1}{2}(1-\cos(\pi))}{\frac{\pi^2}{4}-\pi+1} - \frac{\frac{1}{2}(1-\cos(0))}{1} = \frac{\frac{1}{2}(2)}{\frac{\pi^2}{4}-\pi+1} - 0 = \frac{4}{\pi^2-4\pi+4}$

Example 3:

If w_{t1} and $w_{2t}$are two stochastic processes, then

$d\left(\frac{w_{1t}}{w_{2t}}\right) = \frac{w_{2t}dw_{1t} - w_{1t}dw_{2t} - dw_{1t}dw_{2t}}{w_{2t}^2} + \frac{w_{1t}}{w_{2t}^3}dt$

$d\left(\frac{w_{1t}}{w_{2t}}\right) = d\left(w_{1t}\frac{1}{w_{2t}}\right) = d(w_{1t}w_{2t}^{-1})$

$= w_{1t} d\frac{1}{w_{2t}} + \frac{1}{w_{2t}} dw_{1t} + dw_{1t} d\frac{1}{w_{2t}}$

Let $x_{1t} = w_{1t}$ and $x_{2t} = w_{2t}$ then by using $d\left(\frac{x_1}{x_2}\right) = \frac{x_2dx_1 - x_1dx_2 - dx_1dx_2}{x_2^2} + \frac{x_1}{x_2^3}(dx_2)^2$ then we have

$d\left(\frac{w_{1t}}{w_{2t}}\right) = \frac{w_{2t}dw_{1t} - w_{1t}dw_{2t} - dw_{1t}dw_{2t}}{w_{2t}^2} + \frac{w_{1t}}{w_{2t}^3}(dw_{2t})^2$.

But since $(dw_{2t})^2$= dt and dwdt = 0, then we get

$d\left(\frac{w_{1t}}{w_{2t}}\right) = \frac{w_{2t}dw_{1t} - w_{1t}dw_{2t} - dw_{1t}dw_{2t}}{w_{2t}^2} + \frac{w_{1t}}{w_{2t}^3}dt$

Example 4:

If w_{t1} and $w_{2t}$are two stochastic processes, then

$d\left(\frac{w_{1t}}{w_{2t}}\right)^2 = d\left(\frac{w_{1t}^2}{w_{2t}^2}\right) = \frac{1}{w_{2t}^2}dw_{1t}^2 + w_{1t}^2 d\frac{1}{w_{2t}^2} + dw_{1t}^2 d\frac{1}{w_{2t}^2}$

Let $x_{1t}^2 = w_{1t}^2$ and $x_{2t}^2 = w_{2t}^2$ then by using $d\left(\frac{x_1}{x_2}\right)^2 = \frac{x_2^2dx_1^2 - x_1^2dx_2^2 - dx_1^2dx_2^2}{x_2^4} + \frac{x_1^2}{x_2^6}(dx_2^2)^2$ then we have

$d\left(\frac{w_{1t}}{w_{2t}}\right)^2 = \frac{w_{2t}^2dw_{1t}^2 - w_{1t}^2dw_{2t}^2 - dw_{1t}^2dw_{2t}^2}{w_{2t}^4} + \frac{w_{1t}^2}{w_{2t}^6}(dw_{2t}^2)^2$

Since $dw_t^2 = 2w_{1t}dw_t + dt$, $(dw_t)^2$ =dt and $dtdw_t$= 0 then

$d\left(\frac{w_{1t}}{w_{2t}}\right)^2 = \frac{w_{2t}^2(2w_{1t}dw_{1t}+dt) - w_{1t}^2(2w_{2t}dw_{2t}+dt)}{w_{2t}^4} - \frac{(2w_{1t}dw_{1t}+dt)(2w_{2t}dw_{2t}+dt)}{w_{2t}^4} + \frac{w_{2t}^2}{w_{2t}^6}(2w_{2t}dw_{2t} + dt)^2$

$d\left(\frac{w_{1t}}{w_{2t}}\right)^2 = \frac{w_{2t}^2(2w_{1t}dw_{1t}+dt) - w_{1t}^2(2w_{2t}dw_{2t}+dt)}{w_{2t}^4} - \frac{[4w_{1t}w_{2t}dw_{1t}dw_{2t}+(dt)^2]}{w_{2t}^4} + \frac{w_{2t}^2}{w_{2t}^6}(4w_{2t}^2dt + dt^2)$

Example 5:

Let $f(x_{1t}, x_{2t}) = \frac{x_{1t}}{x_{2t}}$ and x_{1t}=w_{1t} , $x_{2t} = w_{2t}$, $\alpha = 0$ and $\beta = 1$.

Then find $\int_a^b f(w_{1t}, w_{2t}) d(f(w_{1t}, w_{2t}))$.

Solution: since x_{1t} =w_{1t} and $x_{2t} = w_{2t}$ then $f(x_{1t}, x_{2t})$=$f(w_{1t}, w_{2t})$= $\frac{w_{1t}}{w_{2t}}$

$\int_a^b f(w_{1t}, w_{2t}) df(w_{1t}, w_{2t}) = \int_a^b \frac{w_{1t}}{w_{2t}} d\left(\frac{w_{1t}}{w_{2t}}\right) = \frac{1}{2}\left[\frac{w_{1t}}{w_{2t}}\right]^2\Big|_a^b$

$= \frac{1}{2}\left[\frac{w_{1t}^2(b)}{w_{2t}^2(b)} - \frac{w_{1t}^2(a)}{w_{2t}^2(a)}\right]$.

Example 6:

Let x_{1t}=w_{1t} ; $x_{2t} = w_{2t}$, $\alpha = 0$ and $\beta = 1$.

Then if:

$f(x_{1t}, x_{2t}) = \frac{x_{1t}}{x_{2t}}$, find $\int_a^b df(w_{1t}, w_{2t})$

solution : since x_{1t} =w_{1t} and $x_{2t} = w_{2t}$ then $f(x_{1t}, x_{2t})$=$f(w_{1t}, w_{2t})$= $\frac{w_{1t}}{w_{2t}}$ $\int_a^b df(w_{1t}, w_{2t})$ = $\int_a^b d\left(\frac{w_{1t}}{w_{2t}}\right) = \int_a^b [\frac{1}{w_{2t}} dw_{1t} + w_{1t} d\frac{1}{w_{2t}} + \beta_1\beta_2 dt]$

$= \int_a^b \frac{1}{w_{2t}} dw_{1t} + \int_a^b w_{1t} d\frac{1}{w_{2t}} + \int_a^b \beta_1\beta_2 dt$

$\int_a^b \frac{1}{w_{2t}} dw_{1t} = \frac{w_{1t}(b)}{w_{2t}(b)} - \frac{w_{1t}(a)}{w_{2t}(a)} - \int_a^b w_{1t} d\frac{1}{w_{2t}} - \int_a^b \beta_1\beta_2 dt$ ... (23)

$\int_a^b w_{1t} d\frac{1}{w_{2t}} = \frac{w_{1t}(b)}{w_{2t}(b)} - \frac{w_{1t}(a)}{w_{2t}(a)}$ ... (24)

From (23) and (24) we get

$$\int_a^b \frac{1}{w_{2t}} dw_{1t} = \frac{w_{1t}(b)}{w_{2t}(b)} - \frac{w_{1t}(a)}{w_{2t}(a)} - [\frac{w_{1t}(b)}{w_{2t}(b)} - \frac{w_{1t}(a)}{w_{2t}(a)}] - \int_a^b \beta_1\beta_2 dt$$

$$= -\int_a^b \beta_1\beta_2 dt \quad \dots (25)$$

From (24) and (25) then we have.

$$\int_a^b d(\frac{w_{1t}}{w_{2t}}) = [-\int_a^b \beta_1\beta_2 dt] + [\frac{w_{1t}(b)}{w_{2t}(b)} - \frac{w_{1t}(a)}{w_{2t}(a)}] + \int_a^b \beta_1\beta_2 dt = \frac{w_{1t}(b)}{w_{2t}(b)} - \frac{w_{1t}(a)}{w_{2t}(a)}$$

Conclusion

This paper has discussed two techniques for some stochastic differential equation, product rule and quotient rule, taking into consideration the Brownian motion as a wiener process which served as a basis for the second term in a stochastic process. We find the form of the first order, second order then we generalized to m-order. By solving some examples and find the exact solution for many order in quotient stochastic differential equation to explain the rule.

References

[1] Arnold L. (1974), **''Stochastic Differential Equation; Theory and Applications ''**, John Wiley and Sone, Inc.

[2] Akinbo B.J, Faniran T, Ayoola E.O. (2015),” **Numerical solution of stochastic differential equation**”, international Journal of advance research in science, engineering and technology .Vol. (2),Issue 5.

[3] Saver T. (2006), **"Numerical solution of Stochastic Differential Equations in Finance"**. Handbook of computational finance, Springer (2012)

[4] Kloeden P, Platen E. (1992)**," numerical solution of stochastic differential equations"**. Springer, Berlin

[5] E. Platen (1999)**," An introduction to numerical methods for stochastic differential equation"**, Acta Numerica, volume 8, page197-246

[6] J. H. Kim (2005), **"on Fuzzy stochastic differential equations"**, Journal of Korean Mathematical Society, volume 42, pages 153-169

[7] Christos H.Skiadas (2009),” **Exact solution of stochastic differential equation: Gomertz, Generalized logistic and Revised Exponential**”, Methodol Comput Appl Probab (2010) 12:261–270 DOI 10.1007/s11009-009-9145-3

[8] Burrage P. M.(1999), **"Runge-Kutta Method for stochastic Differential Equation** ", Ph.D. Thesis, Department of Mathematics, Queensland University, Brisbane, Queensland, Australia.

[9] Bernt ksendal,(2005)**, ‘Stochastic Differential Equations, An Introduction with Applications"**, Sixth Edition, Springer International Edition.

[10] Allen.E.(2007)," **modeling with Itõ stochastic differential equation**’ ‘Texas Tech University, USA.

[11] Lawrence C. Evans, (2013)**. "An Introduction to Stochastic Differential equation".**

[12] Sasha Cyganowski; Peter Kloeden; Jerzy Ombach (2002), **"Elementary Probability to Stochastic Differential Equation with Maple"**. - Berlin; Heidelberg; New York; Barcelona; Hong Kong; London; Milan; Paris; Tokyo: Springer.

[13] A.N. Kolmogorov published in (1933). **"The Basic Notation of Stochastic Calculus" .**

[14] Shuhong Liu, (2019)." **STOCHASTIC CALCULUS AND STOCHASTIC DIFFERENTIAL EQUATIONS".**

دراسة صيغة ايتو التكاملية لبعض المعادلات التفاضلية التصادفية:
(المعادلات التفاضلية التصادفية الكسرية)

عبد الغفور جاسم سالم ، علي فواز علي

قسم الرياضيات ، كلية علوم الحاسوب والرياضيات ، جامعة الموصل ، الموصل ، العراق

الملخص

يتضمن هذا البحث دراسة بعض المعادلات التفاضلية التصادفية (المعادلات التفاضلية كدالة بشكل دالة كسرية). وباستخدام دالة f(x(t),t) صيغة ايتو التكاملية التي تحقق خاصية الضرب, حاصل الضرب تم ايجاد صيغة ايتو لبعض الدوال الكسرية ابتداء من الدرجة الاولى والثانية ومن ثم ايجاد الصيغة العامة لحاصل القسمة (للدالة الكسرية)، وتم تعزيز هذه الصيغ ببعض الامثلة لتوضيح الطريقة.