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Using Crank-Nicolson Numerical Method to solve Heat-Diffusion Problem

Omar Abdullah Ajeel , Awni M. Gaftan

Department of Mathematics, College of Computer Science and Mathematics, Tikrit University, Tikrit, Iraq

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Corresponding Author:

Name: Omar Abdullah Ajeel

E-mail:

omar.alajeel1@gmail.com

awny.muhammed@tu.edu.iq

Tel:

ABSTRACT

The current study aimed to use the Crank-Nicolson numerical method to solve Heat-Diffusion Problem in comparison with the ADI method. In this paper, the general formula of the Crank-Nicolson Numerical Method was derived and applied to solve the heat diffusion. The same problem then has been solved using ADI numerical method. The results of the Crank-Nicolson numerical method were compared with that of the ADI numerical method. The comparison results revealed that Crank-Nicolson is more accurate than the results of ADI at the initial steps of the problem solution.

استخدام طريقة Crank-Nicolson العددية في حل مسائل انتشار الحرارة

عمر عبد الله عجيل ، عوني محمد كفطان

قسم الرياضيات ، كلية علوم الحاسوب والرياضيات ، جامعة تكريت ، تكريت ، العراق

الملخص

قمنا في هذا البحث باشتقاق الصيغة العامة لطريقة Crank-Nicolson العددية واستخداماتها في حل مسائل الانتشار الحراري مع مقارنة النتائج التي حصلنا عليها مع طريقة ADI العددية مع الحل المضبوط لتلك المسالة وكانت النتائج بان طريقة Crank-Nicolson تكون أكثر دقه في المراحل الأولى من الحل بينما طريقة ADI أكثر دقه في المراحل المتأخرة من الحل وذلك لطبيعة أسلوب الحل المعتمد في طريقة ADI والتي هي بالأساس دمج بين طريقة Implicit, Explicit والتي تتأثر بالشروط الحدودية أسرع من طريقة Crank-Nicolson.

Introduction

The heat-diffusion equation has played a great role in some physical phenomena. These equations were used to describes the operation of heat transfer and heat distribution [1]. The Heat Diffusion equation is one kind of partial differential equation, this equation belongs to parabolic equations that are linear equations [2].that are used to develop some techniques to design the suitable simulation method as in the natural operations. The Heat Diffusion equation describes two kinds of this phenomena, diffusion and heater loading, so these equations have a good tool to describe these kinds [3-4] , as well as,

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many creeping flow problems are governed by some types of partial differential equation [5-6]. The Crank-Nicolson method is one of the finite differences methods that were used in numerical solutions of heat equations and a symmetric partial differential equation [7], these method was developed by John Crank and Phyllis Nicolson in the middle of twenty century, the research depends on the variable of time (t) and this method is used to solve many of problems such as heat diffusion, elasticity, gas diffusion and, creeping flow [8] as well as, this method was used with other Numerical Method (finite differences, finite elements) to obtain the effecting methods to solve the heat diffusion methods but the results of this technique are not accurate in some of the problems [9], The Crank-Nicolson Method is a mix from implicit and explicit method and it is the basis of ADI method [10].The results of this method are more accurate than other methods and it has more stability than any other method [11], The ADI method is one of the numerical methods where its technique is applied alternatively: first using the implicit method and the second using the explicit method. This alternating technique means we can find the values on X-axis in the first step and find the value on Y-axis in the second step[12], The current study aimed to use the Crank-Nicolson numerical method to solve Heat-Diffusion Problem in comparison with the ADI method.

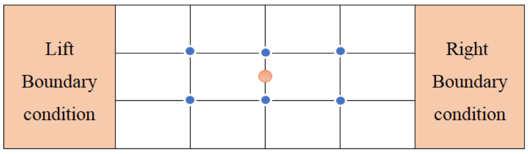
The main steps solution by the Crank-Nicolson Method are:

Step1: dividing the region by vertical and horizontal levels, to obtain the mesh.

Step2: find value of node points (i.e find u_1^1 and u_2^1)

Step3: find the midpoints (i.e find the m.p of u_1^1 , u_2^1)

Step4: repeat the steps (2,3) on other levels and so on.



Initial condition

Midpoints by Crank-Nicolson

Derivation of the Crank-Nicolson formula

Know the heat equation $\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}$ ....(1) where a is constant [13] to obtain the general formula of Crank-Nicolson, we substitute the second partial derivative u_{xx} with the arithmetic average to approximate their central differences at k and $K+1$ [14]

From the general formulas:

$$\frac{\partial u}{\partial t}=\frac{u_j^{k+1}-u_j^k}{\Delta t} \quad ...(2)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{2}\left[\frac{u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}}{h^2}+\frac{u_{j+1}^{K}-2u_j^k+u_{j-1}^k}{h^2}\right] \quad ...(3)$$

Substituting(2) and (3) into equation (1), we obtain

$$\frac{1}{2}\left(\frac{u_{j+1}^{K}-2u_j^k+u_{j-1}^k}{h^2}+\frac{u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}}{h^2}\right)=\frac{u_j^{k+1}-u_j^k}{\Delta t} \quad ...(4)$$

Step2: multiply equation (3) by $2\Delta t$

$$2(u_j^{k+1}-u_j^k)=a\frac{\Delta t}{h^2}(u_{j+1}^{K}-2u_j^k+u_{j-1}^k+u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}) \quad ...(5)$$

Step3: simplify the formula in step2, we assume $r=a\frac{\Delta t}{h^2}$ $2(u_j^{k+1}-u_j^k)=r(u_{j+1}^{K}-2u_j^k+u_{j-1}^k+u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1})$ ... (6)

$$2u_j^{k+1}-ru_{j-1}^{k+1}+2ru_j^{k+1}-ru_{j+1}^{k+1}=2u_j^k+ru_{j-1}^k-2ru_j^k+ru_{j+1}^K$$

$$-ru_{j-1}^{k+1}+(2+2r)u_j^{k+1}-ru_{j+1}^{k+1}=ru_{j-1}^k+(2-2r)u_j^k+ru_{j+1}^K$$

$$-ru_{j-1}^{k+1}+2(1+r)u_j^{k+1}-ru_{j+1}^{k+1}=ru_{j-1}^k+2(1-r)u_j^k+ru_{j+1}^k \quad ...(7)$$

The equation (7) is representing the general form of Crank-Nicolson [15]

Drawbacks of the Crank-Nicolson method

The main drawback of this method is producing a complex system of linear equations in spite of that his solution is more efficient. and this needs more steps for a solution

Crank-Nicolson Algorithm

1- Make a mesh on the region and put the boundary values and initial conditions.

2- Substitute the value of (k=0 and j=1,2,3) where k represents the time periods (the time divisions on the y axis) and j represents the nodal points on the x axis in general from of Crank-Nicolson formula equation (7).

3- Substitute the boundary values and initial conditions and the value of r in equation (7) to obtain three equations by solving the general equation with substitutions.

4- Solve the equation to find u_1^1, u_2^1 *and* u_3^1 and iterate or repeat the previous steps at $k=1$.

Now, the following example explains this algorithm

Example: let us have rectangular plate by the dimensions (2*4m) and it is considered on x and y axis at the original point:

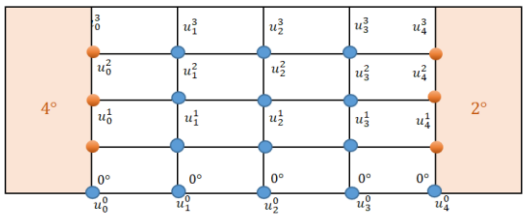
Where $\Delta t=2\ sec,\ \Delta x=2.5cm\ ,\ a=0.625.$

Initial condition $u(x\ ,t)=0°\quad 0<x<10\quad t=0.$

Boundary conditions $u(\,0\,,t)=4°\ ,\ u(\,10\,,t\,)=2°$

Solation:

Make the mesh and substitute the values on shape



Initial condition

The Crank-Nicolson general formula

$$-ru_{j-1}^{k+1}+2(1+r)u_j^{k+1}-ru_{j+1}^{k+1}=ru_{j-1}^k+2(1-r)u_j^k+ru_{j+1}^k$$

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To find r

$r = a\frac{\Delta t}{\Delta x^2} \rightarrow r = 0.625\frac{2}{2.5^2} = 0.2$

We substitute the value of j = 1,2,3 and k = 0 and the value of r = 0.2 into Crank-Nicolson general formula

$-0.2u_0^1 + 2.4u_1^1 - 0.2u_2^1 = 0.2u_0^0 + 1.6u_1^0 + 0.2u_2^0$ ... (1)

$-0.2u_1^1 + 2.4u_2^1 - 0.2u_3^1 = 0.2u_1^0 + 1.6u_2^0 + 0.2u_3^0$ ... (2)

$-0.2u_2^1 + 2.4u_3^1 - 0.2u_4^1 = 0.2u_2^0 + 1.6u_3^0 + 0.2u_4^0$ ... (3)

Substituting initial and boundary values

$-0.2(4) + 2.4u_1^1 - 0.2u_2^1 = 0.2(0) + 1.6(0) + 0.2(0)$ ... (4)

$-0.2u_1^1 + 2.4u_2^1 - 0.2u_3^1 = 0.2(0) + 1.6(0) + 0.2(0)$ ... (5)

$-0.2u_2^1 + 2.4u_3^1 - 0.2(2) = 0.2(0) + 1.6(0) + 0.2(0)$ ... (6)

Hence, we have

$2.4u_1^1 - 0.2u_2^1 = 0.8$ ... ... (7)

$-0.2u_1^1 + 2.4u_2^1 - 0.2u_3^1 = 0$ ... ... (8)

$-0.2u_2^1 + 2.4u_3^1 = 0.4$ ... (9)

Multiply equation (8) by 12 and solve it with equation (7) we have

$-2.4u_1^1 + 28.8u_2^1 - 2.4u_3^1 = 0$

$2.4u_1^1 - 0.2u_2^1 = 0.8$

$28.6u_2^1 - 2.4u_3^1 = 0.8$ ... ... (10)

from (10) and adding to equation (9)

$28.6u_2^1 - 2.4u_3^1 = 0.8$

$-0.2u_2^1 + 2.4u_3^1 = 0.4$

$28.4u_2^1 = 1.2$

$u_2^1 = 0.0422535211$

Substituting u_2^1 value in equation (8), we obtain u_1^1

$2.4u_1^1 - 0.2(0.0422535211) = 0.8$

$2.4u_1^1 = 0.8084507042$

$u_1^1 = 0.3368544601$

Also Substituting u_2^1 value in equation (9), we obtain u_3^1

$-0.2(0.0422535211) + 2.4u_3^1 = 0.4$

$2.4u_3^1 = 0.4084507042$

$u_3^1 = 0.1701877934$

Continue to Substitute from j = 1,2,3 and k = 1 and r = 0.2 in general formula

$-0.2u_0^2 + 2.4u_1^2 - 0.2u_2^2 = 0.2u_0^1 + 1.6u_1^1 + 0.2u_2^1$ ... (11)

$-0.2u_1^2 + 2.4u_2^2 - 0.2u_3^2 = 0.2u_1^1 + 1.6u_2^1 + 0.2u_3^1$ ... (12)

$-0.2u_2^2 + 2.4u_3^2 - 0.2u_4^2 = 0.2u_2^1 + 1.6u_3^1 + 0.2u_4^1$ ... (13)

Substitute initial and boundary values and u_1^1, u_2^1, u_3^1 values

$-0.2(4) + 2.4u_1^2 - 0.2u_2^2 = 0.2(4) + 1.6(0.3368544601) + 0.2(0.0422535211)$ ... (14)

$-0.2u_1^2 + 2.4u_2^2 - 0.2u_3^2 = 0.2(0.3368544601) + 1.6(0.0422535211) + 0.2(0.1701877934)$ ... (15)

$-0.2u_2^2 + 2.4u_3^2 - 0.2(2) = 0.2(0.0422535211) + 1.6(0.1701877934) + 0.2(2)$ ... (16)

Hence, we have

$2.4u_1^2 - 0.2u_2^2 = 2.1474178404$ ... ... (17)

$-0.2u_1^2 + 2.4u_2^2 - 0.2u_3^2 = 0.1690140845.$ ... ... (18)

$-0.2u_2^2 + 2.4u_3^2 = 1.0807511737$ ... ... (19)

Multiply equation (18) by 12 and solve it with equation (17) we have

$-2.4u_1^2 + 28.8u_2^2 - 2.4u_3^2 = 2.028169014$

$2.4u_1^2 - 0.2u_2^2 = 2.1474178404$

$28.6u_2^2 - 2.4u_3^2 = 4.1755868544$ ... ... (20)

from (20) and adding to equation (19)

$28.6u_2^2 - 2.4u_3^2 = 4.1755868544$

$-0.2u_2^2 + 2.4u_3^2 = 1.0807511737$

$28.4u_2^2 = 5.2563380281$

$u_2^2 = 0.1850823249$

Substituting u_2^2 value in equation (17), we obtain u_1^2

$2.4u_1^2 - 0.2(0.1850823249) = 2.1474178404$

$2.4u_1^2 = 2.1844343054$

$u_1^2 = 0.9101809606$

Also Substituting u_2^1 value in equation (19), we obtain u_3^2

$-0.2(0.1850823249) + 2.4u_3^2 = 1.0807511737$

$2.4u_3^2 = 1.1177676387$

$u_3^2 = 0.4657365161$

Continue to Substitute from j = 1,2,3 and k = 2 and r = 0.2 in general formula

$0\ .2u_0^3 + 2.4u_1^3 - 0.2u_2^3 = 0.2u_0^2 + 1.6u_1^2 + 0.2u_2^2$ ... (21)

$-0.2u_1^3 + 2.4u_2^3 - 0.2u_3^3 = 0.2u_1^2 + 1.6u_2^2 + 0.2u_3^2$ ... (22)

$-0.2u_2^3 + 2.4u_3^3 - 0.2u_4^3 = 0.2u_2^2 + 1.6u_3^2 + 0.2u_4^2$ ... (23)

Substitute initial and boundary value and u_1^2, u_2^2, u_3^2 values

$-0.2(4) + 2.4u_1^3 - 0.2u_2^3 = 0.2(4) + 1.6(0.9101809606) + 0.2(0.1850823249)$ ... (24)

$-0.2u_1^3 + 2.4u_2^3 - 0.2u_3^3 = 0.2(0.9101809606) + 1.6(0.1850823249) + 0.2(0.4657365161)$ ... (25)

$-0.2u_2^3 + 2.4u_3^3 - 0.2(2) = 0.2(0.1850823249) + 1.6(0.4657365161) + 0.2(2)$ ... (26)

Hence, we have

$2.4u_1^3 - 0.2u_2^3 = 3.0933060019$ ... ... (27)

$-0.2u_1^3 + 2.4u_2^3 - 0.2u_3^3 = 0.5713152152$ ... ... (28)

$-0.2u_2^3 + 2.4u_3^3 = 1.5821948907$ ... ... (29)

Multiply equation (28) by 12 and solve it with equation (27) we have

$-2.4u_1^3 + 28.8u_2^3 - 2.4u_3^3 = 6.8551825824$

$2.4u_1^3 - 0.2u_2^3 = 3.0933060019$

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$28.6u_2^3 - 2.4u_3^3 = 9.9490885843 \ldots\ldots\ldots (30)$
from (30) and adding to equation (29)
$8.6u_2^3 - 2.4u_3^3 = 9.9490885843$
$-0.2u_2^3 + 2.4u_3^3 = 1.5821948907$

$28.4u_2^3 = 11.531283475$
$u_2^3 = 0.4060311083$
Substituting u_2^3 value in equation (1), we obtain u_1^3
$2.4u_1^3 - 0.2(0.4060311083) = 3.0933060019$
$2.4u_1^3 = 3.1745122236$
$u_1^3 = 1.3227134265$
Also Substituting u_2^1 value in equation (29), we obtain u_3^3
$-0.2(0.4060311083) + 2.4u_3^3 = 1.5821948907$
$2.4u_3^3 = 1.6634011124$
$u_3^3 = 0.6930837968$

Table 1: explain the difference between the Crank-Nicolson result and the ADI result, with exact solution

Crank-Nicolson	ADI Method [16]	Exact solution [16]
0.3368544601	2	0.49
0.0422535211	0.13	0.12
0.1701877934	0.21	0.199
0.9101809606	1.001	0.99
0.1850823249	0.299	0.275
0.4657365161	0.311	0.300
1.3227134265	1.250	1.199
0.4060311083	0.530	0.530
0.6930837968	0.700	0.700

Conclusions

1-The Crank-Nicolson method is more accurate in primary first levels than the ADI method, but ADI is more accurate in the last steps.
2- The stability of the Crank-Nicolson method is superior to the ADI method because the ADI depends on two methods (explicit and implicit) where their stabilities are less than the stability of the Crank-Nicolson method.
3-The total time to reach the final results of the Crank-Nicolson method is less than the time of the ADI method.

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