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The Stability and Catastrophic Behavior of Finite Periodic Solutions in Non-Linear Differential Equations

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ABSTRACT

This study focuses on the stability and catastrophic behavior of finite periodic solutions in non-linear differential equations. The occurrence of folding surfaces and their relationship with saddle-node bifurcations are explored, being classified as fold and butterfly types of catastrophes. Additionally, the application of catastrophe theory is discussed to analyze the qualitative changes in solutions with the change in system parameters.

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الاستقرار والسلوك الكارثي للحلول الدورية المحدودة في المعادلات التفاضلية الغير خطية

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الملخص

تركز هذه الدراسة على استقرار وسلوك الكوارث للحلول الدورية المحدودة في المعادلات التفاضلية غير الخطية. نستكشف حدوث الأسطح المطوية وعلاقتها بتشعبات نقطة العجرية، ونصنفها كأنواع من الكوارث الطي والفراشة. بالإضافة إلى ذلك، نناقش تطبيق نظرية الكوارث لتحليل التغيرات النوعية في الحلول مع تغير معلمات النظام.

الكلمات المفتاحية: نموذج كارثة الطي؛ كارثة من نمط الفراشة؛ المعادلات التفاضلية غير الخطية؛ الحلول الدورية.

1. Introduction

The limit cycle is an isolated closed orbit in a dynamical system, which is stable (or attractive) if all nearby paths get close to it. If not, it is considered to be unstable. Equations have a difficulty in describing several aspects of the discontinuous jumping phenomena. The catastrophe theory (CT) can explain these aspects. In the book under review (abbreviated as ZCT), Just What is Catastrophe Theory? Zeeman compares Newton and Thom. According to Thom, CT "needs to be viewed as a broad morphological theory. A novel mathematical technique called 633 ZCT and Zeeman CT are used to explain how forms evolve in nature". Thorn and Zeeman introduced the elementary catastrophe theory (ECT). In brief, ECT investigates a smooth real-valued map as a function of a state x and a parameter, commonly referred to as a "potential function." Three elementary catastrophic types, namely the Fold, Cusp, and Butterfly Catastrophes, as named by Thom are now existing after studying their theory in this study. The Fold Catastrophic Model was designed to assess stability through graphing the non-linear differential equations Fold Model and its bifurcation set. The projection of the folding part of the Fold Catastrophic Model onto the control parameter is always accompanied by a saddle-node bifurcation. The collapse mode of rock bursts is critical for both practical and theoretical analysis applications in catastrophe theory. The examination of catastrophic issues, such as equilibrium points, catastrophic manifold, capacitance, and phenomenon jump, has been of great interest for a long time because of its increasing applications in physical, biological, and social sciences. Some writers, such as [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], and the authors made important contributions to the examination of various topics, such as points of balance, catastrophic models, recurring patterns, stability and instability, and phenomena linked to forced vibrations. This study aims to determine periodic solutions in a non-linear differential equation and assess their stability and semi-stability. The significance of the saddle-node bifurcation and its classification as a Fold, Cusp, and butterfly mutation in the context of Fold, Cusp, and butterfly surface folding lies in finding the number of stable, semi-stable, and unstable periodic solutions. Applying catastrophe theory can provide insights into the stability and catastrophic behavior of non-linear differential equations by analyzing the qualitative changes that occur in the solutions as the system parameters vary. Catastrophe theory, developed by mathematician René Thom, is a branch of mathematics that studies sudden and dramatic changes in the behavior of systems as a result of small changes in their parameters. Catastrophe theory enhances the comprehension of stability and catastrophic phenomena in the studied equations by providing a geometric and qualitative framework to analyze and understand the behavior of non-linear systems. It focuses on the study of critical points, bifurcations, and the changes in system dynamics as parameters vary. Furthermore, as indicated in the study of [7], the occurrence of catastrophic phenomena, such as folds, cusps, butterflies, and other similar types is contingent upon the level of the non-linearity present in the differential equation.

2. Types of Catastrophe (CT)

CT searches for degenerating the stationary points of a potential function (PF) that are zero in the first derivative as well as one or higher derivatives. When the parameters are perturbed somewhat, PF can be expanded as a Taylor series to reveal the degeneracy of stationary points. The stability analysis of periodic solutions provides insights into the long-term behavior of nonlinear differential equations (for some applications of CT on nonlinear differential equations see [6]).

Understanding the stability properties helps assess the susceptibility of periodic solutions to small perturbations and provides indications of the potential for catastrophic behavior.

The following different local forms of elementary catastrophe [13] occur for four or fewer variables:

1. Fold-type catastrophes,
2. Cusp-type catastrophes,
3. Swallowtail-type catastrophes,
4. Butterfly-type catastrophes.

The following types of catastrophe for the single variable potential functions are studied:

a. Fold catastrophe

A catastrophe which can occur for one control factor u_1 and one behavior axis ζ. It is the universal unfolding of the singularity $F(\zeta) = \zeta^3$ and has the following equation:

$$F(\zeta, u_1) = \zeta^3 + u_1\zeta$$

Where u_1 is the first parameter and ζ is the independent variable. When $u_1 < 0,$ the stable and unstable extrema exist for the function F. The system can follow the stable lowest point if parameter u_1 is gradually increased. Yet, the stable and unstable extrema collide and destroy each other at $u_1 = 0$. The bifurcation point is where this occurs. There is no longer a reliable solution at $u_1 > 0$.

b. Cusp catastrophe

A cusp catastrophe can occur for two control factors $u_1,\ u_2$, and one behavior axis ζ. The cusp catastrophe is the universal unfolding of the singularity $f(\zeta) = \zeta^{\,4}$ and has the following equation

$$F(\zeta, u_1, u_2) = \zeta^4 + u_1\zeta^2 + u_2\zeta$$

When investigating the effect of adding the second parameter (u_2) to the control plane on a fold bifurcation, the geometry of cusp type catastrophe is highly prevalent. When the values of u_1 and u_2 are changed, a set of points known as a bifurcation is established in the $(u_1, u_2\ )$ space, where semi-stability appears and stability disappears. This in turn causes a stable periodic solution to abruptly change to a different result (For more details see [13], and for some applications see [6]).

c. Butterfly catastrophe (for more information about Butterfly bifurcation see [3])

$$F(\zeta, u_1, u_2, u_3\ , u_4) = \frac{1}{6}\zeta^6 + \frac{1}{4}u_1\zeta^4 + \frac{1}{3}u_2\zeta^3 + \frac{1}{2}u_3\zeta^2 + u_{4\zeta}\ .$$

Where $u_1, u_2, u_3\ and\ u_4$ are parameters (real numbers).

3. Systems Arising from non-linear differential equation NLDE

Suppose the following form of NLDE

$$y'' = -\omega_0^2 y + \alpha f(x, y, y') \tag{1}$$

Where α is the ε- parameter and f is of period $\frac{2\pi}{\omega}$ concerning x, the linear form of Eq. (1) is not interesting because the catastrophic behavior appears only in the foregoing non-linear differential equation, and then the process continues to get the approximate solution of (1) for this purpose:

$$\text{Let } y' = v, \tag{2}$$

The next equation results from (1) and (2)

$$v' = y'' = -\omega_0^2 y + \alpha f(x, y, y') \tag{3}$$

Then, the solutions of equations (1) and (3) are:

$$y = a(x)sin\,(\omega x) + b(x)cos\,(\omega x)$$

(4)

$$v = \omega[a(x)cos\,(\omega x) - b(x)sin\,(\omega x)]\,,$$

Where $a(x)$ and $b(x)$ are very slowly varying functions, the following conditions (see [12]) must be satisfied (4):

$$a'sin\,(\omega x) - b'cos\,(\omega x) = 0 \tag{5}$$

$$a'cos\,(\omega x) - b'sin\,(\omega x) = \frac{\alpha}{\omega}[\beta y + f(x, y, y')] \tag{6}$$

$$\alpha\beta = \omega^2 - \omega_0^2 \tag{7}$$

The depended system below yields from (5), (6) and (7):

$$a' = \frac{\alpha}{\omega}\{\beta y + f(x, y, y')\}\cos\,(\omega x)$$

(8)

$$b' = -\frac{\alpha}{\omega}\{\beta y + f(x, y, y')\sin\,(\omega x)\}$$

Integrating Eqs. (8) with respect to x, for $0 < x < 2\pi/\omega$, there exists:

$$a' = \beta b + \mu a - \{X_2 ar^2 + X_4 ar^4 + \cdots + X_{2n} ar^{2n}\}$$

(9)

$$b' = -\beta a + \mu b - \{X_2 br^2 + X_4 br^4 + \cdots + X_{2n} br^{2n}\} - B$$

Where μ, β, B and $X_2, X_4, \dots, X_{2n}$ are the real parameters and $r = \sqrt{a^2 + b^2}$ is the Amplitude. What we desired as the typical system, and what we obtained from the general form (1).

4. Catastrophic Manifold (CM)

In this section, the stationary points of the system (9) have to be found.

Let $a' = b' = 0$, with simple simplifications (see [11]), there is:

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$$[\mu r - (X_2 r^3 + X_4 r^5 + \cdots + X_{2n} r^{2n+1})]^2 + \beta^2 r^2 - B^2 = 0 \qquad (10)$$

When using polar coordinate transformations:

$a = r\cos\ \theta, b = r\sin\theta$. Putting $\zeta = r^2$ and if the appropriate change of coordinates is performed, Eq. (10) can be reduced to the standard form of some types of catastrophes. In addition, some standard forms of (10) may be found (see [12]) for CM:

$$\zeta^{m-1} + u_1\zeta^{m-3} + u_2\zeta^{m-4} + \cdots + u_m = 0$$

This is the desired Eqn., where $m = 2n + 1$. A function F' is defined so that the non-linear dynamic model can be found as follows after integration with respect to ζ:

$$F'(\zeta) = -(\zeta^{m-1} + u_1\zeta^{m-3} + u_2\zeta^{m-4} + \cdots + u_m) \qquad (11)$$

The following equation is the canonical form for the potential function:

$$F(\zeta, u_1, u_2, \dots) = \frac{1}{m}\zeta^m + \frac{u_1}{m-2}\zeta^{m-2} + \cdots + u_m\zeta \qquad (12)$$

The averaged system (8) produces the following equation, where F is the potential function of the butterfly-type catastrophe, if the integer $n = 1$ is put, then $m = 3$, so the result is:

$$F(\zeta, u_1) = \frac{1}{3}\zeta^3 + u_1\zeta \qquad (13)$$

The stationary points of F are provided by

$$\frac{\partial F}{\partial \zeta} = \zeta^2 + u_1 = 0 \qquad (14)$$

Here, F and ζ are considered to be functions of the control variables, in this case, u_1. The non-linear dynamic model is defined as follows:

$$F(\zeta) = -(\zeta^2 + u_1) \qquad (14a)$$

Also, let us look into the Lipsanos function of this dynamic. Construct a function: $(\zeta, u_1) = \frac{1}{3}\zeta^3 + u_1\zeta$, via Fold catastrophe [15].

Someone saw that (14 a) is a Lyapunov function with

$$\frac{dF}{dt} = -(\zeta^2 + u_1)^2 < 0 \Leftrightarrow \zeta^2 + u_1 \neq 0 \qquad (15)$$

Therefore, in this section, the non-linear dynamical solution (14 a) is asymptotically stable.

For the cusp catastrophe (see [12]), for which, if $n = 1$, then $m = 3$.

The condition of three limit cycles is

$$\triangle = 4u_1^3 + 27u_2^2 < 0 \qquad (16)$$

For one or three limit cycles, the region's boundary is specified as:

$$4u_1^3 + 27u_2^2 = 0 \qquad (17)$$

It is demonstrated that the saddle-node bifurcation is a fold type catastrophe.

Furthermore, the following propositions exist:

Proposition 4.1: When Δ is less than zero in equation (16), there are two stable and one unstable periodic solution in the non-linear differential equation.

Proposition 4.2: Every time the folding of the Fold type catastrophe occurs, the saddle-node bifurcation always follows.

5. Conclusion

This study demonstrated that the non-linear differential equation NLDE has two stable and one unstable periodic solution when the value of Δ in equation (16) is negative. Additionally, it revealed that whenever a Fold type Catastrophe occurs, it is always accompanied by a saddle-node bifurcation. Moreover, when a fold type catastrophe takes place, the aforementioned differential equation has one stable and one unstable solution.

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