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Stability conditions of limit cycle for Gompertz Autoregressive model

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ABSTRACT

In this paper, we suggest Gompertz Autoregressive model by using the cumulative distribution function of Gompertz distribution and, the aim of this paper is studying and finding the stability conditions of a limit cycle for the Gompertz Autoregressive model with period, $q > 1$ with giving some examples for Gompertz AR (1) to explain the orbital stable or the orbital unstable with plots the trajectories with different initial values.

1- Introduction

In this field, we proposed one of a nonlinear autoregressive model, which is Gompertz autoregressive model with order p and denoted by Gompertz AR(p) model. it depend on the cumulative distribution function for the Gompertz distribution . In this non- linear autoregressive model, we will study and find the stability conditions for limit cycle by using the local linearization approximation method when period $q > 1$ with giving some examples to Gompertz AR(1) and show when the model be an orbital stable and orbital unstable. we will study only the stability conditions for limit when period q by using the state space .

Many searchers have been able to study the stability conditions of the limit cycle for many of non- linear time series models. In (1977), Oda and Ozaki studied exponential autoregressive model [1] . In (1988) Priestley M.B. studied the non-stability and non-linear time series[2]. In 2007 Mohammad and Salim studied the stability of logistic autoregressive model,[3]. In (2018), Salim and Ahmed studied Stability of a Non-Linear Exponential Autoregressive Model [4]. In (2019), Salim and Youns studied Study of Stability of Non-Linear Model with Hyperbolic Secant function, [5]. In (2020), Mohammad and Hamdi and Khaleel studied On Stability Conditions of Pareto Autoregressive Model, [6].

In this paper we study and find the stability conditions of a limit cycle for some nonlinear (Gompertz Autoregressive model) and give many examples in order to explain the orbital stable or the orbital unstable with plots the trajectories with different initial values .

2- Concepts and Definitions

In 1825, Benjamin Gompertz introduced the Gompertz distribution. A random variable X has the Gompertz distribution with two parameters, the first one λ is called scale parameter and the second γ is called shape parameter.The probability density function (p.d.f) of Gompertz distribution is

$$f(X;\gamma,\lambda) = \gamma e^{\lambda x}.e^{\frac{-\gamma}{\lambda}(e^{\lambda x}-1)} \quad , \quad X > 0 \quad \gamma,\lambda > 0$$

And the cumulative distribution function (c.d.f) of Gompertz distribution is

$F(X;\gamma,\lambda) = 1 - e^{\frac{-\gamma}{\lambda}(e^{\lambda x}-1)}$, $X > 0$ $\gamma,\lambda > 0$ ... (2.1)

Figure (2.1) shows the graph of (c.d.f) of Gompertz distribution with different values of λ and Figure (2.2) shows the graph of (c.d.f) of Gompertz distribution with different values of γ

The smooth jump from 0 to 1 in the under graph of the cumulative distribution function characterized the nonlinear behavior. It is useful to define and suggest the Gompertz autoregressive model which is one of nonlinear time series [7,8].

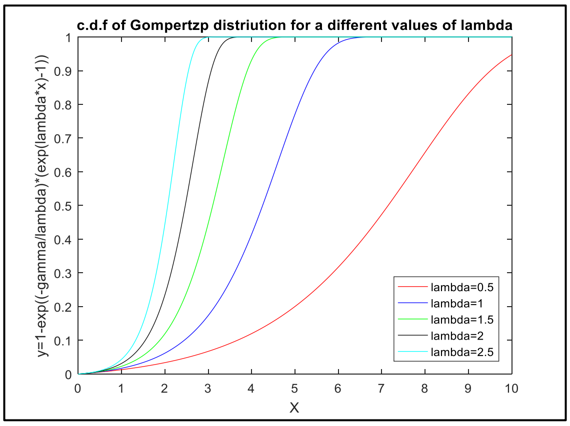

Fig. (2.1): graph (c.d.f) of Gompertz dist. With different values of λ

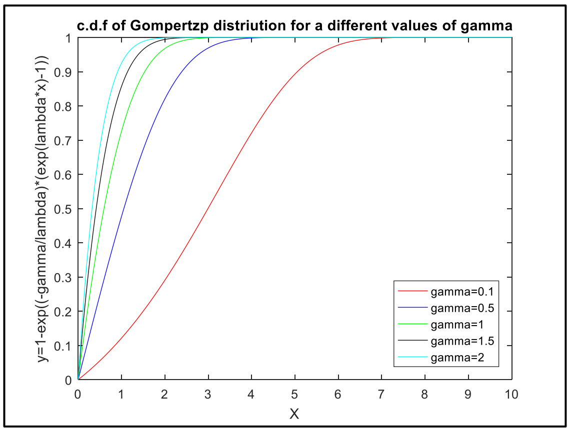

Fig. (2.2): graph (c.d.f) of Gompertz dist. With different values of γ

Definition 2.1

Let T be a finite positive integer. A k-dimensional vector X^* is called periodic point with period T if $X^* = f^T(X^*)$ and $X^* \neq f^j(X^*)$ for $1 \leq j < T$

Here X^* is a fixed point of f^T, we say that X^* is a periodic point with period T for some T≥1. And the ordered set $\{X^*, f(X^*), f^2(X^*), \dots, f^{T-1}(X^*)\}$ is called a T-cycle. We say that X_0 is eventually periodic if there is a positive integer n such that $X^* = f^n(X_0)$ is periodic. We say that X_0 is asymptotically periodic if there exists periodic point X^* for which $\|f^n(X_0) - f^T(X)\| \to 0$ $as\ n \to \infty$. [9]

Definition 2.2

A limit cycle of a model $X_T = f(X_{T-1}, X_{T-2}, \dots, X_{T-p})$ where f is nonlinear function is defined as an closed isolated trajectory $y_T, y_{T+1}, y_{T+2}, \dots, y_{T+q} = y_T$

Where the period $q > 1$ be a smallest positive integer such that $y_{T+q} = y_T$. Closed means that if the initial value $(y_1, y_2, y_3, \dots, y_p)$ belongs to the limit cycle, then$(y_{1+kq}, y_{2+kq}, \dots, y_{p+kq}) = (y_1, y_2, y_3, \dots, y_p)$ for any $k \in Z^+$. By Isolated we mean that every trajectory be sufficiently closed to the limit cycle approaches to it for $T \to \infty$ or $T \to -\infty$ If it approaches to the limit cycle for $T \to \infty$, then the limit cycle is , but if it approaches to the limit cycle for $T \to -\infty$, then the limit cycle is unstable. [6],[10] .

Definition 2.3

By an attractor for f we mean a compact set A such that the $B = \{x: \lim_{n\to\infty} \inf_{y\in A} \|f^n(x) - y\| = 0\}$ have a positive Lebesgue measure and A is a minimal with respect to this property. The set B is called the Basin of attraction for A and it is some time denoted by B(A) . if the attractor is a set of T points $\{x_1, x_2, \dots, x_T\}$ such that $f(x_n) = x_{n+1}$, n=1,2,… T-1 and $f(x_T) = x_1$ then we call the attractor A limit cycle and if T=1 then we call it a limit point.[3],[9]

Definition 2.4

A singular point ψ of a (proposed) model $X_T = f(X_{T-1}, X_{T-2}, \dots, X_{T-p})$ where f a nonlinear function is defined to be a point for which every trajectory of the model beginning sufficiently closed to the singular point ψ approaches to it either for $T \to \infty$ or $T \to -\infty$. If it approaches to a singular point for $T \to \infty$, then it is stable singular point, and if it approaches to a singular point for $T \to -\infty$, then it is unstable singular point. [11],[12] .

Definition 2.5 (Gompertz AR(p): The proposed model.

Let $\{x_t\}$ be a discrete time series then the Gompertz AR(p) model(the proposed model) is defined by

$$x_t = \sum_{i=1}^{p}\left[\alpha_i + \beta_i\left(1 - e^{\frac{-\gamma}{\lambda}(e^{\lambda x_{t-1}}-1)}\right)\right]x_{t-i} + Z_t \quad \dots (2.2)$$

Where $Z_t \sim iidN(0, \sigma_z^2)$

Where $\{Z_t\}$ be a white noise process, γ and λ are shape and scale parameters respectively, $\{\alpha_i\}$ and $\{\beta_i\}$ are constants $i = 1,2,3, \dots, p$.

In the following proposition we find the stability condition of a limit cycle when Gompertz AR (1) has a limit cycle of period > 1 .

PROPOSITION 3.1

If the following Gompertz AR (1) model has a limit cycle of period $q > 1$

$$x_t = \left[\alpha_1 + \beta_1\left(1 - e^{\frac{-\gamma}{\lambda}(e^{\lambda x_{t-1}}-1)}\right)\right]x_{t-1} + z_t \quad \dots (3.1)$$

Then the model (3.1) is orbital stable if

$$\left|\prod_{j=1}^{q}\left[\alpha_1 + \beta_1 - \beta_1 e^{\frac{-\gamma}{\lambda}(e^{\lambda x_{t+j-1}}-1)}\left(1 - \gamma x_{t+j-1}\, e^{\lambda x_{t+j-1}}\right)\right]\right| < 1 \dots (3.2)$$

Proof:

Let $x_t, x_{t+1}, x_{t+2}, \dots, x_{t+q} = x_t$ be a limit cycle of period $q > 1$, near each point of a limit cycle x_s suppose ψ_s be the radius of a neighborhood whose center is the point x_s such that $\psi_s^n \to 0$ for $n \geq 2$ and for $s = t, t+1, \dots, t+q$, by replacing x_s in (3.1) by $x_s + \psi_s$ for $s = t, t-1$ after suppressing a white noise process we get:

$$x_t + \psi_t = \left[\alpha_1 + \beta_1 \left(1 - e^{\frac{-\gamma}{\lambda}\left[e^{\lambda(x_{t-1}+\psi_{t-1})} - 1\right]}\right)\right] (x_{t-1} + \psi_{t-1}) \quad \dots (3.3)$$

We can approximate this term $e^{\frac{-\gamma}{\lambda}\left[e^{\lambda(x_{t-1}+\psi_{t-1})} - 1\right]}$ to get

$$e^{\frac{-\gamma}{\lambda}\left[e^{\lambda(x_{t-1}+\psi_{t-1})} - 1\right]} = e^{\frac{-\gamma}{\lambda}\left[e^{\lambda x_{t-1}} e^{\lambda\psi_{t-1}}\right] + \frac{\gamma}{\lambda}} \quad \dots (3.4)$$

By using Taylor expansion for $e^{\lambda\psi_{t-1}}$ we get

$$e^{\lambda\psi_{t-1}} = 1 + \lambda\psi_{t-1} + \frac{(\lambda\psi_{t-1})^2}{2!} + \cdots \cong 1 + \lambda\psi_{t-1} \quad \dots (3.5)$$

By substituting (3.5) in (3.4) we get

$$= e^{\frac{-\gamma}{\lambda}(1+\lambda\psi_{t-1})\, e^{\lambda x_{t-1}} + \frac{\gamma}{\lambda}}$$

$$= e^{\frac{-\gamma}{\lambda} e^{\lambda x_{t-1}} - \gamma\psi_{t-1} e^{\lambda x_{t-1}} + \frac{\gamma}{\lambda}}$$

$$= e^{\frac{-\gamma}{\lambda} e^{\lambda x_{t-1}}} \, . \, e^{-\gamma\psi_{t-1} e^{\lambda x_{t-1}}} \, . \, e^{\frac{\gamma}{\lambda}}$$

$$= e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)} \; e^{-\gamma\psi_{t-1} e^{\lambda x_{t-1}}} \quad \dots (3.6)$$

By using Taylor expansion for $e^{-\gamma\psi_{t-1} e^{\lambda x_{t-1}}}$ we get

$$e^{-\gamma\psi_{t-1} e^{\lambda x_{t-1}}} = 1 - \gamma\psi_{t-1} e^{\lambda x_{t-1}} + \frac{\left(\gamma\psi_{t-1} e^{\lambda x_{t-1}}\right)^2}{2!} - \cdots \cong 1 - \gamma\psi_{t-1} e^{\lambda x_{t-1}} \quad \dots (3.7)$$

By substituting (3.7) in (3.6) we get

$$e^{\frac{-\gamma}{\lambda}\left[e^{\lambda(x_{t-1}+\psi_{t-1})} - 1\right]} = e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)} \left(1 - \gamma\psi_{t-1} e^{\lambda x_{t-1}}\right) \quad \dots (3.8)$$

By substituting (3.8) in (3.3) we get

$$x_t + \psi_t = \left(\alpha_1 + \beta_1 \left[1 - e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)} \left(1 - \gamma\psi_{t-1} e^{\lambda x_{t-1}}\right)\right]\right) (x_{t-1} + \psi_{t-1})$$

$$= \left(\alpha_1 + \beta_1 \left[1 - e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right] + \beta_1\gamma\psi_{t-1} e^{\lambda x_{t-1}} e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right) (x_{t-1} + \psi_{t-1})$$

$$= \left[\left(\alpha_1 + \beta_1 \left[1 - e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right]\right) + \left(\beta_1\gamma\psi_{t-1} e^{\lambda x_{t-1}} e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right)\right] (x_{t-1} + \psi_{t-1})$$

$$= \left(\alpha_1 + \beta_1 \left[1 - e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right]\right) x_{t-1} + \left(\alpha_1 + \beta_1 \left[1 - e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right]\right) \psi_{t-1} + \left(\beta_1\gamma\psi_{t-1} e^{\lambda x_{t-1}} e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right) x_{t-1} + \left(\beta_1\gamma\psi_{t-1} e^{\lambda x_{t-1}} e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right) \psi_{t-1}$$

but $\left(\alpha_1 + \beta_1 \left[1 - e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right]\right) x_{t-1} = x_t$

$$\psi_t = \left(\alpha_1 + \beta_1 \left[1 - e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right]\right) \psi_{t-1} + \left(\beta_1\gamma\psi_{t-1} e^{\lambda x_{t-1}} e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right) x_{t-1} + \left(\beta_1\gamma\, e^{\lambda x_{t-1}} e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right) \psi_{t-1}^2$$

But $\psi_{t-1}^n \to 0$ for all $n \geq 2$ this implies to all the term

$$\left(\beta_1\gamma\, e^{\lambda x_{t-1}} e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}} - 1\right)}\right) \psi_{t-1}^2 = 0$$

$$\psi_t = \left[\alpha_1 + \beta_1 - \beta_1 e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}-1}\right)} \left(1 - \gamma x_{t-1} e^{\lambda x_{t-1}}\right)\right] \psi_{t-1} \quad \dots (3.9)$$

equation (3.9) is a difference equation of variable coefficients of the first order and it's difficult to solve it exactly but we discuss the convergence of the equation (3.9) to zero by checking the ratio $\left|\frac{\psi_t}{\psi_{t+q}}\right|$, the difference equation (3.9) is stable ($\lim_{t\to\infty} \psi_t = 0$) if $\left|\frac{\psi_t}{\psi_{t+q}}\right| < 1$

Let $T(x_{t-1}) = \left[\alpha_1 + \beta_1 - \beta_1 e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}-1}\right)} \left(1 - \gamma x_{t-1} e^{\lambda x_{t-1}}\right)\right]$ then we can write

$\psi_{t+1} = T(x_t)\psi_t$ Consequently

$$\psi_{t+q} = T(x_{t+q-1})\psi_{t+q-1} = T(x_{t+q-1})T(x_{t+q-2})\psi_{t+q-2} = T(x_{t+q-1})T(x_{t+q-2})T(x_{t+q-3})\psi_{t+q-3}$$

And after q iteration we get

$\psi_{t+q} = \prod_{j=1}^{q} T(x_{t+q-j}).\psi_t$, then $\left|\frac{\psi_{t+q}}{\psi_t}\right| = \left|\prod_{j=1}^{q} T(x_{t+q-j})\right|$

Finally the difference equation (3.9) is stable if $\left|\prod_{j=1}^{q} T(x_{t+q-j})\right| < 1$

Finally, the limit cycle (if it exists) of Gompertz AR (1) model is orbital stable if

$$\left|\prod_{j=1}^{q}\left[\alpha_1 + \beta_1 - \beta_1 e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t+j-1}-1}\right)} \left(1 - \gamma x_{t+j-1} e^{\lambda x_{t+j-1}}\right)\right]\right| < 1 .$$

The following proposition is generalized to proposition (3.1) when the model order p and $p > 1$ in this case will write the model in state space as follows

$$X_t = \begin{bmatrix} \alpha_1 + \beta_1\left(1 - e^{\frac{-\gamma}{\lambda}\left[e^{\lambda x_{t-1}-1}\right]}\right) & \cdots & \alpha_{p-1} + \beta_{p-1}\left(1 - e^{\frac{-\gamma}{\lambda}\left[e^{\lambda x_{t-1}-1}\right]}\right) & \alpha_p + \beta_p\left(1 - e^{\frac{-\gamma}{\lambda}\left[e^{\lambda x_{t-1}-1}\right]}\right) \\ 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{bmatrix} X_{t-1} + e_t \quad \dots (3.10)$$

Where x_t, x_{t-1}, e_t are known vectors on the space R^P as follows

$X_t = (x_t, x_{t-1}, \dots, x_{t-P+1})^T$, $X_{t-1} = (x_{t-1}, x_{t-2}, \dots, x_{t-P})^T$, $e_t = (Z_t, 0,0 \dots, 0)^T$ where e_t is a white noise . and the matrix elements depend on the random variable x_{t-1} that means the model in (3.10) can be expressed as follows

$X_t = T(x_{t-1})X_{t-1} + e_t$ or $X_{t+1} = T(x_t)X_t + e_{t+1}$

Proposition 3.2:

A limit cycle with period q and $q > 1$ if it exists for Gompertz AR(p) is orbitally stable if and only if all the eigenvalues of Matrix A have absolute values less than One .

where $A = A_q A_{q-1} \dots . A_1 = \prod_{j=1}^{q} A_j$ and

$$\mathbf{A}_j = \begin{bmatrix} a_{1,1}^{(j)} & a_{1,2}^{(j)} & \cdots & a_{1,p-1}^{(j)} & a_{1,p}^{(j)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad j = 1, \dots, q$$

$$a_{1,1}^{(j)} = \left[\alpha_1 + \beta_1 - \beta_1 e^{\frac{-\gamma}{\lambda}\left(e^{\lambda x_{t-1}-1}\right)} \left(1 - \gamma x_{t-1} e^{\lambda x_{t-1}}\right)\right]$$

$$a_{1,k}^{(j)} = \alpha_k + \beta_k \left(1 - e^{\frac{-\gamma}{\lambda}\left[e^{\lambda x_{t-1}-1}\right]}\right), \quad k = 2,3 \dots, p \quad \dots (3.11)$$

Proof :

Let the Gompertz AR(P) has represented in state space and has a limit cycle with period q and q> 1 and known as $x_t, x_{t-1}, \dots, x_{t+q} = x_t$ which is an isolated closed trajectory and by using the same hypotheses in the previous proof with substituting in (3.10) we get :

$$X_t + \psi_{t+1} = \begin{bmatrix} \alpha_1 + \beta_1(1 - e^{\frac{-\gamma}{\lambda}[e^{\lambda x_{t-1}-1}]}) & \cdots & \alpha_{p-1} + \beta_{p-1}(1 - (e^{\frac{-\gamma}{\lambda}[e^{\lambda x_{t-1}-1}]}) & \alpha_p + \beta_p(1 - (e^{\frac{-\gamma}{\lambda}[e^{\lambda x_{t-1}-1}]}) \\ 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{bmatrix} X_t + \psi_t$$

Where ψ_{t+1} , ψ_t are vectors known as

$$\psi_{t+1} = \begin{bmatrix} \psi_{t+1} \\ \psi_t \\ \vdots \\ \psi_{t-p+2} \end{bmatrix} , \quad \psi_t = \begin{bmatrix} \psi_t \\ \psi_{t-1} \\ \vdots \\ \psi_{t-p+1} \end{bmatrix}$$

with some the easy calculations we get

$$\psi_{t+1} = \begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \cdots & a_{1,p-1}^{(1)} & a_{1,p}^{(1)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \psi_t \quad \dots (3.12)$$

Where this matrix denoted by A_1 and write as follows

$$A_1 = \begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \cdots & a_{1,p-1}^{(1)} & a_{1,p}^{(1)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$a_{1,1}^{(1)} = \left[\alpha_1 + \beta_1 - \beta_1 e^{\frac{-\gamma}{\lambda}(e^{\lambda x_t}-1)}(1 - \gamma x_t e^{\lambda x_t})\right]$$

$$a_{1,k}^{(1)} = \alpha_k + \beta_k \left(1 - e^{\frac{-\gamma}{\lambda}[e^{\lambda x_t}-1]}\right) , k = 2,3 \dots , p$$

We can write the equation (3.12) as $\psi_{t+1} = A_1 \psi_t$ that is mean

$\psi_{t+2} = A_2 \psi_{t+1}$ and by repeating this operation q times we get

$$\psi_{t+q} = A_q \psi_{t+q-1} = A_q A_{q-1} \dots A_1 \psi_t$$

$$\psi_{t+q} = A_q A_{q-1} \dots A_1 \psi_t \quad \dots (3.13)$$

$$A_j = \begin{bmatrix} a_{1,1}^{(j)} & a_{1,2}^{(j)} & \cdots & a_{1,p-1}^{(j)} & a_{1,p}^{(j)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} , \quad j = 1,2,3, \dots q$$

$$a_{1,1}^{(j)} = \left[\alpha_1 + \beta_1 - \beta_1 e^{\frac{-\gamma}{\lambda}(e^{\lambda x_{t+j-1}}-1)}(1 - \gamma x_{t+j-1} e^{\lambda x_{t+j-1}})\right]$$

$$a_{1,k}^{(j)} = \alpha_k + \beta_k \left(1 - e^{\frac{-\gamma}{\lambda}[e^{\lambda x_{t+j-1}}-1]}\right) , k = 2,3 \dots , p$$

We can write (3.13) as follows $\psi_{t+q} = \prod_{j=1}^{q} A_j \, \boldsymbol{\psi_t}$ and the product of the matrix A_j , $j = 1,2 \dots q$ is the matrix A , where $\psi_{t+q} = A\psi_t$, $A_j \to 0$ as $j \to \infty$, if the eigenvalues for A_j matrix less than 1 then the limit cycle with period q for the Gompertz (P) is orbital stable .

4- Application

We apply the proposition (3.1) to the following examples with an arbitrary values of parameters to checking the stability of limit cycle.

Example 4.1: Let Gompertz AR (1) model is given by

$$x_t = \left[-1.8 - 1.7\left(1 - e^{\frac{-0.7}{0.6}(e^{0.6 x_{t-1}} - 1)}\right)\right] x_{t-1} + z_t$$

Has non-zero singular point $\psi = -2.9968$ and a limit cycle of period 4 which is {0.055,0.032,0.047,-0.014}. we can calculate that by using the following condition (3.2)

$$\left|\prod_{j=1}^{4}\left[-0.8 - 1.7 + 1.7e^{\frac{-0.7}{0.6}(e^{0.6 x_{t+j-1}}-1)}(1 - 0.7 x_{t+j-1} e^{0.6 x_{t+j-1}})\right]\right| = 12.2281 > 1$$

The condition (3.2) does not satisfy, and the limit cycle is orbital unstable. note the Fig (4.1.1) with different initial values.

Example 4.2: Consider the following is Gompertz AR(1) model

$$x_t = \left[-1.3 - 1.6\left(1 - e^{\frac{-0.9}{0.8}(e^{0.8 x_{t-1}} - 1)}\right)\right] x_{t-1} + z_t$$

Has a non-zero singular point $\psi = -1.9626$ and a limit cycle of period 2 which is 0.27, -0.31 we can calculate that by using

$$\left|\prod_{j=1}^{2}\left[-1.3 - 1.6 + 1.6\, e^{\frac{-0.9}{0.8}(e^{0.8 x_{t+j-1}}-1)}(1 - 0.9 x_{t+j-1} e^{0.8 x_{t+j-1}})\right]\right| = 0.8305 < 1$$

The condition (3.2) is satisfying therefore the limit cycle is orbital stable .the Fig(4.2.1) shows the stability of limit cycle with different initial values.

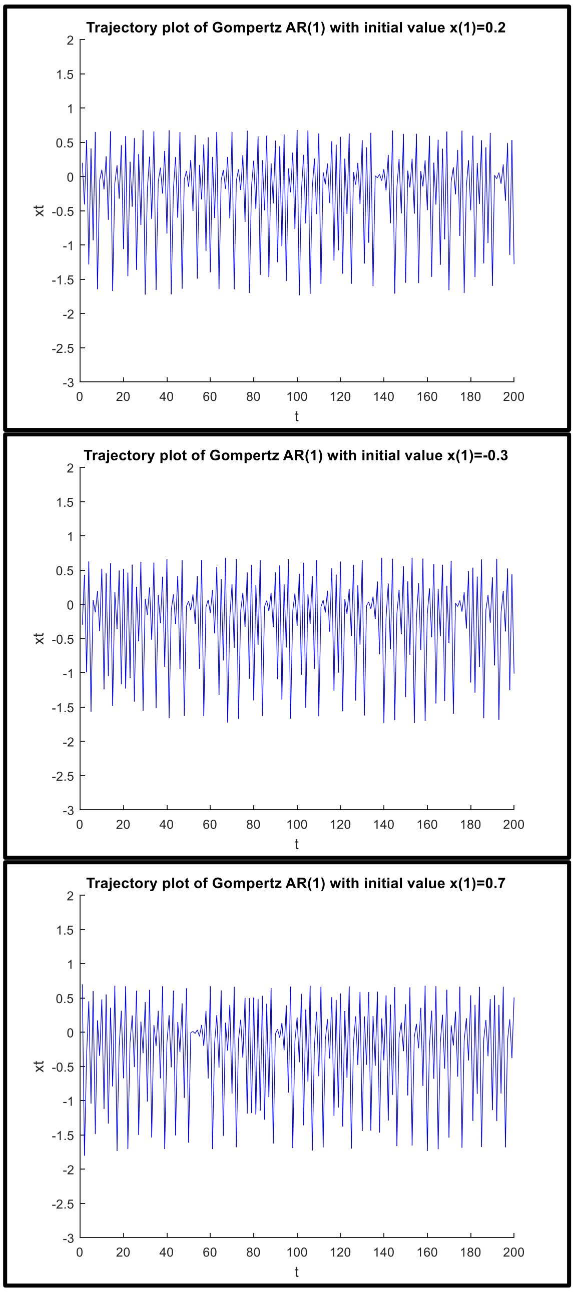

Fig. (4.1.1) shows the orbitally unstable with different initial values

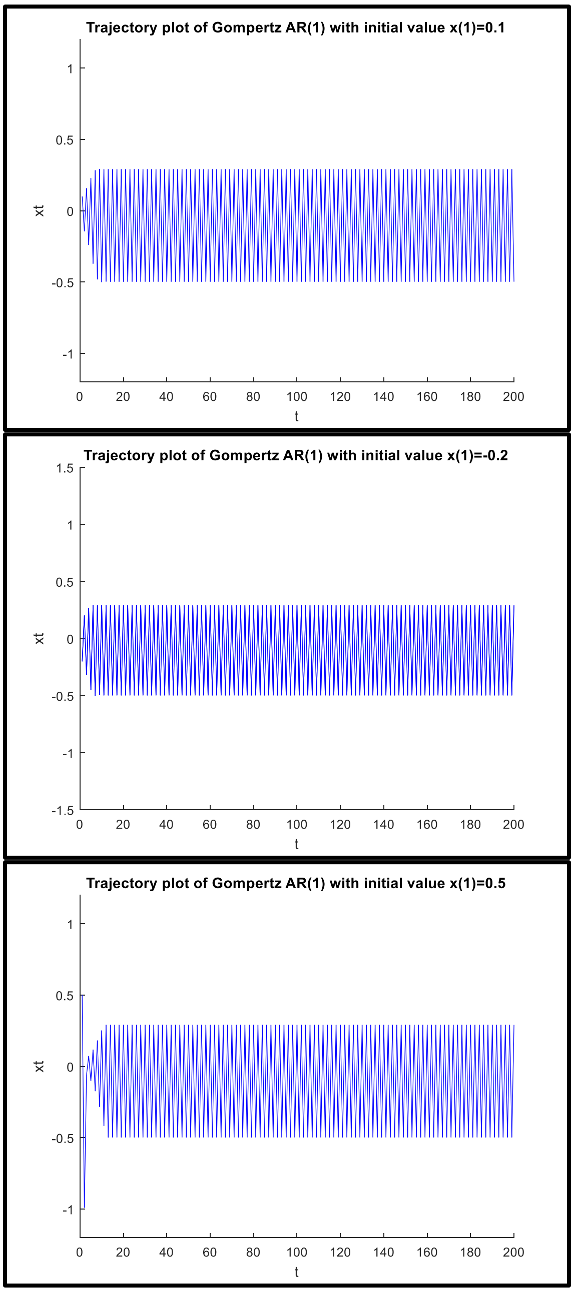

Fig. (4.2.1): shows the orbitally stable with different initial values

5- Conclusion

In this paper, we study and find the stability conditions of limit cycle for Gompertz autoregressive model by using two propositions.

The first one is the proposition (3.1), we used the local linearization approximation technique to find the stability conditions of limit cycle when the period > 1 . Also we took two examples with its plots to explain if the Gompertz AR(1) is an orbital stable or unstable .

The second proposition (3.2), it is generalized for the first proposition where we used the state space to studying the stability conditions of limit cycle .

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شروط استقرارية دورة النهاية لأنموذج جومبيرتز للأنحدار الذاتي

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الملخص

في هذه الورقة تم اقتراح انموذج جديد للمتسلسلات الزمنية اللاخطية وهو انموذج جومبيرتز للانحدار الذاتي والذي استخدمنا الدالة التوزيعية (التراكمية) لتوزيع جومبيرتز الاحصائي لبناء الانموذج .وان الهدف من هذه الورقة هو دراسة و إيجاد شروط استقرارية دورة النهاية عند الدورة q , $q > 1$ لأنموذج جومبيرتز للانحدار الذاتي مع اعطاء بعض الامثلة لأنموذج جومبيرتز من الرتبة الاولى لتوضيح إن الانموذج مستقر مدارياً ام غير مستقر مدارياً مع رسم مسارات دورة النهاية للأمثلة مع قيم ابتدائية مختلفة .