Geometry of concircular curvature tensor of Nearly Kahler manifold

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Abstract

In this paper, we study the necessary condition where a nearly Kahler manifold of flat concircular tensor has been found. And the relationship between these invariants and additional properties of symmetry concircular tensor, as well as geometrical sense of the reference (manipulation) in zero of these invariants are studied.

Let *M* - smooth manifold of dimension *2n*; $C^{\infty}(M)$ - algebra of smooth functions on *M*; $X(M)$ - the module of smooth vector fields on manifold of *M*; $g = \langle\cdot,\ \rangle$ - Riemannian metrics; ∇ – Riemannian connection of the metrics *g* on *M*; *d* - the operator of exterior differentiation. In the further all manifold, Tensor fields, etc. objects are assumed smooth a class C^{∞}.

Keywords: Nearly kahler manifolds, concircular curvature tensor .

Definition 1: **[5]**

Almost Hermition (is shorter, AH) structure on a manifold M the pair (J, g) ,where J-almost complex structure.

$(J^2 = id)$ on $M, g = <.,.>$

$-(pseudo)$ *Riemannian metric on M . In this case* $< JX, JY > = < X, Y >; X, Y \in X(M)$. Endomorphism J is called structural endomorphism. Manifold which is fixed almost manifold M equivalently to the task of G-structure above M with structural U(n) ,the elements of space which.

Definition 2: **[5]**

Almost Hermition structure (J, g) on manifold of M is called nearly Kahler (more shortly ,NK-) structure if M the identity is carried out (if M satisfies the identity)

$$\nabla_X(J)Y + \nabla_Y(J)X = 0 ; X, Y \in X(M). \quad (1)$$

Proposition 3:

Let (*M, J, g*) - *AH*- manifold.

Invariant concircular transformations metric is tensor C type (3, 1) determined by the formula

$$C(X,Y)Z = R(X,Y)Z - \frac{K}{n(n-1)}\{\langle X,Z\rangle Y - \langle Y,Z\rangle X\}. \quad (2)$$

And called concircular curvature tensor was introduced will be Reminded Yano**[8]** on *n*-dimensional Riemannian manifold.

Where *R*–the Riemann curvature tensor, g =⟨.,.⟩ is the Riemannian metric and K-is the scalar curvature X,Y,Z,W ∈ X(M) ;

where X(M) is the Lie algebra of C^{∞}vector fields on M.

This tensore is invariant under concircular transformations,

i.e. with conformal transformations of space keeping a harmony of functions.

Definition 4: **[6]**

Aconcircular curvature tensor on AH-manifold M is a tensor of type (4,0) and satisfied the relation $e^{-2f}\bar{C}(X,Y,Z.W)$=C(X,Y,Z,W),which is defined as the form:

$$C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{k}{n(n-1)}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \quad (3)$$

Where R is the Riemannian curvature tensor ,g is the Riemannian metric and K-is the scalar curvature X,Y,Z,W ∈ X(M) .

Where X(M) is the Lie algebra of C^{∞}vector fields on M.

Let's consider properties tensor concircular curvature:

Remark 5:-

The concircular curvature tensor satisfies all the properties of algebraic curvature tensor:

1) C (*X,Y,Z,W*) = – *C*(*Y, X , Z,W*) ;
2) C(X,Y,Z,W)= - C(X,Y,W,Z) ;
3) *C*((*X,Y,Z,W*) + *C*(*Y, Z, X, W*) + *C*(*Z, X, Y, W*) = 0 ;
4) *C(X,Y,Z,W)=C(Z,W,X,Y)=C(Z,W,X,Y); X,Y,Z,W* ∈ *X(M).* (4)

Proof:- we shall prove(1) and similarly prove the others

1) $C(X,Y,Z.W)=R(X,Y,Z,W)-\frac{1}{2(n-1)}[g(X,W)S(Y,Z) - g(X,Z)S(Y,W)]- -\frac{1}{2(n-1)}[g(Y,Z)S(X,W) - g(Y,W)S(X,Z)] = -R(X,Y,Z,W)+\frac{1}{2(n-1)}[-g(X,W)S(Y,Z) + g(X,Z) + S(Y,W)] +\frac{1}{2(n-1)}[-g(Y,Z)S(X,W) + g(Y,W)S(X,Z)] = -C(Y,X,Z,W)$

Covarient – tensor concircular curvatare C type (3,1) have form

$$C(X,Y)Z = R(X,Y)Z - \frac{K}{n(n-1)}\{< X,Z > Y - < Y,Z > X\} \quad (5)$$

Where R-is the Riemannian curvature tensor and K-is the scalar curvature, X,Y,Z ∈ $X(M)$

By definition of a spectrum tensor.

$C(X,Y)Z= C_0(X,Y)Z + C_1(X,Y)Z + C_2(X,Y)Z + C_3(X,Y)Z + C_4(X,Y)Z + C_5(X,Y)Z + C_6(X,Y)Z + C_7(X,Y)Z;$ X,Y,Z ∈ X(M). (6)

tensor $C_0(X,Y)Z$ as nonzero. The component can have only components of the form $\{C_0{}^a{}_{bcd}, C_0{}^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\} = \{C^a{}_{bcd}, C^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\}$;

tensor $C_1(X,Y)Z$ - components of the form $\{C_1{}^a{}_{bc\hat{d}}, C_1{}^{\hat{a}}{}_{\hat{b}\hat{c}d}\} = \{C^a{}_{bc\hat{d}}, C^{\hat{a}}{}_{\hat{b}\hat{c}d}\}$;

tensor $C_2(X,Y)Z$ - components of the form $\{C_2{}^a{}_{b\hat{c}d}, C_2{}^{\hat{a}}{}_{\hat{b}c\hat{d}}\} = \{C^a{}_{b\hat{c}d}, C^{\hat{a}}{}_{\hat{b}c\hat{d}}\}$;

tensor $C_3(X,Y)Z$ - components of the form $\{C_3{}^a{}_{b\hat{c}\hat{d}}, C_3{}^{\hat{a}}{}_{\hat{b}cd}\} = \{C^a{}_{b\hat{c}\hat{d}}, C^{\hat{a}}{}_{\hat{b}cd}\}$;

tensor $C_4(X,Y)Z$ - components of the form$\{C_4{}^{a}{}_{\hat{b}cd}, C_4{}^{\hat{a}}{}_{b\hat{c}\hat{d}}\} = \{C^{a}{}_{\hat{b}cd}, C^{\hat{a}}{}_{b\hat{c}\hat{d}}\}$;
tensor $C_5(X,Y)Z$ - components of the form $\{C_5{}^{a}{}_{\hat{b}c\hat{d}}, C_5{}^{\hat{a}}{}_{b\hat{c}d}\} = \{C^{a}{}_{\hat{b}c\hat{d}}, C^{\hat{a}}{}_{b\hat{c}d}\}$;
tensor $C_6(X,Y)Z$ - components of the form $\{C_6{}^{a}{}_{\hat{b}\hat{c}d}, C_6{}^{\hat{a}}{}_{bc\hat{d}}\} = \{C_6{}^{a}{}_{\hat{b}\hat{c}d}, C_6{}^{\hat{a}}{}_{bc\hat{d}}\}$;
tensor $C_7(X,Y)Z$ - components of the form$\{C_7{}^{a}{}_{\hat{b}\hat{c}\hat{d}}, C_7{}^{\hat{a}}{}_{bcd}\} = \{C^{a}{}_{\hat{b}\hat{c}\hat{d}}, C^{\hat{a}}{}_{bcd}\}$.
Tensors $C_0 = C_0(X,Y)Z,\ C_1 = C_1(X,Y)Z,\ \ldots, C_7 = C_7(X,Y)Z$.
The basic invariants concircular *AH*-manifold will be named.

Definitidcon 6:-

AH - manifold for which C_i=0, *AH*- manifold of class C_i, i = 0, 1, …, 7.

Theorem 7:

1) *AH*- manifold of class C_0 characterized by identity
C(*X,Y*)*Z*–*C*(*X,JY*)*JZ*–*C*(*JX,Y*)*JZ*–*C*(*JX,JY*)*Z*–*JC*(*X,Y*)*JZ*–*JC*(*X,JY*)*Z*– *JC*(*JX,Y*) *Z*
+ *JC*(*JX,JY*)*JZ* = 0, *X,Y,Z* ∈ *X*(*M*). (7)

2) *AH* - manifold of class C_1 characterized by identity
C(*X,Y*)*Z*+*C*(*X,JY*)*JZ*–*C*(*JX,Y*)*JZ*+*C*(*JX,JY*)*Z*+*JC*(*X,Y*)*JZ*–*JC*(*X,JY*)*Z*– *JC*(*JX,Y*) *Z*
– *JC*(*JX,JY*)*JZ* = 0 , *X,Y,Z* ∈ *X*(*M*). (8)

3) *AH* - manifold of class C_2 characterized by identity
C(*X,Y*)*Z*–*C*(*X,JY*)*JZ*+*C*(*JX,Y*)*JZ*+*C*(*JX,JY*)*Z*–*JC*(*X,Y*)*JZ*–*JC*(*X,JY*)*Z*+*JC*(*JX,Y*)*Z*
–*JC*(*JX,JY*)*JZ*=0, *X,Y,Z* ∈ *X*(*M*). (9)

4) AH - manifold of class C_3 characterized by identity
C(*X,Y*)*Z*+*C*(*X,JY*)*JZ*+*C*(*JX,Y*)*JZ*–*C*(*JX,JY*)*Z*–*JC*(*X,Y*)*JZ*+*JC*(*X,JY*)*Z*+*JC*(*JX,Y*)*Z*
+*JC*(*JX,JY*)*JZ*=0; *X,Y,Z* ∈ *X*(*M*). (10)

5) *AH*- manifold of class C_4 characterized by identity
C (*X,Y*)*Z*+*C* (*X,JY*)*JZ*+*C* (*JX,Y*)*JZ*–*C* (*JX,JY*)*Z*+*JC* (*X,Y*)*JZ*–*JC* (*X,JY*)*Z*–*JC* (*JX,Y*)*Z*
–*JC* (*JX,JY*)*JZ*=0; *X,Y,Z* ∈ *X*(*M*). (11)

6) *AH* - manifold of class C_5 characterized by identity
C(X,Y)Z–C(X,JY)JZ+C(JX,Y)JZ+C(JX,JY)Z+JC(X,Y)JZ+JC(X,JY)Z–JC(JX,Y)Z
+*JC*(*JX,JY*)*JZ*=0; *X,Y,Z*∈*X*(*M*). (12)

7) *AH*- manifold of class C_6 characterized by identity
C(*X,Y*)*Z*+*C*(*X,JY*)*JZ*–*CK*(*JX,Y*)*JZ*+*C*(*JX,JY*)*Z*+*JC*(*X,Y*)*JZ*–*JC*(*X,JY*)*Z*+*JC*(*JX,Y*)*Z*
+*JC*(*JX,JY*)*JZ*=0; *X,Y,Z*∈*X*(*M*). (13)

8) *AH* - manifold of class C_7 characterized by identity
C (*X,Y*)*Z*–*C* (*X,JY*)*JZ*–*C* (*JX,Y*)*JZ*–*C* (*JX,JY*)*Z*+*JC* (*X,Y*)*JZ*+*JC* (*X,JY*)*Z*+*JC* (*JX,Y*)*Z*
–*JC* (*JX,JY*)*JZ*=0; *X,Y,Z*∈*X*(*M*). (14)

Proof:-

1) Let *AH*- manifold of class C_0 , the manifold of class C_0 characterized by a condition
$C_0{}^{a}{}_{bcd} = 0$, or $C^{a}_{\hat{b}cd} = 0$
i.e. $[C(\varepsilon_c, \varepsilon_d)\varepsilon_b]^a \varepsilon_a$.

As σ - a projector on $D_J^{\sqrt{-1}}$, that $\sigma \circ \{C(\sigma x, \sigma y)\sigma z\} = 0$;
i.e (id-$\sqrt{-1}$ J)$\{C(X - \sqrt{-1}\, JX, Y - \sqrt{-1}\, JY)(Z - \sqrt{-1}\, JZ)\}$ = 0.
Removing the brackets can be received: i.e.
C(X,Y)Z-C(X,JY)-C(JX,Y)JZ-C(JX,JY)Z-JC(X,Y)JZ-JC(X,JY)Z-JC(JX,Y)+JC(JX.JY)JZ-$\sqrt{-1}\,\{C(X,Y)JZ + C(X,JY)Z + C(JX,Y)Z - C(JX,JY)JZ\}$
$\{JC(X,Y)Z - JC(X,JY)JZ - JC(JX,Y)JZ - JC(JX,,JY)Z\}$=0
i.e
1) C(X,Y)Z-C(X,JY)JZ-C(JX,Y)JZ-C(JX,JY)Z-JC(X,Y)JZ-JC(X,JY)Z-JC(JX,Y)Z+JC(JX.JY)JZ=0 (15)
2) *C*(*X,Y*)*JZ*+*C*(*X,JY*)*Z*+*C*(*JX,Y*)*Z*–*C*(*JX,JY*)*JZ*+*JC*(*X,Y*)*Z*–*JC*(*X,JY*)*JZ*–*JC*(*JX,Y*)*JZ*–*JC*(*JX,JY*)*JZ*=0 (16)
These equalities (15) and (16) are equivalent. The second equality turns out from the first replacement *Z* on *JZ*.
Thus *AH* - manifold of class C_0 characterized by identity
C(*X,Y*) *Z* – *C*(*X,JY*)*JZ* –*C*(*JX,Y*)*JZ* – *C*(*JX,JY*)*Z* – *JC*(*X,Y*)*JZ* –*JC*(*X,JY*) *Z* – *JC*(*JX,Y*) *Z* + *JC*(*JX,JY*)*JZ* = 0, *X,Y,Z* ∈ *X*(*M*). (17)
Similarly considering *AH* - manifold of classes C_1-C_7 can be received the 2,3,4,5,6,7 and 8.

Theorem 8:

We have the following inclusion relations
1) $C_0 = C_3 = C_5 = C_6$.
2) $C_1 = -\, C_2$.

Proof:-

1) we shall prove $C_5 = C_6$ and similarly prove the others.
For an example we shall prove equality $C_5 = C_6$.
Let (*M*, *J*, *g*) is AH- manifold of class C_5 , i.e $C^{a}_{\hat{b}c\hat{d}}$.Then according to (4) we have $C^{a}_{\hat{b}\hat{c}d} = 0$, i.e. The AH- manifold is manifold of class C_6 Back, let *M* - AH- manifold of class C_6, then $C^{a}_{\hat{b}\hat{c}d}$, so, according to (4) and $C^{a}_{\hat{b}c\hat{d}}$ =0.
Thus, classesC_5andC_6 $AH - manifold\ coincide.$
2) prove inclusion $C_1 = -C_2$.Let (the M, J, g)-AH-manifold of a class C_2,i.e. take place equality $C^{a}_{bc\hat{d}} = C^{a}_{b\hat{d}c} = o$. According to property (4) we have:
$C^{a}_{\hat{b}cd} + C^{a}_{cd\hat{b}} + C^{a}_{d\hat{b}c} = 0$, i.e $C^{a}_{\hat{b}cd} = 0$.
This the AH-manifold of a class $C_1 = -C_2$ is AH-manifold.
Putting (Folding) equality (8) and (9) we shall receive identity describing *AH- manifold* of class C_1=− C_2.
C(X,Y)Z+C(JX,JY)Z+JC(X,Y)JZ-JC(JX,JY)JZ=0; X,Y,Z∈ $X(M)$ (18)
From equality (7), (10), (12), (13) we shall receive the identity describing *AH- manifold* of classes C_0= C_3=$C_5 = C_6 = C_7$.
C(X,Y)Z+JC(JX,JY)JZ = 0 ; X,Y.Z∈ X(M). (19)

Definition 9: [7]

Let (M, J, g) be NK- manifold of dimension 2n , K - tensor conharmonic curvature.

that components tensor Riemann- Christoffel on space of the adjoint, G-structure will be Reminded **[5]** look like:

$$1)\ R^{a}_{bcd} = R^{a}_{b\hat{c}\hat{d}} = 0;\ 2)\ R^{a}_{bc\hat{d}} = -R^{a}_{b\hat{d}c} = A^{ad}_{bc} - B^{adh}B_{hbc};$$
$$3)\ R^{\hat{a}}_{\hat{b}cd} = R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}} = 0;\ 4)\ R^{\hat{a}}_{\hat{b}c\hat{d}} = -R^{\hat{a}}_{\hat{b}\hat{d}c} = -A^{bd}_{ac} + B^{bdh}B_{hac};$$
$$5)\ R^{a}_{\hat{b}c\hat{d}} = R^{a}_{\hat{b}\hat{c}d} = 0;\ 6)\ R^{a}_{\hat{b}cd} = 2B^{abh}B_{hcd};\ 7)\ R^{\hat{a}}_{bcd} = 0;$$
$$8)\ R^{\hat{a}}_{bc\hat{d}} = R^{\hat{a}}_{b\hat{c}d} = 0;\ 9)\ R^{\hat{a}}_{b\hat{c}\hat{d}} = 2B^{cdh}B_{hab};\ 10)\ R^{\hat{a}}_{bcd} = 0.$$
(20)

and the components of Ricci tensor S on space of the adjoint G-structure look like:

$$1)\ S_{ab} = 0;\ 2)\ S_{\hat{a}\hat{b}} = 0;\ 3)\ S_{\hat{a}b} = S_{b\hat{a}} = A^{ac}_{bc} + 3B^{ach}B_{bch}. \quad (21)$$

At last scalar curvature χ nearly Kahler manifolds in the space of the adjoint G-structure is calculated under the formula

$$\chi = 2A^{ab}_{ab} + 6B^{abc}B_{abc}. \qquad (22)$$

Theorem 10:-

The components of the concircular tensor of NK-manifold in the adjoint G-structure space are given by the following forms

$$1)\ C_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + \frac{K}{2n(n+1)}\tilde{\delta}^{ad}_{bc} \qquad (23)$$
$$2)\ C_{a\hat{b}c\hat{d}} = -A^{ac}_{bd} - \frac{K}{2n(n+2)}\tilde{\delta}^{ac}_{bd} \qquad (24)$$

And the others conjugate to the above components or equal to zero .

Proof :

By using definition(9) we compute the components of concircular tensor as the following :

1) Put i = a , j = b , k = c , $l = d$

$$C_{abcd} = R_{abcd} - \frac{K}{4n(n+1)}\{g_{ad}\, g_{bc} - g_{ac}g_{bd}\} = 0.$$

2) Put i = $\hat{a}$, j = b , k = c , I = d

$$C_{\hat{a}bcd} = R_{\hat{a}bcd} - \frac{K}{4n(n+1)}\{g_{\hat{a}d}g_{bc} - g_{\hat{a}c}g_{bd}\} = 0.$$

3) Put i = a , j = $\hat{b}$, k = c and l = d

$$C_{a\hat{b}cd} = R_{a\hat{b}cd} - \frac{K}{4n(n+1)}\{g_{ad}g_{\hat{b}c} - g_{ac}g_{\hat{b}d}\} = 0.$$

4) Put i = a , j = b , k = $\hat{c}$, and l = d

$$C_{ab\hat{c}d} = R_{ab\hat{c}d} - \frac{K}{4n(n+1)}\{g_{ad}g_{b\hat{c}} - g_{a\hat{c}}g_{bd}\} = 0.$$

5) Put i = a, j = b , k = c and l= $\hat{d}$

$$C_{abc\hat{d}} = R_{abc\hat{d}} - \frac{K}{4n(n+1)}\{g_{a\hat{d}}g_{bc} - g_{ac}g_{b\hat{d}}\} = 0.$$

6) Put i = $\hat{a}$, j = $\hat{b}$, k = c , l = d

$$C_{\hat{a}\hat{b}cd} = R_{\hat{a}\hat{b}cd} - \frac{K}{4n(n+1)}\{g_{\hat{a}b}g_{\hat{b}c} - g_{\hat{a}c}g_{\hat{b}d}\} = 0.$$

7) Put i = $\hat{a}$, j = b , k = $\hat{c}$ and l = d

$$C_{\hat{a}b\hat{c}d} = R_{\hat{a}b\hat{c}d} - \frac{K}{4n(n+1)}\{g_{\hat{a}b}g_{b\hat{c}} - g_{\hat{a}\hat{c}}g_{bd}\}$$
$$C_{\hat{a}b\hat{c}d} = -A^{ac}_{bd} - \frac{K}{4n(n+1)}\delta^{a}_{d}\delta^{c}_{b} - \frac{K}{4n(n+1)}\{\delta^{a}_{d}\delta^{c}_{b} + 2\delta^{a}_{b}\delta^{c}_{d}\}$$
$$C_{\hat{a}b\hat{c}d} = A^{ac}_{bd} - \frac{K}{4n(n+1)}\{2\delta^{a}_{d}\delta^{c}_{b} + 2\delta^{a}_{b}\delta^{c}_{d}\}$$
$$C_{\hat{a}b\hat{c}d} = -A^{ac}_{bd} - \frac{K}{2n(n+1)}\tilde{\delta}^{ac}_{bd},$$

where $\tilde{\delta}^{ac}_{bd} = \delta^{a}_{b}\delta^{c}_{d} + \delta^{a}_{d}\delta^{c}_{b}$ is the kroneker delta of the second type.

8) Put i=$\hat{a}$, j=b , k=c and l=$\hat{d}$

$$C_{\hat{a}bc\hat{d}} = R_{\hat{a}bc\hat{d}} - \frac{K}{4n(n+1)}\{g_{\hat{a}\hat{d}}g_{bc} - g_{\hat{a}c}g_{b\hat{d}}\}$$
$$C_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + \frac{K}{4n(n+1)}\delta^{a}_{c}\delta^{d}_{b} + \frac{K}{4n(n+1)}\{\delta^{a}_{c}\delta^{d}_{b} + 2\delta^{a}_{b}\delta^{d}_{c}\}$$
$$C_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + \frac{K}{2n(n+1)}\tilde{\delta}^{ad}_{bc}$$

By using the properties of concircular tensor we obtained :

$C_{\hat{a}bc\hat{d}} = C_{\hat{a}b\hat{c}d}$ as follows :

$C_{\hat{a}b\hat{d}c} = -C_{\hat{a}bc\hat{d}}$

$C_{\hat{a}b\hat{d}c} = -A^{ad}_{bc} - \frac{K}{2n(n+1)}\tilde{\delta}^{ad}_{bc}$

Therefore,

$$C_{\hat{a}b\hat{c}d} = -A^{ac}_{bd} - \frac{K}{2n(n+1)}\tilde{\delta}^{ac}_{bd}$$

In above theorem me calculated components concircular tensor curvature on space of the adjointe G-structure for NK-manifolds and we $C_1 and C_2$ have only other components concircular curvature tensor are equal to zero .

i.e for NK-manifold only two concircular curvature tensor donts equal zero.

C_1 with component $\{C^{a}_{bc\hat{d}}, C^{a}_{b\hat{c}d}\}$ and C_2 with component $\{C^{a}_{\hat{b}\hat{c}d}, C^{\hat{a}}_{\hat{b}c\hat{d}}\}$.

In the theory of *almost Hermitian* structures, there is a principle of classification of such structures on differential-geometric invariants of the second order (symmetry properties of Riemann-Christoffel tensor). The principle put forward by A. Gray and generated in a number of their works are put depented on the basis (**[1]**, **[2]**, **[3]**), according to which key to understanding of differential -geometrical properties Kahler manifolds identities with which satisfies them Riemann curvature tensor are:

$R_1: \langle R(X,Y)Z,W\rangle = \langle R(JX,JY)Z,W\rangle;$

$R_2: \langle R(X,Y)Z,W\rangle$
$= \langle R(JX,JY)Z,W\rangle$
$+ \langle R(JX,Y)JZ,W\rangle$
$+ \langle R(JX,Y)Z,JW\rangle;$

$R_3: \langle R(X,Y)Z,W\rangle = \langle R(JX,J\,Y)JZ,JW\rangle;$

Definition 11:- **[5]**

AH -structures, tensor R which satisfies to identity Ri, are called the structures of class R_i. If $\theta \subset AH$ - any subclass of AH-structures designation $\theta \cap R_i$=θwhere i = 1, 2, 3 .**[4]**

well - known that $C \subset R_1 \subset R_2 \subset R_3$ **[4]** So it is natural to expect that among AH-manifolds for differential - geometrical and topological properties closest to the kahler manifold class, manifold class, R_1manifold class R_2 and at last manifold of class R_3. The manifold of class R_2 while having no special items were introdused into consideration by A.Gray's in connection with studying nearly Kahler manifold whereas $NK \subset R_2$, and were considered Gray and Vanhecke **[3]** ,Watson and Vanhecke, and other authors.

Let (M, J, g) - nearly Kahler manifold of dimension 2n,C concircular curvature tensor.

Definition 12:-

The manifold (M, J, g) refers to as manifold of a class:

1. $\bar{C}_1$ if $< C(X,Y)Z, W >=< C(X,Y)JZ, JW >$;
2. $\bar{C}_1$ if $< C(X,Y)Z, W >=< C(JX,JY)Z, W > + < C(JX,Y)JZ, W > + < C(JX,Y)Z, JW >$;
3. $\bar{C}$ if $< C(X,Y)Z, W >=< C(JX,JY)JZ, JW >$.
(25)

Note 13.

From equation (19) follows that *NK* - manifold of class $C_0 = C_3 = C_5 = C_6$ are also manifold of a class $\bar{C}_3$.

Sense of the specified identities of curvature it is most transparent it is shown in terms of a spectrum concircular curvature tensor.

Theorem 14:-

Let $\theta = (J, g = <\times, \times>)$ is nearly Kahler structure. Then the following statements are equivalent:

(1) θ- Structure of a class $\bar{C}_3$;

(2) $C_{(0)} = 0$; and

(3) On space of the adjoint G-structure identities $C^a_{bcd} = 0$ are fair.

proof.

Let θ - structure of a class $\bar{C}_3$. Obviously, it is equivalent to identity $C(X,Y)Z + JC(JX,JY)JZ = 0$; $X,Y,Z \in X(M)$. By definition of a spectrum tensor

$C(X,Y)Z = C_{(0)}(X,Y)Z + C_{(1)}(X,Y)Z + C_{(2)}(X,Y)Z + C_{(3)}(X,Y)Z + C_{(4)}(X,Y)Z + C_{(5)}(X,Y)Z + C_{(6)}(X,Y)Z + C_{(7)}(X,Y)Z$; $X,Y,Z \in X(M)$

$J\circ C(JX,JY)JZ = J\circ C_{(0)}(JX,JY)JZ + J\circ C_{(1)}(JX,JY)JZ + J\circ C_{(2)}(JX,JY)JZ + J\circ C_{(3)}(JX,JY)JZ + J\circ C_{(4)}(JX,JY)JZ + J\circ C_{(5)}(JX,JY)JZ + J\circ C_{(6)}(JX,JY)JZ + J\circ C_{(7)}(JX,JY)JZ = C_{(0)}(X,Y)Z - C_{(1)}(X,Y)Z - C_{(2)}(X,Y)Z + C_{(3)}(X,Y)Z - C_{(4)}(X,Y)Z + C_{(5)}(X,Y)Z + C_{(6)}(X,Y)Z - C_{(7)}(X,Y)Z$; $X, Y, Z \in X(M)$.

Putting term by these identities, will be received:

C(X,Y)Z+JC(JX,JY)JZ= $\{C_{(0)}(X,Y)Z + C_{(3)}(X,Y)Z + C_{(5)}(X,Y)Z + C_{(6)}(X,Y)Z\}$

With means, the identity $C(X,Y)Z + JC(JX,JY)JZ = 0$ is equivalent to that

$$C_{(0)}(X,Y)Z + C_{(3)}(X,Y)Z + C_{(5)}(X,Y)Z + C_{(0)}(X,Y)Z$$

and this identity is equivalent to identities $C_{(0)} = C_{(3)} = C_{(5)} = C_{(6)} = 0$.

According to properties (4), the received identities on space of the adjoint G-structure are equivalent to relations:

$C^a_{bcd} = C^a_{b\hat{c}\hat{d}} = C^a_{\hat{b}c\hat{d}} = 0$.

By virtue of materiality tensor C and its properties(4) received relations which are equivalent to relations $C^a_{bcd} = 0$,i.e. identity $C_{(0)}(X,Y)Z = 0$.

The opposite, according to (19), obviously

Theorem 15:-

Let $\theta = (J, g = <\cdot,\cdot>)$ is nearly Kahler structure, then the following statements are equivalent:

(1) θ- Structure of a class $\bar{C}_2$;

(2) $C_{(0)} = C_{(7)} = 0$; and

(3) On space of the attached *G*-structure identities $C^a_{bcd} = C^a_{\hat{b}\hat{c}\hat{d}} = 0$ are fair.

Proof:

Let θ - structure of a class $\bar{C}_2$. We shall copy identity $\bar{C}_2$ in the following form.

With everyone composed this identity will be painted according to definition of a spectrum tensor:

1) $C(X, Y)Z = C_{(0)}(X,Y)Z + C_{(1)}(X,Y)Z + C_{(2)}(X,Y)Z + C_{(3)}(X,Y)Z + C_{(4)}(X,Y)Z + C_{(5)}(X,Y)Z + C_{(6)}(X,Y)Z + K_{(7)}(X,Y)Z$;

2) $C(JX,JY)Z = C_{(0)}(JX,JY)Z + C_{(1)}(JX,JY)Z + C_{(2)}(JX,JY)Z + C_{(3)}(JX,JY)Z + C_{(4)}(JX,JY)Z + C_{(5)}(JX,JY)Z + C_{(6)}(JX,JY)Z + C_{(7)}(JX,JY)Z = -C_{(0)}(X,Y)Z + C_{(1)}(X,Y)Z + C_{(2)}(X,Y)Z - C_{(3)}(X,Y)Z - C_{(4)}(X,Y)Z + C_{(5)}(X,Y)Z + C_{(6)}(X,Y)Z - C_{(7)}(X,Y)Z$.

3) $C(JX,Y)JZ = C_{(0)}(JX,Y)JZ + C_{(1)}(JX,Y)JZ + C_{(2)}(JX,Y)JZ + C_{(3)}(JX,Y)JZ + C_{(4)}(JX,Y)JZ + C_{(5)}(JX,Y)JZ + C_{(6)}(JX,Y)JZ + C_{(7)}(JX,Y)JZ = -C_{(0)}(X,Y)Z - C_{(1)}(X,Y)Z + C_{(2)}(X,Y)Z + C_{(3)}(X,Y)Z + C_{(4)}(X,Y)Z + C_{(5)}(X,Y)Z - C_{(6)}(X,Y)Z - C_{(7)}(X,Y)Z$

4) $JC(JX,Y)Z = JC_{(0)}(JX,Y)Z + JC_{(1)}(JX,Y)Z + JC_{(2)}(JX,Y)Z + JC_{(3)}(JX,Y)Z + JC_{(4)}(JX,Y)Z + JC_{(5)}(JX,Y)Z + JC_{(6)}(JX,Y)Z + JC_{(7)}(JX,Y)Z = -C_{(0)}(X,Y)Z - C_{(1)}(X,Y)Z + C_{(2)}(X,Y)Z + C_{(3)}(X,Y)Z - C_{(4)}(X,Y)Z - C_{(5)}(X,Y)Z + C_{(6)}(X,Y)Z + C_{(7)}(X,Y)Z$

Substituting these decomposition in the previous equality, we shall receive:

C(X,Y)Z-C(JX,JY)Z-C(JX,Y)JZ+JC(JX,Y)Z=

$2\{C_{(0)}(X,Y)Z + C_{(3)}(X,Y)Z + C_{(6)}(X,Y)Z + C_{(7)}(X,Y)Z\}$

This identity is equivalent to that

$C_{(0)}(X,Y)Z = C_{(3)}(X,Y)Z = C_{(5)}(X,Y)Z = C_{(7)}(X,Y)Z = 0$

and these identities on space of the adjoint *G-structure* are equivalent to identities

$C^a_{bcd} = C^a_{b\hat{c}\hat{d}} = C^a_{\hat{b}c\hat{d}} = C^a_{\hat{b}\hat{c}d} = C^a_{\hat{b}\hat{c}\hat{d}}$.

By virtue of materiality tensor C and his properties (4), the received relations are equivalent to relations: $C^a_{bcd} = C^a_{\hat{b}\hat{c}\hat{d}}$, i.e. to identities $C_{(0)}(X,Y)Z = C_{(0)}(X,Y)Z$.

Back, let for *NK*- manifold identities $C_{(0)}(X,Y)Z = C_{(7)}(X,Y)Z = 0$ are executed.

Then from (10) and (17) have:

C(X,Y)Z-C(X,JY)JZ-C(JX,Y)JZ-C(JX,JY)Z= 0

i.e.

C(X,Y)Z=C(JX,Y)JZ=C(JX,JY)Z=C(X,JY)Z

In the received identity instead of C(X,JY)Z we shall put the value received from (17) replacement $Y \to JY$ and $Z \to JZ$, i.e C(X,JY)JZ= -JC(JX,Y)Z.

Then:

C(X,Y)Z=C(JX,JY)+C(JX,Y)JZ-JC(JX,JY)Z

i.e.

Thus, the manifold satisfies to identity $\bar{C}_2$.

The following theorem is similarly proved.

Theorem 16:-

Let θ= (*J*, *g* = <×, ×>) is nearly Kahler structure. Then the following statements are equivalent:

(1) θ -structure of a class $\bar{C}_1$;

(2) $C_{(0)} = C_{(4)} = C_{(7)} = 0$;

(3) On space of the attached *G*-structure identities$C^{a}_{\hat{b}cd} = C^{a}_{\hat{b}cd} = C^{a}_{\hat{b}\hat{c}\hat{d}}$ are fair.

proof :

Let θ- structure of a class $\bar{C}_1$. Obviously, it is equivalent to identity

<*C(X, Y)Z*, *W* > = < *C(X, Y)JZ*, *JW* >

and we get *C(X, Y)Z* + *JC(X, Y)JZ* = 0 ; *X* , *Y* , *Z* ∈ *X*(*M*) .

By definition of a spectrum tensor

1) *C(X, Y)Z* = $C_{(0)}(X,Y)Z$ + $C_{(1)}$ (*X, Y)Z* + $C_{(2)}$ (*X, Y)Z* + $C_{(3)}$ (*X, Y)Z* + $C_{(4)}$ $(X,Y)Z$ + $C_{(5)}$ (*X, Y)Z* + $C_{(6)}$ (*X, Y)Z* + $C_{(7)}$ (*X, Y)Z* ; *X* , *Y* , *Z* ∈ *X*(*M*).

2) *J*∘ *C(X, Y)JZ* = *J*∘ $C_{(0)}(X,Y)JZ$+ *J*∘ $C_{(1)}(X,Y)JZ$ + *J* ∘ $C_{(2)}(X,Y)JZ$ + *J* ∘ $C_{(3)}(X,Y)JZ$ + *J* ∘$C_{(4)}(X,Y)JZ$ + *J* ∘ $C_{(5)}(X,Y)JZ$ + *J*∘ $C_{(6)}(X,Y)JZ$ + *J*∘ $C_{(7)}(X,Y)JZ$

=− $C_{(0)}(X,Y)Z$−$C_{(1)}$ $(X,Y)Z$−$C_{(2)}$ $(X,Y)Z$− $C_{(3)}$ $(X,Y)Z$+$C_{(4)}$ $(X,Y)Z$−$C_{(5)}$ $(X,Y)Z$− $C_{(6)}$ $(X,Y)Z$ + $C_{(7)}$ $(X,Y)Z$; *X* , *Y* , *Z* ∈ *X*(*M*).

Putting (1) and (2) in

C(X, Y)Z + *JC(X, Y)JZ* means, this identity is equivalent to that $C_{(0)}(X,Y)Z +$ $C_{(4)}(X,Y)Z$ + $C_{(7)}(X,Y)Z$ = 0 .

And this identity is equivalent to identities $C_{(0)}$ = $C_{(4)}$ = $C_{(7)}$ = 0

According to properties (4), The received identities in space of the adjoint *G*- structure are equivalent to relation $C^{a}_{\hat{b}cd}$ = $C^{a}_{\hat{b}cd}$ = $C^{a}_{\hat{b}\hat{c}\hat{d}}$ = 0 .

Corollary 17:-

Let θ = (*J*, *g* = <×, ×>) is nearly Kahler structure. Then the following inclusions of classes $\bar{C}_1 \subset \bar{C}_2 \subset \bar{C}_3$are fair.

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هندسة تنسر الانحناء الدائري لمنطوي الهرميتي التقريبي

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الملخص

تم إيجاد مركبات تنسر الانحناء الدائري في مستوى المنطوي الهرميتي التقريبي. والعلاقة بين المركبات اللامتغيرة لهذا التنسر , بالإضافة الى خواص التماثل لتنسر الانحناء الدائري , مع دراسة المعنى الهندسي للمركبات هذا التنسر عندما تساوي صفر.