On Contra *b-I-* Continuous and Contra *b-I-* open Functions

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Abstract
In this paper, we study the concepts of contra b-I-continuity and contra b-I-openness in ideal topological spaces, and obtain several characterizations and some properties of two functions. Also, we investigate its relationship with other types of functions.

Key words: Ideal topological spaces, b - I -open sets, b - I -closed sets, contra b - I -continuous functions, contra b - I -open functions.

1. Introduction
The notion of b-open sets was first introduced and investigated by Andrijevic' [2]. By using b-open set, Nasef [23] and [1] defined and studied contra b-continuity in topological spaces.

In 2005, Caksu Guler and G. Aslim [5] have introduced the notion of b - I -open sets and b - I -continuous function and used them to obtain a decomposition of continuity. In this paper, we introduce two new classes of functions called contra b - I -continuous functions and contra b - I -open functions in ideal topological spaces. Some characterizations and several basic properties these classes of functions are obtained and relationships between new two classes and other of function are established.

2. Preliminaries
Throughout this paper, for a subset A of a topological space (X,τ), $cl(A)$ and $int(A)$ denoted the closure and the interior of A, respectively. An ideal on a topological space (X,τ) is defined as a nonempty collection I of subsets of X satisfying the following two conditions:

(1) If $A \in I$ and $B \subseteq A$, then $B \in I$;

(2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

An ideal topological space is a topological space (X,τ) with an ideal I on X, and is denoted (X,τ,I). For a subset $A \subseteq X$, the set

$A^*(I)=\{x \in X : U \cap A \notin I$ for every $U \in \tau$ with $x \in U\}$

is called the local function of A with respect to I and τ [17]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. It is well known that $cl^*(A)=A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

We recall some known definition.

Definition 2.1: A subset A of a topological space (X,τ) is said to be

(a) Semi open [18] if $A \subseteq cl(int(A))$ and semi closed if $int(cl(A)) \subseteq A$.

(b) Preopen [19] if $A \subseteq int(cl(A))$ and preclosed if $cl(int(A)) \subseteq A$.

(c) $\alpha-$open [24] if $A \subseteq int(cl(int(A)))$ and $\alpha-$closed if $cl(int(cl(A))) \subseteq A$.

(d) $\beta-$open [20] if $A \subseteq cl(int(cl(A)))$ and $\beta-$closed if $int(cl(int(A))) \subseteq A$.

(e) b-open [2] if $A \subseteq cl(int(A)) \cup int(cl(A))$

Definition 2.2: A subset A of an ideal topological space (X,τ,I) is said to be

(a) I-open [16] if $A \subseteq int(A^*)$ and I-closed [17] if $cl(A^*) \subseteq A$.

(b) Semi-I-open[11] if and semi-I-closed if $int^*(cl(A)) \subseteq A$.

(c) Pre-I-open[8] if $A \subseteq cl^*(int(A))$ $A \subseteq int(cl^*(A))$ and pre-I-closed if $cl(int^*(A)) \subseteq A$.

(d) α - I - open [11] or pre- semi - I - open if $A \subseteq int(cl^*(int(A)))$ and α - I - closed or pre-semi- I - closed if $cl(int^*(cl(A))) \subseteq A$.

(e) β - I -open or semi pre-I-open [11] if $A \subseteq cl(int(cl^*(A)))$ and β - I -closed or semi pre-I-closed if $int(cl(int^*(A))) \subseteq A$.

(f) Strong β - I -open [12] if $A \subseteq cl^*(int(cl^*(A)))$.

(g) b - I -open [5] if $A \subseteq cl^*(int(A)) \cup int(cl^*(A))$ and b - I -closed [3] if $int(cl^*(A)) \cap cl^*(int(A)) \subseteq A$.

(h) b - I -clopen if A is b - I -open and b - I -closed.

Definition 2.3: A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is said to be

(a) Contra-continuous [7] (briefly c-c) if $f^{-1}(U)$ is open set in (X,τ) for each closed set U of (Y,σ),

(b) Contra-pre continuous [15] (briefly c-pre-c) if $f^{-1}(U)$ is pre open in (X,τ) for each closed set U of (Y,σ),

(c) Contra-semi continuous [10] (briefly c-semi-c) if $f^{-1}(U)$ is semi open in (X,τ) for each closed set U of (Y,σ),

(d) Contra-β-continuous [6] (briefly c-β-c) if $f^{-1}(U)$ is $\beta-$open in (X,τ) for each closed set U of (Y,σ),

(e) Contra-α-continuous [14] (briefly c-α-c) if $f^{-1}(U)$ is α-open in (X,τ) for each closed set U of (Y,σ),

(f) Contra-b-continuous [23] (briefly c-b-c) if $f^{-1}(U)$ is b-open in (X,τ) for each closed set U of (Y,σ).

(g) b-continuous [2] (briefly b-c) if $f^{-1}(U)$ is b-open in (X,τ) for each open set U of (Y,σ).

Definition 2.4: A function $f:(X,\tau,I)\to(Y,\sigma)$ is said to be

(a) b-I-continuous [5] (briefly b-I-c) if $f^{-1}(U)$ is b-I-open set in (X,τ,I) for each open set U of (Y,σ),

(b) Contra I-continuous [22] (briefly c-I-c) if $f^{-1}(U)$ is I-open set in (X,τ,I) for each closed set U of (Y,σ),

(c) Contra pre-I-continuous [25] (briefly c-pre-I-c) if $f^{-1}(U)$ is pre-I-open in (X,τ,I) for each closed set U of (Y,σ),

(d) Contra semi-I-continuous [22] (briefly c-semi-I-c) if $f^{-1}(U)$ is semi-I-open in (X,τ,I) for each closed set U of (Y,σ),

(e) Contra β-I-continuous [4] (briefly c-β-I-c) if $f^{-1}(U)$ is β-I-open in (X,τ,I) for each closed set U of (Y,σ),

(f) Contra α-I-continuous [27] (briefly c-α-I-c) if $f^{-1}(U)$ is α-I-open in (X,τ,I) for each closed set U of (Y,σ),

(g) Contra strong β-I-continuous [25] (briefly c-strong β-I-c) if $f^{-1}(U)$ is strong β-I-open in (X,τ,I) for each closed set U of (Y,σ).

Proposition 2.5: [5] For an ideal topological space (X,τ,I) and $A\subseteq X$, we have the following:

(1) If $I=\phi$, then A is b-I-open if and only if A is b-open.

(2) If $I=P(x)$, then A is b-I-open if and only if $A\in\tau$.

(3) If $I=N$, then A is b-I-open if and only if A is b-open, where N is the ideal of all nowhere dense sets.

3. contra b-I-continuous functions

We have introduced the following definition

Definition 3.1: A function $f:(X,\tau,I)\to(Y,\sigma)$ is said to be contra b-I-continuous if $f^{-1}(V)$ is b-I-open set in (X,τ,I) for every closed set V of (Y,σ).

Example 3.2: Let $X=\{a,b,c\}$, $\tau=\{\phi,X,\{a\},\{b\},\{a,b\}\}$ and $I=\{\phi,\{b\}\}$. Let $f:(X,\tau,I)\to(X,\tau)$ be defined by $f(a)=c$, $f(b)=a$ and $f(c)=b$. Then f is contra b-I-continuous.

Definition 3.3: [21] Let A be a subset of a space (X,τ,I). Then the set $\cap\{U\in\tau : A\subset U\}$ is called the kernel of A and denoted by $Ker(A)$.

Lemma 3.4: [13] The following properties hold for subsets A,B of a space X:

1. $x\in Ker(A)$ if and only if $A\cap F\neq\phi$ for any closed set F of X with $x\in F$;

2. $A\subset Ker(A)$ and $A=Ker(A)$ if A is open in X;

3. If $A\subset B$ then $Ker(A)\subset Ker(B)$.

Definition 3.5: [3] Let N be a subset of a space (X,τ,I) and let $x\in X$. Then N is called b-I-neighborhood of x, if there exists a b-I-open set U containing x such that $U\subset N$.

Theorem 3.6: The following statements are equivalent for a function $f:(X,\tau,I)\to(Y,\sigma)$:

(1) f is contra b-I-continuous;

(2) For each $x\in X$ and each closed set V of Y with $f(x)\in V$, there exists a b-I-open set U of X containing x such that $f(U)\subset V$;

(3) For each $x\in X$ and each closed set V of Y with $f(x)\in V$, $f^{-1}(V)$ is a b-I-neighborhood of x;

(4) For each open set V of Y, $f^{-1}(V)$ is b-I-closed;

(5) $f({}_{bI}cl(A))\subset Ker(f(A))$ for every subset A of X;

(6) ${}_{bI}cl(f^{-1}(B))\subset f^{-1}(Ker(B))$ for every subset B of Y.

Proof: $(1)\to(2)$: Let $x\in X$ and V be a closed set in Y such that $f(x)\in V$. Since f is contra b-I-continuous, $f^{-1}(V)$ is b-I-open in X. By putting $U=f^{-1}(V)$ which is containing x, we have $f(U)\subset V$.

$(2)\to(3)$: Let V be a closed set in Y and $f(x)\in V$. Then by (2), there exists a b-I-open set U of X containing x such that $f(U)\subset V$. So $x\in U\subset f^{-1}(V)$. Hence $f^{-1}(V)$ is a b-I-neighborhood of x.

$(3)\to(1)$: It is obvious.

$(1)\leftrightarrow(4)$: It is obvious.

$(1)\rightarrow(5)$: Let A be any subset of X. Suppose that $y\notin Ker(f(A))$. Then by lemma 3.4, there exists a closed set V of Y containing y such that $f(A)\cap V=\phi$. Thus, we have $A\cap f^{-1}(V)=\phi$ and ${}_{bI}cl(A)\cap f^{-1}(V)=\phi$. Therefore, we obtain $f({}_{bI}cl(A))\cap V=\phi$ and $y\notin f({}_{bI}cl(A))$. This implies that $f({}_{bI}cl(A))\subset Ker(f(A))$.

$(5)\rightarrow(6)$: Let B be any subset of Y. By (5) and lemma 3.4, we have $f({}_{bI}cl(f^{-1}(B)))\subset Ker(f(f^{-1}(B)))\subset Ker(B)$ and ${}_{bI}cl(f^{-1}(B))\subset f^{-1}(Ker(B))$.

$(6)\rightarrow(4)$: Let V be any open set of Y. Then by lemma 3.4, we have ${}_{bI}cl(f^{-1}(V))\subset f^{-1}(Ker(V))=f^{-1}(V)$ and ${}_{bI}cl(f^{-1}(V))=f^{-1}(V)$. This shows that $f^{-1}(V)$ is b-I-closed in (X,τ,I).

Remark 3.7: In fact contra b-I- continuity and b-I- continuity are independent notions. Example 3.2 above shows that contra b-I- continuous function need not be b-I- continuous while the reverse is show in the following example.

Example 3.8: A b-I- continuous function does not imply contra b-I- continuous function. Let $X=Y=\{a,b,c\}$, $\tau=\{\phi,X,\{a\},\{c\},\{a,c\}\}$, $I=\{\phi,\{b\}\}$ and $\sigma=\{\phi,Y,\{b,c\}\}$. Let $f:(X,\tau,I)\rightarrow(Y,\sigma)$ be defined by $f(a)=b$, $f(b)=a$ and $f(c)=c$. Then f is b-I- continuous but not contra b-I- continuous, since $\{b,c\}$ is open in (Y,σ) and $f^{-1}(\{b,c\})=\{a,c\}$ is not b-I-closed in (X,τ,I).

Proposition 3.9: For a subset of an ideal topological space, the following conditions hold:

1. Every open set is b-I-open [5].
2. Every semi-I- open set is b-I-open [5].
3. Every pre-I- open set is b-I-open [5].
4. Every b-I-open set is β-I-open[5].
5. Every b-I-open set is b-open [3].
6. Every α-I- open set is b-I-open.
7. Every α- open set is b-I-open.
8. Every pre open set is b-I-open.
9. Every b-I-open set is β-open.
10. Every b-I-open set is strong β-I-open.

Proof: It is obvious.

Proposition 3.10: The following statements hold for a function $f:(X,\tau,I)\rightarrow(Y,\sigma)$:

1. Every contra-continuous function is contra b-I- continuous.
2. Every contra I- continuous function is contra b-I- continuous.
3. Every contra α-I- continuous function is contra b-I- continuous.
4. Every contra α- continuous function is contra b-I- continuous.
5. Every contra semi-I- continuous function is contra b-I- continuous.
6. Every contra pre-I- continuous function is contra b-I- continuous.
7. Every contra-pre continuous function is contra b-I- continuous.
8. Every contra b-I- continuous function is contra b- continuous.
9. Every contra b-I- continuous function is contra β-I- continuous.
10. Every contra b-I- continuous function is contra β- continuous.
11. Every contra b-I- continuous function is contra strong β-I- continuous.

Proof: This follows from Proposition 3.9.

The converse of Proposition 3.10 is not true in general. The following counter examples show the cases.

Example 3.11: Let $X=Y=\{a,b,c,d\}$, $\tau=\{\phi,X,\{b\},\{a,d\},\{a,b,d\}\}$, $\sigma=\{\phi,Y,\{a,c\}\}$ and ideal $I=\{\phi,\{b\}\}$.

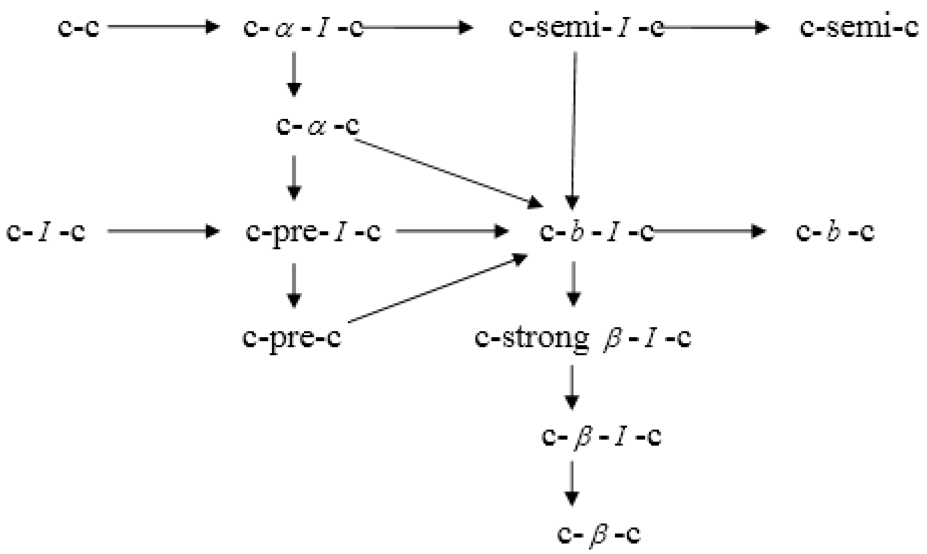
(a) Let $f:(X,\tau,I)\rightarrow(Y,\sigma)$ be the identity function. Then f is contra b-I- continuous but not contra-continuous (resp. contra I- continuous, contra α-I- continuous, contra α- continuous, contra semi-I- continuous), since $\{b,d\}$ is closed in (Y,σ) and $f^{-1}(\{b,d\})=\{b,d\}$ is b-I-open but not open (resp. I-open, α-I-open, α-open, semi-I- open) in (X,τ,I).

(b) Let $f:(X,\tau,I)\rightarrow(Y,\sigma)$ be defined by $f(a)=f(c)=b$ and $f(b)=a$ $f(d)=d$. Then f is contra b-I- continuous but not contra pre-I- continuous (resp. contra-pre continuous), since $\{b,d\}$ is closed in (Y,σ) and $f^{-1}(\{b,d\})=\{a,c,d\}$ is b-I-open but not pre-I-open (resp. pre open) in (X,τ,I).

(c) Let $f:(X,\tau,I)\rightarrow(Y,\sigma)$ be defined by $f(a)=a$, $f(b)=b$, $f(c)=d$ and $f(d)=c$. Then f is contra b- continuous but not contra b-I- continuous, since $\{b,d\}$ is closed in (Y,σ) and $f^{-1}(\{b,d\})=\{b,c\}$ is not b-I-open in (X,τ,I).

(d) Let $f:(X,\tau,I)\rightarrow(Y,\sigma)$ be defined by $f(a)=b$, $f(b)=a$, $f(c)=d$ and $f(d)=c$. Then f is contra strong β-I- continuous (resp. contra β-I- continuous, contra β- continuous) but not contra b - I -continuous, since $\{b,d\}$ is closed in (Y,σ) and $f^{-1}(\{b,d\})=\{a,c\}$ is strong β - I -open (resp. β - I -open, β -open) but not b - I -open in (X,τ,I).

Remark 3.12: We have the following implications:



Theorem 3.13: If a function $f:(X,\tau,I)\to(Y,\sigma)$ is contra b - I - continuous and Y is regular, then f is b - I - continuous.

Proof: Let $x\in X$ and V be an open set in Y with $f(x)\in V$. Since Y is regular, there exists an open set W in Y such that $f(x)\in W\subseteq cl(W)\subseteq V$. Since f is contra b - I -continuous, by theorem 3.6 there exists a b - I -open set U in X with $x\in U$ such that $f(U)\subseteq cl(W)$. Then $f(U)\subseteq cl(W)\subseteq V$. Hence f is b - I - continuous.

Corollary 3.14: If a function $f:(X,\tau,I)\to(Y,\sigma)$ is contra b - I - continuous and Y is regular, then f is b - continuous.

Proof: The proof follows from theorem 3.13 and every b - I -open set is b -open.

Theorem 3.15: (1) A function $f:(X,\tau,\{\phi\})\to(Y,\sigma)$ is contra b - I - continuous if and only if it is contra b - continuous.

(2) A function $f:(X,\tau,P(X))\to(Y,\sigma)$ is contra b - I - continuous if and only if it is contra - continuous.

(3) A function $f:(X,\tau,N)\to(Y,\sigma)$ is contra b - I - continuous if and only if it is contra b - continuous.

Proof: This follows from Proposition 2.5.

Lemma 3.16: For a subset A of (X,τ,I), the following are equivalent:

(1) A is b - I -open.

(2) A is b -open and strong β - I -open.

Proof: $(1)\to(2)$. It is obvious.

$(2)\to(1)$. Let A be b -open and strong β - I -open. Then, we have

$$A\subseteq cl(int((A))\bigcup int(cl(A))$$
$$\subseteq cl(int(cl^{*}(int(cl^{*}(A)))))\bigcup int(cl(cl^{*}(int(cl^{*}(A)))))$$
$$\subseteq cl(int(cl(int(cl(A)))))\bigcup int(cl(cl(int(cl(A)))))$$
$$\subseteq cl(int(cl(int(A))))\cup A\subset int(cl(int(cl(A))))\cup A)$$
$$\subseteq cl*((int(A))\cup A\bigcup int(A^{*}\cup A)$$
$$\subseteq cl^{*}(int(A))\bigcup int(cl^{*}(A))$$

Hence A is b - I -open.

Theorem 3.17: The following are equivalent for a function $f:(X,\tau,I)\to(Y,\sigma)$:

1. f is contra b - I - continuous;
2. f is contra b - continuous and contra strong β - I - continuous.

Proof: This follows from lemma 3.16.

Recall that for a function $f:(X,\tau)\to(Y,\sigma)$, the subset $\{(x,f(x)):x\in X\}\subseteq X\times Y$ is called the graph of f and is denoted by $G(f)$.

Theorem 3.18: A function $f:(X,\tau,I)\to(Y,\sigma)$ is contra b - I - continuous if and only if the graph function $g:X\to X\times Y$ defined by $g(x)=(x,f(x))$ for each $x\in X$ is contra b - I - continuous.

Proof: "Necessity". Let f is contra b - I - continuous. Now let $x\in X$ and let W be a closed subset of $X\times Y$ containing $g(x)$. Then $W\cap(\{x\}\times Y)$ is closed in $\{x\}\times Y$ containing $g(x)$. Also $\{x\}\times Y$ is homeomorphic to Y. Hence $\{y\in Y\,|\,(x,y)\in W\}$ is a closed subset of Y. Since f is contra b - I -continuous, $\cup\{f^{-1}(y)\in Y\,|\,(x,y)\in W\}$ is a b - I -open subset of (X,τ,I). Further, $x\in\cup\{f^{-1}(y)\in Y\,|\,(x,y)\in W\}\subset g^{-1}(w)$. Hence $g^{-1}(w)$ is b-I-open. Then g is contra b-I-continuous.

"Sufficiency". Suppose that g is contra b-I-continuous and let F be a closed subset of Y. Then $X\times F$ is a closed subset of $X\times Y$. Since g is contra b-I- continuous, $g^{-1}(X\times F)$ is a b-I-open subset of X. Also, $g^{-1}(X\times F)=f^{-1}(F)$. Hence f is contra b-I- continuous.

Definition 3.19: A space (X,τ,I) is said to be b-I-T_1 if for each pair of distinct points x and y in X, there exist b-I-open sets U and V containing x and y, respectively, such that $y\notin U$ and $x\notin V$.

Definition 3.20: A space (X,τ,I) is said to be b-I-T_2 if for each pair of distinct points x and y in X, there exist b-I-open sets U and V containing x and y, respectively, such that $U\cap V=\phi$.

Theorem 3.21: Let $f:(X,\tau,I)\to(Y,\sigma)$ be a contra b-I- continuous injective function. If (Y,σ) is a Urysohn space, then (X,τ,I) is a b-I-T_2 space.

Proof: Let x_1 and x_2 be a pair of distinct points in X. Then $f(x_1)\neq f(x_2)$. Since (Y,σ) is Urysohn space, there exist open sets U and V of Y such that $f(x_1)\in U$, $f(x_2)\in V$ and $cl(U)\cap cl(V)=\phi$. Since f is contra b-I- continuous at x_i for $i=1,2$, there exists b-I-open sets A and B in X such that $x_1\in A$, $x_2\in B$ and $f(A)\subseteq cl(U)$, $f(B)\subseteq cl(V)$. Then, $f(A)\cap f(B)=\phi$, so $A\cap B=\phi$. Hence, X is a b-I-T_2 space.

Definition 3.22: An ideal topological space (X,τ,I) is said to be b-I-connected [3] if X is not the union of two disjoint b-I-open subsets of X.

Theorem 3.23: A contra b-I- continuous image of a b-I-connected space is connected.

Proof: Let $f:(X,\tau,I)\to(Y,\sigma)$ be a contra b-I-continuous function from a b-I-connected space X onto a space Y. Assume that Y is disconnected. Then $Y=A\cup B$ where A and B are non-empty clopen sets in Y with $A\cap B=\phi$. Since f is contra b-I- continuous, we have that $f^{-1}(A)$ and $f^{-1}(B)$ are b-I-open non-empty sets in X with $f^{-1}(A)\cup f^{-1}(B)=f^{-1}(A\cup B)=f^{-1}(Y)=X$ and $f^{-1}(A)\cap f^{-1}(B)=$ $f^{-1}(A\cap B)=f^{-1}(\phi)$. This means that X is not b-I-connected, which is a contradiction. Then Y is connected.

Theorem 3.24: Let (X,τ,I) be b-I-connected space and (Y,σ) be a T_1-space. If $f:(X,\tau,I)\to(Y,\sigma)$ is a contra b-I- continuous, then f is a constant function.

Proof: Since Y is a T_1-space, $B=\{f^{-1}(\{y\})\,|\,y\in Y\}$ is a disjoint b-I-open partition of X. If $|B|\geq 2$, then X is the union of two non-empty b-I-open sets. Since X is b-I-connected, $|B|=1$, therefore f is constant.

Theorem 3.25: If $f:(X,\tau,I)\to(Y,\sigma)$ is a contra b-I- continuous function from a b-I-connected space (X,τ,I) onto any space Y, then Y is not a discrete space.

Proof: suppose that Y is discrete. Let A be a proper non-empty clopen set in Y. Then $f^{-1}(A)$ is a proper non-empty b-I- clopen subset of X, which contradicts the fact that X is b-I-connected.

Definition 3.26: The graph $G(f)$ of a function $f:(X,\tau,I)\to(Y,\sigma)$ is said to be contra b-I-closed in $X\times Y$ if for each $(x,y)\in(X\times Y)\setminus G(f)$, there exist a b-I-open set U in X containing x and a closed set V in Y containing y such that $(U\times V)\cap G(f)=\phi$.

Lemma 3.27: The graph $f:(X,\tau,I)\to(Y,\sigma)$ is contra b-I-closed in $X\times Y$ if and only if for each $(x,y)\in(X\times Y)\setminus G(f)$, there exist a b-I-open set U in X containing x and a closed set V in Y containing y such that $f(U)\cap V=\phi$.

Theorem 3.28: If $f:(X,\tau,I)\to(Y,\sigma)$ is a contra b-I- continuous function and Y is a Urysohn space, then $G(f)$ is contra b-I-closed in $X\times Y$.

Proof: Let $(x,y)\in(X\times Y)\setminus G(f)$. Then $y\neq f(x)$ and there exist open set V,W of Y such that $f(x)\in V, y\in W$ and $cl(V)\cap cl(W)=\phi$. Since f is contra b-I- continuous, there exist a b-I-open set U in X containing x such that $f(U)\subseteq cl(V)$. Therefore, we obtain $f(U)\cap cl(W)=\phi$. This shows that $G(f)$ is contra b-I-closed.

Theorem 3.29: If $f:(X,\tau,I)\to(Y,\sigma)$ is a contra b-I- continuous function and (Y,σ) is T_2, then $G(f)$ is contra b-I-closed.

Proof: Let $(x,y)\in(X\times Y)\setminus G(f)$. Then $y\neq f(x)$. Since Y is T_2, there exists open set V in Y such that $f(x)\in V$ and $y\notin V$. Since f is contra b-I-continuous, there exists a b-I-open set U in X containing x such that $f(U)\subseteq cl(V)$. Therefore, $f(U)\cap(Y-V)=\phi$ and $Y-V$ is a closed set of Y

containing y. This shows that $G(f)$ is contra b-I-closed.

Theorem 3.30: Let the graph of a function $f:(X,\tau,I)\to(Y,\sigma)$ be contra b-I-closed. If f is injective function, then X is b-I-T_1.

Proof: Let x and y be any two distinct points in X. Since f is injection, then we have $(x,f(y))\in(X\times Y)\setminus G(f)$. Since $G(f)$ is contra b-I-closed, there exist a b-I-open set U of X and a closed set V of Y such that $(x,f(y))\in U\times V$ and $f(U)\cap V=\phi$. Therefore, we have $U\cap f^{-1}(V)=\phi$ and $y\notin U$. Thus $y\in X-U$ and $x\notin X-U$ and $X-U$ is b-I-open set in X. This implies that X is b-I-T_1.

Definition 3.31: An ideal topological space (X,τ,I) is said to be b-I-normal [3] if for each pair of nonempty disjoint closed sets of X, it can be separated by disjoint b-I-open sets.

Definition 3.32: A topological space (X,τ) is said to be ultra normal [26]. if for each pair of nonempty disjoint closed sets of X, it can be separated by disjoint clopen sets.

Theorem 3.33: If $f:(X,\tau,I)\to(Y,\sigma)$ is a contra b-I- continuous, closed injective function and Y is a ultra normal space, then (X,τ,I) is b-I-normal.

Proof: Let F_1 and F_2 be a disjoint closed subsets of X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. But Y is ultra normal, so $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2 of Y, respectively. Since f is contra b-I- continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are b-I-open, with $F_1\subseteq f^{-1}(V_1)$, $F_2\subseteq f^{-1}(V_2)$ and $f^{-1}(V_1)\cap f^{-1}(V_2)=\phi$. Hence, (X,τ,I) is b-I-normal.

Definition 3.34: A topological space (X,τ) is said to be strongly S-closed [7] if every closed cover of X has a finite subcover.

Definition 3.35: An ideal topological space (X,τ,I) is said to be b-I-compact [3] if every b-I-open cover of X has a finite subcover.

Definition 3.36: A subset A of a space (X,τ,I) is said to be b-I-compact relative to X if every cover of A by b-I-open subset of X has a finite subcover.

Theorem 3.37: If $f:(X,\tau,I)\to(Y,\sigma)$ is a contra b-I- continuous and A is b-I-compact relative to X, then $f(A)$ is strongly S-closed in Y.

Proof: Let $\{V_i : i\in I\}$ be any cover of $f(A)$ by closed subsets of the subspace $f(A)$. For each $i\in I$, there exists a closed set F_i of Y such that $V_i=F_i\cap f(A)$. For each $x\in A$, there exists $i(x)\in I$ such that $f(x)\in F_{i(x)}$. Now by theorem 3.6, there exists a b-I-open set U_x of X with $x\in U_x$ such that $f(U_x)\subseteq F_{i(x)}$. Since the family $\{U_x : x\in A\}$ is a cover of A by b-I-open sets of X, there exists a finite subset A_o of A such that $A\subseteq\cup\{U_x : x\in A_o\}$. Therefore, we obtain $f(A)\subseteq\cup\{f(U_x) : x\in A_o\}$, which is a subset of $\cup\{F_{i(x)} : x\in A_o\}$. Thus $f(A)\subseteq\cup\{V_{i(x)} : x\in A_o\}$, and hence $f(A)$ is strongly S-closed.

Corollary 3.38: If $f:(X,\tau,I)\to(Y,\sigma)$ is a contra b-I- continuous surjection and X is b-I-compact, then Y is strongly S-closed.

Theorem 3.39: Let the graph of a function $f:(X,\tau,I)\to(Y,\sigma)$ be contra b-I-closed, then the inverse image of a strongly S-closed set A of Y is b-I-closed in X.

Proof: Assume that A is a strongly S-closed set of Y and $x\notin f^{-1}(A)$. For each $a\in A$, $(x,a)\notin G(f)$. There exists b-I-open set U_a of X and closed set V_a in Y such that $f(U_a)\cap V_a=\phi$. Since $\{A\cap V_a : a\in A\}$ is a closed cover of the subspace A, since A is strongly S-closed, then there exists a finite subset $A_o\subseteq A$ such that $A\subseteq\cup\{V_a : a\in A_o\}$. Set $U=\cap\{U_a : a\in A_o\}$, then U is b-I-open in X and $f(U)\cap A\subseteq f(U_a)\cap[\cup\{V_a : a\in A_o\}]=\phi$. Therefore $U\cap f^{-1}(A)=\phi$ and hence $x\notin bICl(f^{-1}(A))$. This show that $f^{-1}(A)$ is b-I-closed.

Theorem 3.40: Let Y be strongly S-closed space. If $f:(X,\tau,I)\to(Y,\sigma)$ has a contra b-I-closed graph, then f is contra b-I- continuous.

Proof: Let U be an open set of Y and $\{V_i : i\in I\}$ be cover of U by closed sets V_i of U. For each $i\in I$, there exists a closed set K_i of X such that $V_i=K_i\cap U$. Then the family $\{K_i : i\in I\}\cup(Y-U)$ is a closed cover of Y. Since Y is strongly S-closed, there exists a finite subset $I_o\subseteq I$ such that $Y=\cup\{K_i : i\in I\}\cup(Y-U)$. Therefore, we obtain $U=\cup\{V_i : i\in I_o\}$. This shows that U is strongly S-closed. By theorem 3.39, $f^{-1}(U)$ is b-I-closed in X for every open U in Y. Therefore, f is contra b-I- continuous.

Theorem 3.41: If $f:(X,\tau,I)\to(Y,\sigma)$ is contra b-I- continuous, $g:(X,\tau,I)\to(Y,\sigma)$ is contra-I-continuous, and Y is Urysohn, then $E=\{x\in X: f(x)=g(x)\}$ is b-I-closed in X.

Proof: Let $x\in X-E$. Then $f(x)\neq g(x)$. Since Y is Urysohn, there exist open sets V and W such that $f(x)\in V, g(x)\in W$ and $cl(V)\cap cl(W)=\phi$. Since f is contra b-I-continuous, then $f^{-1}(cl(V))$ is b-I-open in X and g is contra-I-continuous, then $g^{-1}(cl(W))$ is open set in X. Let $U=f^{-1}(cl(V))$ and $G=g^{-1}(cl(W))$. Then U and G contain x. Set $A=U\cap G$. A is b-I-open in X. And $f(A)\cap g(A)\subseteq f(U)\cap g(G)\subseteq cl(V)\cap cl(W)=\phi$. Hence $f(A)\cap g(A)=\phi$ and $A\cap E=\phi$ where A is b-I-open therefore $x\notin bICl(E)$. Thus E is b-I-closed in X.

Definition 3.42: An ideal space (X,τ,I) is said to be b-I-space if every b-I-open set of X is open in X.

Theorem 3.43: Let $f:(X,\tau,I)\to(Y,\sigma)$ be a function and let (X,τ,I) be b-I-space, then the following are equivalent:

1. f is contra b-I- continuous.
2. f is contra b- continuous.
3. f is contra pre-I- continuous.
4. f is contra pre- continuous.
5. f is contra semi-I- continuous.
6. f is contra semi- continuous.
7. f is contra α-I- continuous.
8. f is contra α- continuous.
9. f is contra -continuous.

Proof: It is straightforward.

Definition 3.44: A function $f:(X,\tau,I)\to(Y,\sigma,J)$ is said to be b-I-irresolute if $f^{-1}(U)$ is b-I-open in X for every b-J-open U of Y.

Theorem 3.45: Let $f:(X,\tau,I)\to(Y,\sigma,J)$ and $g:(Y,\sigma,J)\to(Z,\rho)$ be two functions such that $g\circ f:(X,\tau,I)\to(Z,\rho)$. Then:

1. $g\circ f$ is contra b-I- continuous, if g is continuous and f is contra b-I- continuous.
2. $g\circ f$ is contra b-I- continuous, if g is contra-continuous and f is b-I- continuous.
3. $g\circ f$ is contra b-I- continuous, if g is contra b-I-continuous and f is b-I- irresolute.
4. $g\circ f$ is contra b-I- continuous, if g is continuous and f is b-I- continuous, and Y is locally indiscrete.

Proof: It is obvious.

Recall that a function $f:(X,\tau)\to(Y,\sigma,J)$ is said to be b-I-open [3] if for each $V\in\tau$, $f(V)$ is b-J-open in Y.

Theorem 3.46: Let $f:(X,\tau,I)\to(Y,\sigma,J)$ be surjective function, b-I- irresolute and b-I-open, and let $g:(Y,\sigma,J)\to(Z,\rho)$ be any function. Then $g\circ f$ is contra b-I- continuous if and only if g is contra b-I-continuous.

Proof: "Necessity". Let $g\circ f$ is contra b-I-continuous and V be a closed subset of Z. Then $(g\circ f)^{-1}(V)=f^{-1}(g^{-1}(V))$ is b-I-open subset of X. Since f is b-I-open, $f(f^{-1}(g^{-1}(V)))$ is b-J-open subset of (Y,σ,J). So $g^{-1}(V)$ is b-J-open in (Y,σ,J). Therefore, g is contra b-I-continuous.

"Sufficiency". Let g is contra b-I-continuous and V be closed subset of Z. Then $g^{-1}(V)$ is b-J-open in (Y,σ,J). Since f is b-I- irresolute, $f^{-1}(g^{-1}(V))=(g\circ f)^{-1}(V)$ is b-I-open in (X,τ,I). Therefore, $g\circ f$ is contra b-I-continuous.

Remark 3.47: The following example shows that composition of any contra b-I-continuous functions need not be contra b-I-continuous function in general.

Example 3.48: Let $X=\{a,b,c,d\}$ with topology $\tau_1=\{\phi,X,\{a\},\{a,b\},\{a,b,c\}\}$, $\tau_2=\{\phi,X,\{a\},\{b,c\},\{a,b,c\}\}$, $\tau_3=\{\phi,X,\{a\},\{a,b,c\}\}$ and ideal $I=\{\phi,\{a\}\}$, $J=\{\phi,\{b\}\}$. Let $f:(X,\tau_1,I)\to(X,\tau_2,J)$ be defined by $f(a)=f(b)=d$, $f(c)=b, f(d)=c$ and $g:(X,\tau_2,J)\to(X,\tau_3)$ be defined by $g(a)=d, g(b)=c, g(c)=b$ and $g(d)=a$. Then f and g are contra b-I-continuous. But $g\circ f$ is not contra b-I-continuous, since $F=\{b,c,d\}$ is closed in (X,τ_3) and $(g\circ f)^{-1}(F)=f^{-1}(g^{-1}(F))=\{c,d\}$ is not b-I-open in (X,τ_1,I).

4. Contra b-I open functions

Definition 4.1: A function $f:(X,\tau)\to(Y,\sigma,J)$ is said to be contra b-I-open if for each closed set U of X, $f(U)$ is b-J-open in (Y,σ,J).

Theorem 4.2: For a function $f:(X,\tau)\to(Y,\sigma,J)$, the following are equivalent:

(1) f is contra b-I- open,

(2) For every $F\in\tau$, $f(F)$ is b-J-closed in (Y,σ,J),

(3) For every subset B of Y and for every open set F of X with $f^{-1}(B)\subseteq F$, there exists a b-J-closed set V of Y with $B\subseteq V$ and $f^{-1}(V)\subseteq F$,
(4) For every subset $y\in Y$ and for every open set F of X with $f^{-1}(y)\subseteq F$, there exists a b-J-closed set V of Y with $y\in V$ and $f^{-1}(V)\subseteq F$.

Proof: The implications $(1)\to(2)$, $(2)\to(3)$ and $(3)\to(4)$ are obvious.

$(4)\to(1)$: Let U be a closed set of X. Then let $y\in Y-f(U)$ and let $F=X-U$. By (4), there exists a b-J-closed set V of Y with $y\in V$ and $f^{-1}(V)\subseteq F$. Then, we see that $y\in V\subseteq Y-f(U)$ and hence $f(U)$ is b-J-open and therefore f is contra b-I- open function.

Remark 4.3: Contra b-I-open and b-I-open functions are independent of each other.

Example 4.4: Let $X=\{a,b,c\}$, $\tau=\{\phi,X,\{a\}\}$, $I=\{\phi,\{b\}\}$. Let $f:(X,\tau)\to(X,\tau,I)$ be defined by $f(a)=b$, $f(b)=c$ and $f(c)=a$. Observe that f is contra b-I-open. But f is not b-I-open function, since $\{a\}$ is open in (X,τ) and $f(\{a\})=\{b\}$ is b-I-closed but not b-I-open in (X,τ,I).

Example 4.5: Let $X=\{a,b,c,d\}$, $\tau_1=\{\phi,X,\{a,c\}\}$, $\tau_2=\{\phi,X,\{b\},\{a,d\},\{a,b,d\}\}$ and ideal $I=\{\phi,\{b\}\}$. Let $f:(X,\tau_1)\to(X,\tau_2,I)$ be defined by $f(a)=f(c)=a$ and $f(b)=f(d)=b$. Then f is b-I-open but not contra b-I-open, since $\{a,c\}$ is open in (X,τ_1) and $f(\{a,c\})=\{a\}$ is b-I-open but not b-I-closed in (X,τ_2,I).

Theorem 4.6: For any bijective function $f:(X,\tau)\to(Y,\sigma,J)$, the following are equivalent:
1. $f^{-1}:(Y,\sigma,J)\to(X,\tau)$ is contra b-I-continuous.
2. f is contra b-I-open.

Proof: It is straightforward.

Definition 4.7: A function $f:(X,\tau)\to(Y,\sigma,J)$ is said to be:
1. Contra -I- open if for each closed subset U of X, $f(U)$ is J- open set in (Y,σ,J).
2. Contra semi -I- open if for each closed subset U of X, $f(U)$ is semi -J- open set in (Y,σ,J).
3. Contra pre -I- open if for each closed subset U of X, $f(U)$ is pre -J- open set in (Y,σ,J).
4. Contra α-I- open if for each closed set U of X, $f(U)$ is α-J- open set in (Y,σ,J).

Remark 4.8: (1) Every contra-I- open function is contra b-I-open.
(2) Every contra semi -I- open function is contra b-I-open.
(3) Every contra pre -I- open function is contra b-I-open.
(4) Every contra α-I- open function is contra b-I- open.

The converse of the above remark need not be true as shown in the following example.

Example 4.9: Let $X=\{a,b,c,d\}$, $\tau_1=\{\phi,X,\{a\}\}$, $\tau_2=\{\phi,X,\{b\},\{a,d\},\{a,b,d\}\}$ and ideal $I=\{\phi,\{b\}\}$.

(a) Let $f:(X,\tau_1)\to(X,\tau_2,I)$ be the identity function. Then f is contra b-I-open but not contra-I- open (resp. contra semi -I- open), since $\{b,c,d\}$ is closed in (X,τ_1) and $f(\{b,c,d\})=\{b,c,d\}$ is b-I-open but not I-open (resp. semi -I- open) in (X,τ_2,I).

(b) Let $f:(X,\tau_1)\to(X,\tau_2,I)$ be defined by $f(a)=b$, $f(b)=a$ and $f(c)=c$ $f(d)=d$. Then f is contra b-I-open but not contra pre-I- open (resp. contra α-I- open), since $\{b,c,d\}$ is closed in (X,τ_1) and $f(\{b,c,d\})=\{a,c,d\}$ is b-I-open but not pre-I-open (resp. α-I- open) in (X,τ_2,I).

Theorem 4.10: If a function $f:(X,\tau)\to(Y,\sigma,J)$ is contra b-I-open and X is regular, then f is b-I-open function.

Proof: Let $y\in Y$ and V be an open subset of X with $f^{-1}(y)\in V$. Since X is regular, there exists an open set W in X such that $f^{-1}(y)\in W\subseteq cl(W)\subseteq V$. Since f is contra b-I-open, by theorem 4.4, there exists a b-J-open set U in (Y,σ,J) with $y\in U$ such that $f^{-1}(U)\subseteq cl(W)$. Then $f^{-1}(U)\subseteq cl(W)\subseteq V$. This shows that f is b-I-open function.

Definition 4.11: An ideal space (X,τ,I) is said to be locally β-I- indiscrete if every b-I-open set of X is closed in X.

Theorem 4.12: If a function $f:(X,\tau)\to(Y,\sigma,J)$ is contra b-I-open and (Y,σ,J) is locally β-I- indiscrete space, then f is open function.

Proof: It is straightforward.

Theorem 4.13: Let $f:(X,\tau)\to(Y,\sigma,J)$ be a function and let (Y,σ,J) be b-I-space, then the following are equivalent:
1. f is contra b-I- open.

2. f is contra b - open.
3. f is contra pre- I - open.
4. f is contra pre- open.
5. f is contra semi- I - open.
6. f is contra semi- open.
7. f is contra α - I - open.
8. f is contra α - open.
9. f is contra -open.

Proof: It is straightforward.

Theorem 4.14: If a function $f:(X,\tau)\to(Y,\sigma,J)$ is contra b-I-open surjective function of an Urysohn space X onto a space (Y,σ,J), then (Y,σ,J) is b-I-T_2.

Proof: Let y_1 and y_2 be a pair of distinct points in Y. Then $x_1 \neq x_2$ and $y_i = f(x_i)$, for $i = 1,2$. Since X is a Urysohn space, there exist open sets U and V of X such that $x_1 \in U$, $x_2 \in V$ and $cl(U)\cap cl(V)=\phi$. Since f is contra b-I-open at y_1 and y_2, there exist b-I-open sets $f(A)$ and $f(B)$ in Y such that $y_1 \in f(A)$, $y_2 \in f(B)$ and $A \subseteq cl(U)$, $B \subseteq cl(V)$. Then $A\cap B=\phi$, so $f(A)\cap f(B)=\phi$. Hence, Y is b-I-T_2.

Theorem 4.15: Let function $f:(X,\tau)\to(Y,\sigma,J)$ and $g:(Y,\sigma,J)\to(Z,\rho,K)$ be two functions such that $g\circ f:(X,\tau)\to(Z,\rho,K)$. Then:

1. $g\circ f$ is contra b - I - open, if f is open and g is a contra b - I - open.
2. f is contra b - I - open, if $g\circ f$ is open and g is contra b - I - continuous injection.

Proof: It is obvious.

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في الدوال المستمرة المضادة – $I-b$ و المفتوحة المضادة – $I-b$

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الملخص

في هذا البحث، نقدم مفهوم الاستمرارية المضادة – $I-b$ و المفتوحة المضادة – $I-b$ في الفضاءات التبولوجية المثالية، وتم التوصل الى عدة مميزات وخصائص لهاتين الدالتين. وأيضا تم التحري عن علاقتهما مع الدوال الاخرى.