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# Quasi–essentially pseudo – prime modules

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## Abstract
Let R be a commutative ring with identity and M be a unitary R-module. In this paper we introduce the concept Quasi-essentially pseudo-prime module as a generalization of a prime module and give of examples, characterizations and some basic properties of this concept. Further more we study the relationships of Quasi-essentially pseudo-prime modules with some classes of modules .

## 1-Introduction
Let R be a commutative ring with identity and M be a unitary R-module. An R-module M is called a prime if $ann_R$ M=$ann_R$ N for every non-zero submodule N of M [8].A proper submodule N of an R-module M is called Quasi-essential submodule in M if N⋂Q≠(0) for each non-zero quasi-prime submodule Q of M [6] where a proper submodule Q of an R-module M is called a quasi-prime if $r_1r_2m$∈Q,m∈M ,$r_1$,$r_2$∈R then either $r_1m$∈Q or $r_2m$∈Q[2]. And a proper submodule N of M is called a primary if r∈R , m∈M and rm∈N then m∈N , or $r^n$∈ [N:M] for some n∈Z $_+$ , where [N:M]={r∈R:rM⊆N}. And M is called primary module if (0) is a primary submodule of M [7]. In this paper we introduce and study first the concept quasi-essentially pseudo-prime submodule because we needed latter and give characterization and some basic properties of it. In the last section of this paper we introduce and study the concept quasi-essentially pseudo-prime module and give examples , characterizations and some properties of it. On the other hand we study the relation of this concept with prime modules, primary modules and quasi-Dedekind modules .

## 2- Quasi-essentially pseudo-prime sub-modules
In this section, we introduce the concept of a Quasi-essentially pseudo-prime submodule as a generalization of primary submodule .

**Definition (2.1):**
A proper submodule N of an R-module M is said to be quasi-essentially pseudo-prime submodule If [N:K] is a primary ideal of R for each a quasi-essential submodule K of M such that N ⊂ K .

Specially an ideal I is a quasi-essentially pseudo-prime ideal of R if and only if I is a quasi-essentially pseudo-prime R-submodule of R .

**Proposition (2.2):**
A proper submodule N of an R-module M is a quasi-essentially pseudo-prime submodule of M if and only if $\sqrt{[N:K]}=\sqrt{[N:rK]}$ for r∈R and for each quasi-essential submodule K of M such that N ⊂ K , rK⊈N .

**Proof:**
It is clear that $\sqrt{[N:K]}\subseteq\sqrt{[N:rK]}$

Now, Let a$\in\sqrt{[N:rK]}$ for each rK⊈ N, where K is a quasi-essential submodule of M .
Hence $a^n r\in[N:K]$ for some n⊂Z$_+$ , then $a^n$rK⊆N, thus $a^n$r ∈[N: K] .
But [N: K] is a primary ideal of R and r∉[N:K]
So, $(a^n)^m$ ∈ [N: K] for some m∈Z$_+$ . Hence a $\in\sqrt{[N:K]}$ there fore $\sqrt{[N:rK]}\subseteq\sqrt{[N:K]}$ . Hence $\sqrt{[N:K]}=\sqrt{[N:rK]}$ .
Conversely, to prove that [N: K] is a primary ideal of R for each a quasi-essential submodule K of M such that N⊂K.
Let ar∈[N: K] and suppose that r∉ [N: K] . Then a ∈[N: K] but [N: K] ⊆ [N: rK] $=\sqrt{[N:K]}$ . Thus a$\in\sqrt{[N:K]}$ and hence N is a quasi-essential pseudo-prime submodule of M .
Recall that an R-module M is multiplication if every submodule N of M is of the form N=IM for some ideal I of R [3]. The following proposition shows that the concepts of a quasi-essentially pseudo-prime submodule and primary submodules are equivalent in the class of multiplication modules .

**Proposition (2.3):**
Let M be a multiplication R-module, and N be a proper submodule of M, then the following statement are equivalent:-
1- N is a quasi-essentially pseudo-prime submodule of M .
2- [N: M] is a primary ideal of R .
3- N is a primary submodule of M .

**Proof:**
(1) →(2): by definition (2-1)

(2) →(3): Since M is multiplication, then by [4, remark (2-15)], we have [K: M] ⊈ [N: M] for each submodule K of M such that N⊂K .
Now, to prove that N is a primary sub-module of M
Let r∈R, and x∈M such that rx∈N and suppose that x∉N. It is clear that the submodule N⊂N+(x) =K and so [K: M] ⊈ [N: M] then there exists s∈[K:M] and s∉[N:M] . Thus sM⊆K, but sM⊈N. But sM⊆K, implies that rsM⊆rK=r(N+(x))⊆N and rs∈[N:M]. since[N:M] is a primary ideal and s∉[N:M],

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$r^n \in [N:M]$ for some $n \in Z_+$ . Therefore N is a primary submodule of M .

(3) →(1): trivial

**Proposition (2.4):**

Let M be an R-module, N is a submodule of M then N is a quasi-essentially pseudo-prime submodule of M if and only if [N:I] is a quasi-essentially pseudo-prime submodule of M for every ideal I of R .

**Proof:**

To prove that [N:I] is a quasi-essentially pseudo-prime submodule of M. We must prove that [[N:I]:K] is a primary ideal of R for each a quasi-essentially submodule K of M such that $[N:I] \subset K$ .

Note that $N \subseteq [N: I] \subset K$. Now , Let $a,b \in R$ such that $ab \in [[N:I]:K]$ and suppose that $a \notin [[N:I]:K]$ then $abKI \subseteq N$ and $aKI \nsubseteq N$. Hence $abI \subseteq [N: K]$. But [N:K] is a primary ideal of R and $aI \nsubseteq [N:K]$ then $b^n \in [N:K]$ for some $n \in Z_+$ , hence $b^nK \subseteq N \subseteq [N:I]$ . Therefore $b^n \in [[N: I]: K]$

Hence [N: I] is a quasi-essentially pseudo-prime submodule of M.

The converse follows by taking I=R .

We end section by studying the be havior of a quasi-essentially pseudo-prime submodules under homomorphic image and inverse image .

**Proposition (2.5):**

Let M and $M'$ be two R-modules and let $\Psi: M \to M'$ be an epimorphism, if N is a quasi-essentially pseudo-prime submodule of $M'$ then $\Psi^{-1}(N)$ is a quasi-essentially pseudo-prime submodule of M .

**Proof:**

To prove $\Psi^{-1}(N)$ is a quasi-essentially submodule of M, we must prove that $[\Psi^{-1}[N]: K]$ is a primary ideal of R, for each a quasi-essential submodule K of M such that $\Psi^{-1}(N) \subset K$ . Let $a,b \in R$ such that $ab \in [\Psi^{-1}(N) : K]$ and so $abK \subseteq \Psi^{-1}(N)$. Hence $\Psi(abK) \subseteq \Psi(\Psi^{-1}(N))$. Thus $ab\Psi(K) \subseteq \Psi(\Psi^{-1}(N))=N$ because $\Psi$ is an epimorphism.Thus $ab \in [N:\Psi(K)]$. Since K is a quasi-essential in M, then by [6, prop.(2.3) (1)] $\Psi(K)$ is a quasi-essential in $M'$. But $\Psi^{-1}(N) \subset K$, so $N=\Psi(\Psi^{-1}(N)) \subset \Psi(K)$ and hence $[N: \Psi(K)]$ is a primary ideal . Hence either $a \in [N:\Psi(K)]$ or $b^n \in [N:\Psi(K)]$ for some $n \in Z_+$ .And so either $a\Psi(K) \subseteq N$ or $b^n\Psi(K) \subseteq N$ Therefore, either $\Psi(aK) \subseteq N$ or $\Psi(b^nK) \subseteq N$. That is either $aK \subseteq \Psi^{-1}(N)$ or $b^nK \subseteq \Psi^{-1}(N)$. Thus $a \in [\Psi^{-1}(N): K]$ or $b^n \in [\Psi^{-1}(N): K]$.

Therefore $[\Psi^{-1}(N): K]$ is a primary ideal for each a quasi-essential submodule K of M such that $\Psi^{-1}(N) \subset K$ . Hence $\Psi^{-1}(N)$ is a quasi-essentially pseudo-prime submodule of M .

**Proposition(2.6):**

Let M, $M'$ be two R-modules, and let $\Psi: M \to M'$ be an epimorphism. If N is a quasi-essentially pseudo-prime submodule of M such that $Ker\Psi \subseteq N$, then $\Psi(N)$ is a quasi-essentially pseudo-prime submodule of $M'$ .

**Proof:**

To prove that $\Psi(N)$ is a quasi-essentially pseudo-prime submodule of $M'$ , we must prove that $[\Psi(N): K']$ is a primary ideal of R for each a quasi-essentially submodule $K'$ of $M'$ such that $\Psi(N) \subset K'$ .

Since $\Psi$ is an epimorphism, then $K'=\Psi(\Psi^{-1}(K'))$ . Let $K=\Psi^{-1}(K')$ which is a quasi-essential submodule of M by [6, prop.(2.3)(2)].Thus $\Psi(K)=K'$ It follows that $\Psi(N) \subset \Psi(K)$ .

To prove that $[\Psi(N) = \Psi(K)]$ is a primary ideal of R. Let $a, b \in R$ such that $ab \in [\Psi(N) = \Psi(K)]$, so $ab\Psi(K) \subseteq \Psi(N)$. Hence for each $x \in K$ , $ab\Psi(x) \subseteq \Psi(N)$ , so that $\Psi(abx)=\Psi(n)$ for some $n \in N$ , implies that $abx-n \in Ker\Psi \subseteq N$, and so $abx \in N$ for each $x \in K$ .Hence $ab \in [N:K]$. But [N: K] is a primary ideal of R, because N is a quasi-essentially pseudo-prime submodule of M . so either $a \in [N: K]$ or $b^n \in [N: K]$ for some $n \in Z_+$ Thus either $aK \subseteq N$ or $b^nK \subseteq N$ and so either $a\Psi(K) \subseteq \Psi(N)$ or $b^n\Psi(K) \subseteq \Psi(N)$.

Therefore, either $a \in [\Psi(N):\Psi(K)]$ or $b^n \in [\Psi(N):\Psi(K)]$. Thus $[\Psi(N):\Psi(K)]$ is a primary ideal of R and $\Psi(N)$ is a quasi-essentially pseudo-prime submodule of $M'$ .

## 3-Quasi-essentially pseudo-prime modules

In this section, we introduce the definition of quasi-essentially pseudo-prime module as a generalization of prime module .

**Definition (3.1):**

An R-module M is said to be a quasi-essentially pseudo-prime module if $ann_R N$ is a primary ideal of R for each non-zero a quasi-essential submodule N of M .

Specially a ring R is called a quasi-essentially pseudo-prime ring if and only if R is a quasi-essentially pseudo-prime R-module .

**Remarks And Examples (3.2):**

1- Every prime R-module is quasi-essentially pseudo-prime R-module, but the converse is not true, as the following example shows:

The Z-module $Z_4$ is a quasi-essentially pseudo-prime Z-module because $Z_4$ is a quasi-essential in $Z_4$ and <2> is a quasi-essential in Z4, and $ann_z Z_4=4Z$ is primary ideal of Z, and $ann_z(2)=2Z$ is a prime ideal, hence it is a primary ideal of Z , but $Z_4$ is not prime because $ann_z Z_4 \neq ann_z(N)$ for each non-zero submodule N of $Z_4$ .

2 – Z as Z-module is quasi-essentially pseudo-prime Z-module .

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3 – The homomorphic image of a quasi-essentially pseudo-prime R-module is not a quasi-essentially pseudo-prime module, as the following example shows:

$\frac{Z}{6Z} \simeq Z_6$ is not a quasi-essentially pseudo-prime Z-module .

The following result gives a characterization for quasi-essentially pseudo-prime modules.

**Preposition (3.3):**

Let M be R-module. Then M is a quasi-essentially pseudo-prime module if and only if (0) is a quasi-essentially pseudo-prime submodule of M .

**Proof:**

Suppose that (0) is quasi-essentially pseudo-prime submodule of M, to prove that M is a quasi-essentially pseudo-prime module. Since (0) is a quasi-essentially pseudo-prime submodule, then [(0): K] is a primary ideal of R, for each a quasi-essential submodule of M, such that (0)⊆K . But [(0):K] = $ann_R K$, hence M is a quasi-essentially pseudo-prime module .

Conversely, suppose that M is a quasi-essentially pseudo-prime module , to prove that (0) is a quasi-essentially pseudo-prime submodule of M . Since M is a quasi-essentially pseudo-prime module , then $ann_R$ K =[(0):K] is a primary ideal of R for each a quasi-essential submodule K of M. Hence (0) is a quasi-essentially pseudo-prime submodule of M .

The following corollaries are direct consequence of proposition (3.3).

**Corollary (3.4):**

Let N be a proper submodule of an R-module M, then N is a quasi-essentially pseudo-prime submodule ,if and only if, $\frac{M}{N}$ is a quasi-essentially pseudo-prime module .

**Corollary (3.5):**

An R-module M is a quasi-essentially pseudo-prime module if and only if $ann_M I$ is a quasi-essentially pseudo-prime submodule of M, for each ideal I of R .

**Corollary (3.6):**

Let M be an R-module. Then M is a quasi-essentially pseudo-prime module, if and only if, $\sqrt{ann_R K} = \sqrt{ann_R rK}$, for each a quasi-essential submodule K of M such that rK≠(0) , r∈R .

**Proof :**

By proposition (3.3) and proposition (2.2) .

Recall that an R-module M is uniform if every non-zero submodule of M is an essential in M [4] . Since every essential submodule is a quasi-essential [6] then we have the following result.

**Proposition (3.7):**

Let M be a finitely generated uniform R-module. Then M is a primary module if and only if M is a quasi-essentially pseudo-prime module.

**Proof:**

The if part, direct.

Conversely, suppose that M is a quasi-essentially pseudo-prime R-module. To prove that M is a primary module, we must prove that (0) is a primary submodule of M . Let rx=0 for r∈R and x∈M and x≠0. Since M is uniform then every submodule of M is a quasi-essential in M. That is (x) ∩ (y) ≠ (0) for any y ∈ M, y ≠ 0 and so there exists a non-zero elements a,b∈ R such that ax = by ≠ 0 . But rx=0, so rax=0. It follows that rax=rby=0 , and so r∈$ann_R$ (by). Hence r∈ $\sqrt{ann_R(by)} = \sqrt{ann_R(y)}$ .On the other hand, since M is finitely generated module, then M = $\sum_{i=1}^{n} Rxi$ for some x1,x2,…,xn∈M. But $ann_R = \bigcap_{i=1}^{n} ann_R(xi)$ so $\sqrt{ann_R M} = \sqrt{\bigcap_{i=1}^{n} ann_R(xi)} = \bigcap \sqrt{ann_R(xi)}$.But r∈$\bigcap_{i=1}^{n} \sqrt{ann_R(xi)}$ so r∈ $\sqrt{ann_R M}$. Thus (0) is a primary submodule of M . Hence M is a primary module .

In the following proposition, we show that the to concepts prime module and a quasi-essentially pseudo-prime module are equivalent:

**Proposition (3.8):**

If M is a uniform R-module, with $ann_R N$ is semi-prime ideal of R for each non-zero submodule N of M then M is a prime, if and only if, M is a quasi-essentially pseudo-prime module.

**Proof:**

The If part is direct .

Conversely, To prove that M is a prime module. We must prove (0) is a prime submodule of M. Let rx=0 for r∈R , x∈M ,x≠0. Since M is a uniform every submodule of M is essential , hence every submodule of M is a quasi-essential in M. Thus (x)∩(y)≠(0) for any y∈M, y≠0 and so there exists a non-zero elements a,b∈R such that ax=ay≠0. But rx=0, so rax=0. Thus, rax=rby=0 and so r∈$ann_R$ (by). Hence by corollary (3.6) r∈ $\sqrt{ann_R(by)} = \sqrt{ann_R(y)}$. Since $ann_R N$ is a semi-prime ideal of R, so r∈$ann_R$(y). Hence r(y) =0 for any y∈M . Thus r∈$ann_R M$. That is (0) is a prime submodule. Therefore M is a prime module.

Recall that an R-module M is a quasi-Dedekind if Hom $(\frac{M}{N}, M)$= (0) for each non-zero submodule N of M [7] .

**Proposition (3.9):**

Let M be a uniform R-module and $ann_R N$ is a semi-prime ideal of R for each non-zero submodule N of M then the following statements are equivalents:

1 – M is a quasi-essentially pseudo-prime module .
2 – M is a prime module .
3 – M is a quasi-Dedekind module .

**Proof:**

(1) $\rightarrow$(2): by proposition (3.8) .

(2) $\rightarrow$(3): by [5, theo.(3.11)] .

(3) $\rightarrow$(1) by [5, theo.(1.7)] and remarks and examples (3.2)(1) .

We end this section by the following results:

**Proposition (3.10):**
If M is a multiplication a quasi-essentially pseudo-Prime R-module then M is a finitely generated module.

**Proof :**
Since M is a quasi-essentially pseudo-prime module, then $ann_R M$ is a primary ideal of R .
Then by [1, prop.(2.7)] M is finitely generated module .

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# المقاسات الاولية الكاذبة جوهرياً ظاهرياً

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**الملخص**

لتكن R حلقة ابدالية بمحايد و M مقاساً احادياً على R . في هذا البحث قدمنا مفهوم المقاس الاولي الكاذب جوهرياً ظاهرياً كأعمام للمقاس الاولي واعطينا العديد من الامثلة والتشخيصات و بعض الخواص الاساسية لهذا المفهوم. اضافة لهذا درسنا العلاقة بين المقاسات الاولية الكاذبة جوهرياً ظاهرياً مع بعض اصناف اخرى من المقاسات .