CERTAIN SUBCLASSES OF MEROMORPHICALLY P-VALENT FUNCTIONS WITH POSITVE OR NEGATIVE COEFICIENTS USING DIFFERENTIAL OPERATOR

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Abstract

In this paper, we have introduced two subclasses $\mathcal{A}R(\lambda,k,p,\zeta,\delta)$ and $\mathcal{A}C(\lambda,k,p,\zeta,\delta)$ of meromorphically p-valent functions with positive and negative coefficients, defined by differential operator in the punctured unit disk $\mathcal{U}^* = \{z: z \in \mathbb{C};\ 0 < |z| < 1\} = \mathcal{U}\backslash\{0\}$ and obtain some sharp results including coefficient inequality, distortion theorem, radii of starlikeness and convexity, closure theorems of these subclasses of meromorphically p-valent functions. We also derive some interesting results for the Hadamard products of functions belonging to the classes $\mathcal{A}R(\lambda,k,p,\zeta,\delta)$ and $\mathcal{A}C(\lambda,k,p,\zeta,\delta)$.

Keywords: Meromorphic Functions, p-valent ,Differential Operator, Hadamard Product (Convolution).

1. Introduction

Let $\Sigma_p^{(A,B)}$ denote the class of functions of the form

$$f(z) = \frac{A}{z^p} + B\sum_{n=0}^{\infty} a_n z^n, \quad (AB \neq 0; p \in \mathbb{N} = \{1,2,\dots\}) \qquad (1.1)$$

Which are analytic and p-valent in the punctured disc

$$\mathcal{U}^* = \{z: z \in \mathbb{C};\ 0 < |z| < 1\} = \mathcal{U}\backslash\{0\}.$$

For the function f(z) in the class $\Sigma_p^{(A,B)}$ given by (1.1).We define the following differential operator

$$I_\lambda^0 f(z) = f(z),$$

$$I_\lambda^1 f(z) = z^p f'(z) + \frac{2}{z^p},$$

$$I_\lambda^2 f(z) = z^p \left(I_\lambda^1 f(z)\right)' + \frac{2}{z^p},$$

and, for k= 0, 1, 2, 3, …, we can write

$$I_\lambda^k f(z) = z^p \left(I_\lambda^{k-1} f(z)\right)' + \frac{2}{z^p},$$

$$= A\frac{1}{z^p} + B\sum_{n=p}^{\infty}[1 + \lambda(n-p)]^k a_n z^n, \qquad (1.2)$$

Where $\lambda \geq 1, k \geq 0$ and $z \in U$.

The differential operator I_λ^k was considered, when A= B= p= 1, by Ghanim and Darus [8,9] and Ghanim et al. [10].

Also let $\Sigma^{(A,B)}(p)$ denote the class of functions of the form

$$f(z) = \frac{A}{z^p} + B\sum_{n=p}^{\infty} a_n z^n, \quad (a_n \geq 0;\ AB \neq 0; p \in \mathbb{N} = \{1,2,\dots\}) \ (1.3)$$

By applying the definition of the above differential operator, we introduce here a new class $\mathcal{A}\Sigma^{(A,B)}(\lambda, k,p,\delta;\zeta,\sigma)$of meromorphically functions, which is defined as follows:

Definition 1.1. A function $f \in \Sigma^{(A,B)}(p)$ of the form (1.2) is said to be in the class $\mathcal{A}\Sigma^{(A,B)}(\lambda,k,p,\delta;\zeta,\sigma)$ if it satisfies the following inequality:

$$Re\left\{\frac{(1-2\zeta)z\left(I_\lambda^k f(z)\right)' - \zeta z^2\left(I_\lambda^k f(z)\right)''}{(\sigma-1)I_\lambda^k f(z) + \sigma z\left(I_\lambda^k f(z)\right)'}\right\} > \delta,$$

$$(z \in \mathcal{U}^*; 0 \leq \delta < p; n \in \mathbb{N}_0; p \in \mathbb{N}) \quad (1.4)$$

For some restricted real parameters ζ, σ.

We note that

(i) $\mathcal{A}\Sigma^{(A,B)}(\lambda,0,1,\delta;\zeta,\sigma) = M_1^{(A,B)}(\delta;\zeta,\sigma)$[3]

$$\left\{\begin{array}{c} f(z) = Az^{-1} \\ +B\sum_{n=1}^{\infty} a_k z^k \in \Sigma^{(A,B)}(1): Re\left\{\frac{(1-2\zeta)zf'(z) - \zeta z^2 f''(z)}{(\sigma-1)f(z) + \sigma z f'(z)}\right\} > \delta, \quad z \in \mathcal{U}^* \end{array}\right\};$$

$$0 \leq \delta < 1. \quad (1.5)$$

For some restricted parameters ζ and σ.

(ii) $\mathcal{A}\Sigma^{(A,B)}(\lambda,0,p,\delta;0,0) = \Sigma_*^{(A,B)}(p,\delta)$

$$= \left\{f(z) = Az^{-1} + B\sum_{n=1}^{\infty} a_k z^k \in \Sigma^{(A,B)}(p): Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \delta, z \in \mathcal{U}^*\right\}; 0 \leq \delta < p. \quad (1.6)$$

Where $\Sigma_*^{(A,B)}(p,\delta)$ is the class of meromorphically p-valent starlike functions of order δ $(0 \leq \delta < p)$ (with positive or negative coefficients depending upon the value of the nonzero constant B);

(iii) $\mathcal{A}\Sigma^{(A,B)}(\lambda,0,p,\delta;1,1) = \Sigma_k^{(A,B)}(p,\delta)$

$$= \left\{f(z) = Az^{-1} + B\sum_{n=1}^{\infty} a_k z^k \in \Sigma^{(A,B)}(p): Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \delta, z \in \mathcal{U}^*\right\}; 0 \leq \delta < p. \quad (1.7)$$

Where $\Sigma_k^{(A,B)}(p,\delta)$ is the class of meromorphically p-valent convex functions of order δ $(0 \leq \delta < p)$ (with positive or negative coefficients depending upon the value of the nonzero constant B) (see Duren[6], Goodman [7] and Srivastava and Owa [18]);

(iv) $\mathcal{A}\Sigma^{(1,-1)}(\lambda,0,1,\delta;1,1) = \Sigma_k(\delta)$, where $\Sigma_k(\delta)$ is the class of meromorphically convex functions of order δ $(0 \leq \delta < p)$ with negative coefficients (see Uralegaddi and Ganigi [19] and Srivastava et.al. [17]);

Some other subclasses of the class $\Sigma^{(A,B)}(p)$were studied (for example) by Cho et al. [4, 5], Altintas et al. [1], Liu [12], Liu and Srivastava [13, 14], Joshi et al, [11], Raina and Srivastava [15] and Aouf and Shammaky [2].

The aim of this paper is to proving a systematic investigation of the various interesting properties and characteristics of functions belonging to the following subclasses $\mathcal{A}R(\lambda,k,p,\zeta,\delta)$ and $\mathcal{A}C(\lambda,k,p,\zeta,\delta)$ of the general class $\mathcal{A}\Sigma^{(A,B)}(\lambda,k,p,\delta;\zeta,\sigma)$ which we introduced above:

$$\mathcal{A}R(\lambda,k,p,\zeta,\delta) = \mathcal{A}\Sigma^{(-1,1)}((\lambda,k,p,\delta;\zeta,\zeta)\left(0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right), \quad (1.8)$$

and

$$\mathcal{A}C(\lambda,k,p,\zeta,\delta) = \mathcal{A}\Sigma^{(-1,1)}((\lambda,k,p,\delta;1,1-\zeta)\left(0 \le \delta < p; 0 \le \zeta \le \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (1.9)$$

We also derive many interesting results for the Hadamard products of functions belonging to the classes $\mathcal{A}R(\lambda,k,p,\zeta,\delta)$and $\mathcal{A}C(\lambda,k,p,\zeta,\delta)$.

Clearly the class $\mathcal{A}R(\lambda,0,p,1,\delta)$ consisting of meromorphically p-valent convex functions f(z) of order δ $(0 \le \delta \le p)$ with positive coefficients, given by

$$f(z) = \frac{-1}{z^p} + B\sum_{n=p}^{\infty} a_n z^n, \quad (a_n \ge 0; p \in \mathbb{N} = \{1,2,\ldots\}). \quad (1.10)$$

Furthermore, since the condition $\zeta \ge \frac{1}{p+1}; (p \in \mathbb{N})$ is not actually a requirement for the definition (1.4), we may set $k = \zeta = 0$ in (1.4) and observe that the class $\mathcal{A}R(\lambda,0,p,0,\delta)$ consisting of meromorphically p-valent starike functions $f(z)$of order δ $(0 \le \delta < p)$ with positive coefficients given by (1.10). On the other hand, the class$\mathcal{A}R(\lambda,0,p,0,\delta)$ consisting of meromorphically p-valent convex functions of order δ $(0 \le \delta < p)$ with negative coefficients (see [2]), given by

$$f(z) = \frac{-1}{z^p} - B\sum_{n=p}^{\infty} a_n z^n, \quad (a_n \ge 0; p \in \mathbb{N} = \{1,2,\ldots\}). \quad (1.11)$$

2. Coefficient Inequalities and Inclusion Properties

We first determine a necessary and sufficient condition for a function $f(z)$, given by (1.10), to be in the class $\mathcal{A}R(\lambda,k,p,\zeta,\delta)$.

Theorem 2.1. Let $f(z) \in \Sigma^{(-1,1)}(p)$ be given by (1.10). Then $f(z) \in \mathcal{A}R(\lambda,k,p,\zeta,\delta)$ if and only if

$$\sum_{n=p}^{\infty}(n+\delta)[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n \le [1+\zeta(p-1)](p-\delta), \quad \left(0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (2.1)$$

Proof. Let $f(z) \in \mathcal{A}R(\lambda,k,p,\zeta,\delta)$ be given by (1.10). Then from (1.10), (1.8) and (1.4), we have

$$Re\left\{\frac{p[1+\zeta(p-1)] - \sum_{n=p}^{\infty} n[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n z^{n+p}}{[1+\zeta(p-1)] + \sum_{n=p}^{\infty} n[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n z^{n+p}}\right\} > \delta,$$

$$\left(0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}, z \in \mathcal{U}^*\right) \quad (2.2)$$

Choosing z to be real and letting $z \to 1^-$through real values, (2.2) yields

$$\frac{p[1+\zeta(p-1)] - \sum_{n=p}^{\infty} n[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n z^{n+p}}{[1+\zeta(p-1)] + \sum_{n=p}^{\infty} n[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n z^{n+p}} \ge \delta$$

$$\left(0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (2.3)$$

Which is equivalent of our condition of the theorem.

Conversely, we assume that the inequality (2.1) holds true. We must show that $f(z) \in \mathcal{A}R(\lambda,k,p,\zeta,\delta)$. Then, if we let $z \in \partial\mathcal{U} = \{z : z \in \mathbb{C}, |z| = 1\}$, we find from (1.10) and (2.1) that

$$\left|\frac{(1-2\zeta)z\left(I_\lambda^k f(z)\right)' - \zeta z^2\left(I_\lambda^k f(z)\right)''}{(\sigma-1)I_\lambda^k f(z) + \sigma z\left(I_\lambda^k f(z)\right)'} - p\right|$$

$$\le \frac{\sum_{n=p}^{\infty}(n+p)[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n |z|^{n+p}}{[1+\zeta(p-1)] + \sum_{n=p}^{\infty}[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n |z|^{n+p}}$$

$$= \frac{\sum_{n=p}^{\infty}(n+p)[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n}{[1+\zeta(p-1)] + \sum_{n=p}^{\infty}[\zeta(n+1)-1][1+\lambda(n-p)]^k a_n}$$

$$\le (p-\delta); \left(0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (2.4)$$

Hence, by maximum modulus theorem, we have $f(z) \in \mathcal{A}R(\lambda,k,p,\zeta,\delta)$. This completes the proof of Theorem 1.1.

Corollary 2.1. Let f defined by (1.10) be in the class $R(\lambda,k,p,\zeta,\delta)$. Then

$$a_n \le \frac{[1+\zeta(p-1)](p-\delta)}{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-p)]^k}, \quad \left(n \ge p; 0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (2.5)$$

The equality in (2.5) is attained for the function $f(z)$ given by

$$f(z) = \frac{-1}{z^p} - \frac{[1+\zeta(p-1)](p-\delta)}{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-p)]^k} z^n; \quad \left(n \ge p; 0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (2.6)$$

Using argument similar to those in the proof of Theorem 2.1, we can prove the following theorem:

Theorem 2.2. Let $f(z) \in \Sigma^{(-1,1)}(p)$ be given by (1.11). Then $f(z) \in \mathcal{A}C(\lambda,k,p,\zeta,\delta)$ if and only if

$$\sum_{n=p}^{\infty}[n(n-\zeta\delta)+\delta(n-\zeta)][1+\lambda(n-p)]^k a_n \le [p(p-\delta)+\zeta\delta(p-1)], \quad \left(0 \le \delta < p; 0 \le \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (2.7)$$

Corollary 2.2. Let f defined by (1.11) be in the class $C(\lambda,k,p,\zeta,\delta)$. Then

$$a_n \le \frac{[p(p-\delta)+\zeta\delta(p-1)]}{[n(n-\zeta\delta)+\delta(n-\zeta)][1+\lambda(n-p)]^k}, \quad \left(n \ge p; 0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (2.8)$$

The equality in (2.5) is attained for the function $f(z)$ given by

$$f(z) = \frac{-1}{z^p} - \frac{[p(p-\delta)+\zeta\delta(p-1)]}{[n(n-\zeta\delta)+\delta(n-\zeta)][1+\lambda(n-p)]^k} z^n; \quad \left(n \ge p; 0 \le \delta < p; \zeta \ge \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \quad (2.9)$$

Theorem 2.3. The class $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$ is closed under linear combination.

Proof. Let $f_j \in \mathcal{AR}(\lambda,k,p,\zeta,\delta)$, where

$$f_j(z) = \frac{-1}{z^p} + \sum_{n=p}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \geq 0; j = 1,2;\ p \in \mathbb{N} = \{1,2,\dots\}). \qquad (2.10)$$

Then it is sufficient to show that the function $w(z)$ defined by

$$w(z) = tf_1(z) + (1-t)f_2(z), \qquad (0 \leq t \leq 1)$$

is also in the class $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$. Since for $0 \leq t \leq 1$, we get

$$f_j(z) = \frac{-1}{z^p} + \sum_{n=p}^{\infty} (ta_{n,1} + (1-t)a_{n,2}) z^n,$$

we observe that

$$\sum_{n=p}^{\infty} (n+\delta)[\zeta(n+1)-1][1+\lambda(n-p)]^k (ta_{n,1} + (1-t)a_{n,2})$$

$$= t\sum_{n=p}^{\infty} (n+\delta)[\zeta(n+1)-1][1+\lambda(n-p)]^k a_{n,1} + (1-t)\sum_{n=p}^{\infty} (n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k a_{n,2}$$

$$\leq [1+\zeta(p-1)](p-\delta), \left(0 \leq \delta < p; \zeta \geq \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}, 0 \leq t \leq 1\right). \qquad (2.11)$$

By Thoerem 1.1, $w \in \mathcal{AR}(\lambda,k,p,\zeta,\delta)$.

Similarly, let $f_j \in \mathcal{AC}(\lambda,k,p,\zeta,\delta)$, where

$$f_j(z) = \frac{-1}{z^p} - \sum_{n=p}^{\infty} a_{n,j} z^n, \qquad (a_{n,j} \geq 0; j = 1,2;\ p \in \mathbb{N} = \{1,2,\dots\}). \qquad (2.12)$$

By using Theorem 2.2 and (2.12), we can prove:

Theorem 2.4. The class $\mathcal{AC}(\lambda,k,p,\zeta,\delta)$ is closed under linear combinations.

Proof. The proof is similar to that of Theorem 2.3 and hence is omitted.

3. Distortion Thoerems

We now state the following growth and distortion theorems for the classes $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$ and $\mathcal{AC}(\lambda,k,p,\zeta,\delta)$.

Theorem 3.1. Let the function $f(z)$ given by (1.10) be in the class $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$. Then

$$\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{[1+\zeta(p-1)](p-\delta)}{(p+\delta)[\zeta(p+1)-1]}\cdot\frac{p!}{(p-m)!}r^{2p}\right\}r^{-(p+m)} \leq |f^{(m)}(z)| \leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{[1+\zeta(p-1)](p-\delta)}{(p+\delta)[\zeta(p+1)-1]}\cdot\frac{p!}{(p-m)!}r^{2p}\right\}r^{-(p+m)}$$

$$\left(0 < |z| = r < 1;\ 0 \leq \delta < p; \zeta \geq \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}, p > m\right). \qquad (3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{-1}{z^p} - \frac{[1+\zeta(p-1)](p-\delta)}{(p+\delta)[\zeta(p+1)-1][1+\lambda(n-p)]^k} z^n; \ (k \in \mathbb{N}_0; p \in \mathbb{N}).$$

Proof: From Theorem 2.1, we have

$$\frac{(p+\delta)[\zeta(p+1)-1]}{p!}\sum_{n=p}^{\infty} n!\, a_n \leq \sum_{n=p}^{\infty}(n+\delta)(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k a_n \leq [1+\zeta(p-1)](p-\delta), \left(0 \leq \delta < p; \zeta \geq \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \qquad (3.2)$$

Which yields

$$\sum_{n=p}^{\infty} n!\, a_n \leq \frac{[1+\zeta(p-1)](p-\delta)p!}{(p+\delta)[\zeta(p+1)-1]}, \qquad \left(0 \leq \delta < p; \zeta \geq \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}\right). \qquad (3.3)$$

Now, by differentiating both sides of (1.10) m times with respect to z we get

$$f^{(m)}(z) = \frac{(p+m-1)!}{(p-1)!}\frac{(-1)^{m+1}}{z^{p+m}} + B\sum_{n=p}^{\infty}\frac{n!}{(n-m)!}a_n z^{n-m}, \qquad (a_n \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0; p > m), \qquad (3.4)$$

then Theorem 3.1 follows from (3.3) and (3.4).

Hence the proof is complete.

Finally, the result is sharp for the function $f(z)$ given by (3.2).

Theorem 3.2. Let the function $f(z)$ given by (1.11) be in the class $\mathcal{AC}(\lambda,k,p,\zeta,\delta)$. Then

$$\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{[p(p-\delta)+\zeta\delta(p-1)]}{[p(p-\zeta\delta)+\delta(p-\zeta)]}\cdot\frac{p!}{(p-m)!}r^{2p}\right\}r^{-(p+m)} \leq |f^{(m)}(z)| \leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{[p(p-\delta)+\zeta\delta(p-1)]}{[p(p-\zeta\delta)+\delta(p-\zeta)]}\cdot\frac{p!}{(p-m)!}r^{2p}\right\}r^{-(p+m)}$$

$$\left(0 < |z| = r < 1;\ 0 \leq \delta < p; \zeta \geq \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}, p > m\right). \qquad (3.5)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{-1}{z^p} - \frac{[p(p-\delta)+\zeta\delta(p-1)]}{[p(p-\zeta\delta)+\delta(p-\zeta)][1+\lambda(n-1)]^k} z^p; \quad (k \in \mathbb{N}_0; p \in \mathbb{N}). \qquad (3.6)$$

Now, putting $\zeta = 0$ in Theorem 3.2, we have

Corollary3.2. Let the function $f(z)$ given by (1.11) be in the class $\mathcal{AC}(\lambda,k,p,0,\delta) = \mathcal{AC}(\lambda,k,p,\delta)$. Then

$$\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(p-\delta)}{(p+\delta)}\cdot\frac{p!}{(p-m)!}r^{2p}\right\}r^{-(p+m)} \leq |f^{(m)}(z)| \leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(p-\delta)}{(p+\delta)}\cdot\frac{p!}{(p-m)!}r^{2p}\right\}r^{-(p+m)}$$

$$1\left(0 < |z| = r < 1;\ 0 \leq \delta < p; \zeta \geq \frac{1}{p+1}; k \in \mathbb{N}_0; p \in \mathbb{N}, p > m\right). \qquad (3.7)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=\frac{-1}{z^p}-\frac{(p-\delta)}{(p+\delta)[1+\lambda(n-1)]^k}z^p; \quad (k\in\mathbb{N}_0; p\in\mathbb{N}). \quad (3.8)$$

Remark 1. Putting (i) k= m= 0, (ii) k= 0 and m= 1 in Corollary 3.2, we get the result obtained by Aouf and Shammaky [2, Corollary 4] for the class $\sum_n(p,\delta)$.

4. Radii of Meromorphically p-valentStarlikeness and Convexity

In this section we determine the radii of meromorphically p-valent starlikeness of order $\eta(0\le\eta<p)$ and meromorphically p-valent convexity of order $\eta(0\le\eta<p)$ for functions in the class $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$and $\mathcal{AC}(\lambda,k,p,\zeta,\delta)$.

Theorem 4.1. Let the function$f(z)$ given by (1.10) be in the class $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$. Then

(i) $f(z)$ is meromorphically p-valent starlike of order $\eta(0\le\eta<p)$ in the disc $|z|<r_1$, that is,

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\delta \quad (|z|<r_1;\ 0\le\eta<p; p\in\mathbb{N}), \quad (4.1)$$

where

$$r_1=inf_{n\ge p}\left\{\frac{(p-\eta)(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{(n+\eta)[1+\zeta(p-1)](p-\delta)}\right\}^{\frac{1}{(n+p)}} \quad (4.2)$$

(ii) $f(z)$ is meromorphically p-valent convex of order $\eta(0\le\eta<p)$ in the disc $|z|<r_2$, that is,

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\delta \quad (|z|<r_2;\ 0\le\eta<p; p\in\mathbb{N}), \quad (4.3)$$

where

$$r_2=inf_{n\ge p}\left\{\frac{p(p-\eta)(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{n(n+\eta)[1+\zeta(p-1)](p-\delta)}\right\}^{\frac{1}{(n+p)}} \quad (4.4)$$

Each of these results is sharp for the function $f(z)$ given by (2.5).

Proof: (i) From the definition (1.10), we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\eta}\right|\le\frac{\sum_{n=p}^{\infty}(n+p)a_n|z|^{n+p}}{2(p-\eta)-\sum_{n=p}^{\infty}(n-p+2\eta)a_n|z|^{n+p}}. \quad (4.5)$$

Thus we have the desired inequality:

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\eta}\right|\le 1, \quad (0\le\eta<p; p\in\mathbb{N}), \quad (4.6)$$

if

$$\sum_{n=p}^{\infty}\frac{(n+\eta)}{(p-\eta)}a_n|z|^{n+p}\le 1. \quad (4.7)$$

Hence, by Theorem 1.1, inequality (4.7) will be true if

$$\frac{(n+\eta)}{(p-\eta)}|z|^{n+p}\le\frac{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{[1+\zeta(p-1)](p-\delta)}, \quad (n\ge p; p\in\mathbb{N};\ k\in\mathbb{N}_0) \quad (4.8)$$

This gives

$$|z|^{n+p}\le\frac{(p-\eta)(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{(n+\eta)[1+\zeta(p-1)](p-\delta)}, \quad (n\ge p; p\in\mathbb{N};\ k\in\mathbb{N}_0) \quad (4.9)$$

$$|z|\le\left\{\frac{(p-\eta)(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{(n+\eta)[1+\zeta(p-1)](p-\delta)}\right\}^{\frac{1}{n+p}}, \quad (n\ge p; p\in\mathbb{N};\ k\in\mathbb{N}_0) \quad (4.10)$$

If we take r_1 to be the infimum value of $|z|$ we get the result.

(ii) In order to prove the second assertion of Theorem 4.1, we have

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\eta}\right|\le\frac{\sum_{n=p}^{\infty}n(n+p)a_n|z|^{n+p}}{2(p-\eta)-\sum_{n=p}^{\infty}(n-p+2\eta)a_n|z|^{n+p}}. \quad (4.11)$$

Thus we have the desired inequality:

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\eta}\right|\le 1, \quad (0\le\eta<p; p\in\mathbb{N}), \quad (4.12)$$

if

$$\sum_{n=p}^{\infty}\frac{n(n+\eta)}{p(p-\eta)}a_n|z|^{n+p}\le 1. \quad (4.13)$$

Hence, by Theorem 1.1, inequality (4.13) will be true if

$$\frac{n(n+\eta)}{p(p-\eta)}|z|^{n+p}\le\frac{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{[1+\zeta(p-1)](p-\delta)}, \quad (n\ge p; p\in\mathbb{N};\ k\in\mathbb{N}_0) \quad (4.14)$$

This gives

$$|z|^{n+p}\le\frac{p(p-\eta)(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{n(n+\eta)[1+\zeta(p-1)](p-\delta)}, \quad (n\ge p; p\in\mathbb{N};\ k\in\mathbb{N}_0) \quad (4.15)$$

$$|z|\le\left\{\frac{p(p-\eta)(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{n(n+\eta)[1+\zeta(p-1)](p-\delta)}\right\}^{\frac{1}{n+p}}, \quad (n\ge p; p\in\mathbb{N};\ k\in\mathbb{N}_0) \quad (4.16)$$

If we take r_2 to be the infimum value of $|z|$ we get the result.

Each result is sharp for the function $f(z)$ given by (2.5)

Similarly, by using (2.7), we can prove

Theorem 4.2. Let the function$f(z)$ given by (1.11) be in the class $\mathcal{AC}(\lambda,k,p,\zeta,\delta)$. Then

(i) $f(z)$ is meromorphically p-valent starlike of order $\eta(0\le\eta<p)$ in the disc $|z|<r_3$, that is,

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\delta \quad (|z|<r_3;\ 0\le\eta<p; p\in\mathbb{N}), \quad (4.17)$$

where

$$r_3=inf_{n\ge p}\left\{\frac{(p-\eta)[n(n-\zeta\delta)+\delta(n-\zeta)][1+\lambda(n-1)]^k}{(n+\eta)[p(p-\delta)+\zeta\delta(p-1)]}\right\}^{\frac{1}{(n+p)}} \quad (4.18)$$

(ii) $f(z)$ is meromorphically p-valent convex of order $\eta(0\le\eta<p)$ in the disc $|z|<r_4$, where

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\delta \quad (|z|<r_2;\ 0\le\eta<p; p\in\mathbb{N}), \quad (4.19)$$

where

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$$r_4 = \inf_{n\geq p}\left\{\frac{p(p-\eta)[n(n-\zeta\delta)+\delta(n-\zeta)][1+\lambda(n-1)]^k}{n(n+\eta)[p(p-\delta)+\zeta\delta(p-1)]}\right\}^{\frac{1}{(n+p)}}. \quad (4.20)$$

Each of these results is sharp for the function $f(z)$ given by (2.9).

5. Convolution Properties

If

$$f_j(z) = -z^{-p} + \sum_{n=p}^{\infty} a_{n,j}z^n \left(a_{n,j} \geq 0; j = 1,2; p \in \mathbb{N}\right), \quad (5.1)$$

and

$$g_j(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,j}z^n \left(a_{n,j} \geq 0; j = 1,2; p \in \mathbb{N}\right), \quad (5.2)$$

We denote by $(f_1 \otimes f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$and $f_2(z)$, that is,

$$(f_1 \otimes f_2)(z) = -z^{-p} + \sum_{n=p}^{\infty} a_{n,1}a_{n,2}z^n, \quad (5.3)$$

Also we denote by $(g_1 \otimes g_2)(z)$ the Hadamard product (or convolution) of the functions $g_1(z)$and $g_2(z)$, that is,

$$(g_1 \otimes g_2)(z) = z^{-p} - \sum_{n=p}^{\infty} a_{n,1}a_{n,2}z^n. \quad (5.4)$$

Theorem 5.1. Let the function$f_j(z)(j = 1,2)$ given by (5.1) be in the class $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$. Then $(f_1 \otimes f_2)(z) \in \mathcal{AR}(\lambda,k,p,\zeta,\rho)$, where

The result is sharp for the functions $f_j(z)(j = 1,2)$ given by

$$f_j(z) = -z^{-p} + \frac{[1+\zeta(p-1)](p-\delta)}{(p+\delta)[\zeta(p+1)-1][1+\lambda(p-1)]^k}(j = 1,2; p \in \mathbb{N};\ k \in \mathbb{N}_0). \quad (5.6)$$

Proof: Employing the technique used earlier by Schild and Silverman [16], we need to find the largest ρ such that

$$\sum_{n=p}^{\infty}\frac{(n+\rho)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{[1+\zeta(p-1)](p-\rho)}a_{n,1}a_{n,1} \leq 1 \quad (5.7)$$

For $f_j(z) \in \mathcal{AR}(\lambda,k,p,\zeta,\delta)(j = 1,2)$. Since $f_j(z) \in \mathcal{AR}(\lambda,k,p,\zeta,\delta)(j = 1,2)$, we readily see that

$$\sum_{n=p}^{\infty}\frac{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{[1+\zeta(p-1)](p-\delta)}a_{n,j} \leq 1, \quad (j = 1,2). \quad (5.8)$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=p}^{\infty}\frac{(n+\delta)[\zeta(n+1)-1][(1+\lambda(n-1)]^k}{[1+\zeta(p-1)](p-\delta)}\sqrt{a_{n,1}a_{n,1}} \leq 1 \quad (5.9)$$

This implies that we only need to show that

$$\frac{\rho = p - \frac{(p+1)[1+\zeta(p-1)](p-\delta)^2}{[\zeta(p+1)-1](p+\delta)^2+[1+\zeta(p-1)](p-\delta)^2}. \quad (5.5)(n+\rho)}{(p-\rho)}a_{n,1}a_{n,1} \leq \frac{(n+\rho)}{(p-\delta)}\sqrt{a_{n,1}a_{n,1}}(n \geq p) \quad (5.10)$$

Or, equivalently, that

$$\sqrt{a_{n,1}a_{n,1}} \leq \frac{(p-\rho)(n+\rho)}{(n+\rho)(p-\delta)}(n \geq p). \quad (5.11)$$

Hence, by the inequality (5.9), it is sufficient to prove that

$$\frac{[1+\zeta(p-1)](p-\delta)}{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k} \leq \frac{(p-\rho)(n+\rho)}{(n+\rho)(p-\delta)}(n \geq p). \quad (5.12)$$

It follows from (5.12) that

$$\rho \leq p - \frac{(n+1)[1+\zeta(p-1)](p-\delta)^2}{[\zeta(n+1)-1][1+\lambda(n-1)]^k(n+\delta)^2+[1+\zeta(p-1)](p-\delta)^2}. \quad (n \geq p) \quad (5.13)$$

Now, defining the function $\chi(n)$ is an increasing function of n. Therefore, we conclude that$\rho \leq$

$$\chi(n) = p - \frac{(p+1)[1+\zeta(p-1)](p-\delta)^2}{[\zeta(p+1)-1](p+\delta)^2+[1+\zeta(p-1)](p-\delta)^2}, \quad (n \geq p) \quad (5.14)$$

which completes the proof of Theorem 5.1.

Using arguments similar to those in the proof of Theorem 5.1, we obtain the following result:

Theorem 5.2.

Let the function$f_1(z)$ given by (5.1) be in the class $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$. Suppose also that the function$f_2(z)$ given by (5.1) be in the class $\mathcal{AR}(\lambda,k,p,\zeta,\varrho)$. Then $(f_1 \otimes f_2)(z) \in \mathcal{AR}(\lambda,k,p,\zeta,\tau)$, where

$$\tau = p - \frac{(p+1)[1+\zeta(p-1)](p-\delta)(p-\varrho)}{[\zeta(p+1)-1](p-\delta)(p-\varrho)+[1+\zeta(p-1)](p-\delta)(p-\varrho)}. \quad (5.15)$$

The result is sharp for the functions $f_j(z)(j = 1,2)$ given

by$f_1(z) = -z^{-p} + \frac{[1+\zeta(p-1)](p-\delta)}{(p+\delta)[\zeta(p+1)-1][1+\lambda(p-1)]^k}(p \in \mathbb{N};\ k \in \mathbb{N}_0)$. (5.16)

And $f_2(z) = -z^{-p} + \frac{[1+\zeta(p-1)](p-\varrho)}{(p+\varrho)[\zeta(p+1)-1][1+\lambda(p-1)]^k}(p \in \mathbb{N};\ k \in \mathbb{N}_0)$. (5.17)

Theorem 5.3 Let the function$g_j(z)(j = 1,2)$ given by (5.2) be in the class $\mathcal{AC}(\lambda,k,p,\zeta,\delta)$. Then $(f_1 \otimes f_2)(z) \in \mathcal{AR}(\lambda,k,p,\zeta,\rho)$, where

$$\rho = p\left\{1 - \frac{[\omega(\zeta)+p][p(p-\delta)+\zeta\delta(p-1)]^2-\zeta(p-1)[p(p-\zeta\delta)+\delta(p-\zeta)]^2}{\omega(\zeta)[p(p-\delta)+\zeta\delta(p-1)]^2+[\omega(\zeta)+2\zeta][p(p-\zeta\delta)+\delta(p-\zeta)]^2}\right\}$$

$$(\omega(\zeta) = [p(1-\zeta)-\zeta]). \quad (5.18)$$

The result is sharp for the functions $g_j(z)(j = 1,2)$ given by

$$g_j(z) = z^{-p} - \frac{[p(p-\delta)+\zeta\delta(p-1)]}{[p(p-\delta)+\zeta\delta(p-1)][1+\lambda(p-1]^k}(j = 1,2; p \in \mathbb{N};\ k \in \mathbb{N}_0). \quad (5.19)$$

Theorem 5.4. Let the function$f_j(z)(j = 1,2)$ given by (5.1) be in the class $\mathcal{AR}(\lambda,k,p,\zeta,\delta)$. Then the function $h(z)$defined by

$$h(z) = -z^{-p} + \sum_{n=p}^{\infty}({a_{n,1}}^2 + {a_{n,2}}^2)z^n, \quad (5.20)$$

belongs to the class $\mathcal{AR}(\lambda,k,p,\zeta,\kappa)$, where$\kappa = p\left(1 - \frac{4[1+\zeta(p-1)](p-\delta)^2}{[\zeta(p+1)-1][1+\lambda(p-1]^k(p+\delta)^2+2[1+\zeta(p-1)](p-\delta)^2}\right)$. (5.21)

The result is sharp for the functions $f_j(z)(j = 1,2)$ given by (5.6).

Proof: Since

$$\sum_{n=p}^{\infty}\frac{\{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k\}^2}{[1+\zeta(p-1)]^2(p-\delta)^2}{a_{n,j}}^2 \leq \left(\sum_{n=p}^{\infty}\frac{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k}{[1+\zeta(p-1)](p-\delta)}a_{n,j}\right)^2 \leq 1, \quad (j=1,2). \quad (5.22)$$

for $f_j(z) \in \mathcal{AR}(\lambda,k,p,\zeta,\delta)(j=1,2)$, we have $\sum_{n=p}^{\infty}\frac{\{(n+\delta)[\zeta(n+1)-1][1+\lambda(n-1)]^k\}^2}{2[1+\zeta(p-1)]^2(p-\delta)^2}\left({a_{n,j}}^2+{a_{n,j}}^2\right) \leq 1. \quad (5.23)$

Therefore, we have to find the largest κ such that

$$\frac{(n+\kappa)}{(p-\kappa)} \leq \sum_{n=p}^{\infty}\frac{(n+\delta)^2[\zeta(n+1)-1][1+\lambda(n-p)]^k}{2[1+\zeta(p-1)](p-\delta)^2} \quad (n \geq p) \quad (5.24)$$

that is, that

$$\kappa \leq p-\frac{2(p+n)[1+\zeta(p-1)](p-\delta)^2}{[\zeta(n+1)-1][1+\lambda(n-p)]^k(n+\delta)^2+2[1+\zeta(p-1)](p-\delta)^2}. \quad (n \geq p) \quad (5.25)$$

Now, defining the function $\varphi(n)$by

$$\kappa(n)=p-\frac{2[1+\zeta(p-1)](p-\delta)^2}{[\zeta(p+1)-1][1+\lambda(n-p)]^k(p+\delta)^2+2[1+\zeta(p-1)](p-\delta)^2}, \quad (n \geq p) \quad (5.26)$$

We obtain that $\kappa(n)$is an increasing function of n. Therefore, we conclude that

$$\kappa \leq \varphi(n) = p\left\{1-\frac{4[1+\zeta(p-1)](p-\delta)^2}{[\zeta(p+1)-1](p+\delta)^2+2[1+\zeta(p-1)](p-\delta)^2}\right\}, \quad (n \geq p) \quad (5.27)$$

which completes the proof of Theorem 5.4.

Theorem 5.5. Let the function$g_j(z)(j=1,2)$ given by (5.2) be in the class $\mathcal{AC}(\lambda,k,p,\zeta,\delta)$. Then the function $h(z)$ defined by

$$h(z)=z^{-p}-\sum_{n=p}^{\infty}({a_{n,1}}^2+{a_{n,2}}^2)z^n, \quad (5.28)$$

belongs to the class $\mathcal{AC}(\lambda,k,p,\zeta,\kappa)$, where$\kappa =$

$$p\left\{1-\frac{[\omega(\zeta)+p][p(p-\delta)+\zeta\delta(p-1)]^2-\zeta(p-1)[p(p-\zeta\delta)+\delta(p-\zeta)]^2[1+\lambda(p-1)]^k}{[\omega(\zeta)+2\zeta][p(p-\zeta\delta)+\delta(p-\zeta)]^2+2\omega(\zeta)[p(p-\delta)+\zeta\delta(p-\zeta)]^2}\right\}$$

$$(\omega(\zeta)=[p(1-\zeta)-\zeta]). \quad (5.29)$$

The result is sharp for the functions $g_j(z)(j=1,2)$ given by (5.19)

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بعض الاصناف من الدوال الميرومورفية ومتعددة التكافؤ بمعاملات موجبة او سالبة باستخدام مؤثر تفاضلي

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الملخص

في هذا البحث نقدم صنفين $\mathcal{AR}(\lambda, k, p, \zeta, \delta)$ و $\mathcal{AC}(\lambda, k, p, \zeta, \delta)$ من الدوال متعددة التكافؤ وميرومورفيك بمعاملات موجبة و سالبةمعرف بواسطة مؤثر تفاضلي في دائرة الوحدة المثقب $\mathcal{U}\backslash\{0\}$ $\mathcal{U}^* = \{z: z \in \mathbb{C};\ 0 < |z| < 1\} =$ ونحصل على بعض النتائج مثل متباينة المعاملات، مبرهنة البعد، نصف قطر التنجيم والتحدب، مبرهنة الانغلاق لهذه الدوال من الاصناف من الدوال متعددة التكافؤ وميرومورفيك. كذلك نشتق بعض النتائج من الضرب الالتفافي لدوال تنتمي الى الصنفين $\mathcal{AR}(\lambda, k, p, \zeta, \delta)$ و $\mathcal{AC}(\lambda, k, p, \zeta, \delta)$.