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Generalization of Fuglede-Putnam Theorem to (*p*, *q*)−Quasiposinormal Operator and (*p*, *q*)− Co-posinormal Operator

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Abstract

In this paper we generalize the Fuglede-Putnam theorem to non-normal operators to posinormal operator and co-posinormal operators. Also we prove this theorem to supra class posinormal operators (called supraposinormal operator) and co-supra class posinormal operators (called cosupraposinormal operator).

Keywords: Fuglede-Putnam Theorem, posinormal operator, positive operator.

Introduction

In this paper we give positive answer to the question that appeared in [1].

Let H be a separable complex Hilbert space and B(H), S_1, and S_2 denote the algebra of all bounded linear operators acting on H, the Hilbert-Schmidt class and the trace class in $\boldsymbol{B}$(H) respectively. It is well known that S_1 is itself a Hilbert space with the inner product

$$< x, y > = \sum < xe_i, ye_i > = Tr(Y^*X)$$
$$= Tr(XY^*)$$

where e_i is any orthonormal basis of H and *Tr*(.) is the natural trace on S_2(H) [2]. The Hilbert-Schmidt norm of $X \in S_1$ is given by $||X||_2 = < X, X >^{1/2}$. Berberian [3] relaxes the hypothesis on A and B by assuming A and B*
hyponormal operators and X to be Hilbert-Schmidt class. An operator $T \in$ B(H) is normal if $TT^* = T^*T$, positive, $T \geq 0$, if $< T_x, x > \geq 0$ for all $x \in$ H, posinormal if there exists a positive operator $P \in$ B(H) such that $TT^* = T^*PT$. Here, P is called an interrupter of T, and co-posinormal if T^* is posinormal i.e $T^*T = TPT^*$. From [4, Theorem 2.1], we know that T is posinormal if and only if $c^2T^*T - TT^* \geq 0$ for some $c > 0$. Let p be $0 < p \leq 1$. An operator $T \in \boldsymbol{B}$(H) is said to be p-hyponormal if $(TT^*)^p \leq (T^*T)^p$,
and p-posinormal if $(TT^*)^p \leq c^2(T^*T)^p$, for some $c > 0$. It is clear that 1-hyponormal and 1-posinormal are hyponormal and posinormal, respectively.

Definition 1. *For a positive integer k and a positive number* $0 < p \leq 1$*, an operator T is said to be* (*p*, *k*)*-quasiposinormal if*

$$(T^*)^k(c^2(T^*T)^p - (TT^*)^p)T^k \geq 0$$
for some $c > 0$ [5].

The Main Results

The Posinormal Operator Case

The basic elementary operator $M_{A,B}$ induced by the operators A and B is defined on S_1(H) by $M_{A,B}(X) = AXB$, and the adjoint of $M_{A,B}$ is given by the formula $M^*_{A,B}(X) = A^*XB^*$ [3].

The familiar Fuglede-Putnam Theorem is as follows [6, Theorem 6.7] and [7, Theorem 12.16]):

Theorem 1. *If A and B are normal operators and if X is an operator such that AX = XB, then A*X = XB*.*

Proposition 1. *Let A,B ∈* $\boldsymbol{B}$(H)*. If* $A \geq 0$ *and* $B \geq 0$*, then* $M_{A,B} \geq 0$.

Proof: Let $X \in S_1$(H),

$$\begin{aligned} < M_{A,B}X, X > &= Tr(AXBX^*) \\ &= Tr(A^{1/2}XBX^*A^{1/2}) \\ &= Tr((A^{1/2}XB^{1/2})(B^{1/2}X^*A^{1/2})) \\ &= Tr((A^{1/2}XB^{1/2})(A^{1/2}XB^{1/2})^*) > 0. \end{aligned}$$

Indeed, $M^{1/2}_{A,B}(X) = A^{1/2}X\,B^{1/2}$.

Proposition 2. *If A and B* ∈* $\boldsymbol{B}$(H)*, A is a (p,q)-quasiposinormal operator and B* is a posinormal operators then* $M_{A,B}$ *is a positive operator.*

Proof : Let $X \in S_1$(H), since B^* is posinormal operator, then there exists a positive *number c* such that $c^2A^*A - AA^* \geq 0$, and we must show that $c^2M^*_{A,B}M_{A,B} - M_{A,B}M^*_{A,B} \geq 0$ Indeed, the formula

$c^2M^*_{A,B}M_{A,B} - M_{A,B}M^*_{A,B}$ $=c^2 A^*AXBB^* - AA^*XB^*B$
$= c^2 A^*AXBB^* - AA^*XB^*B + c^2AA^*XBB^* - c^2AA^*XBB$
$=c^2(A^*A - AA^*)XBB^* + AA^*X(c^2BB^* - B^*B)$

shows that $c^2M^*_{A,B}M_{A,B} - M_{A,B}M^*_{A,B}$ is the sum of two positive operators. Hence $M_{A,B}$ is posinormal.

Lemma 1. *If A is an invertible posinormal operator, then* A^{-1} *is posinormal operator.*

Proof *:* Since A is posinormal, then for some $c > 0$ we have

$$c^2A^*A - AA^* \geq 0$$
$$c^2A^*A \geq AA^*$$
$$A^{-1}(c^2A^*A)(A^*)^{-1} \geq A^{-1}AA^*(A^*)^{-1}$$
$$A^{-1}(c^2A^*A)(A^*)^{-1} \geq I$$

Taking inverses gives

$$A^*\left(\frac{1}{c^2}A^{-1}(A^*)^{-1}\right)A \leq I$$
$$\frac{1}{c^2}A^{-1}(A^*)^{-1} \leq (A^*)^{-1}A^{-1}$$
$$c^2(A^*)^{-1}A^{-1} \geq A^{-1}(A^*)^{-1}$$

This means that A^{-1} is posinormal.

The following theorem with proof can be found in [5, Theorem 2.5]

Theorem 2. *Let T be a* (*p*, *q*)*-quasiposinormal operator. Then*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$$

on $H = \overline{ran(T^k)} + ker(T^{*k})$*, where* T_1 *is p-posinormal operator and* $T_3^k = 0$.

Lemma 2. *Let T be a (p, q)-quasiposinormal operator on a Hilbert space H. If $\lambda \in \mathbb{C}, x \in H$ and $Tx = \lambda x$, then $T^*x = \bar{\lambda}x$.*

Proof : If $x = 0$ then the proof is obvious. If $x \neq 0$, let H_0 be a span of $\{x\}$. Then H_0 is an invariant subspace of T and

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$$

on $H = H_0 + H_0^{\perp}$.

Let Q be the orthogonal projection of H onto H_0. Then $T_1 = TQ$ and $T_1^* = QT^*$, so
$T_1 = \lambda$ and

$$(T_1^*T_1)^p = (QT^*TQ|_{H_0})^p = (QT^*TQ)^p|_{H_0}$$
$$\geq Q(T^*T)^pQ|_{H_0}$$

by Hansen's inequality [7]. On the other hand

$$(T_1T_1^*)^p = (TQT^*|_{H_0})^p = (TQT^*)^p|_{H_0}$$
$$\leq Q(TT^*)^pQ|_{H_0}$$

by Lowner-Heinz's inequality [9, 10]. Hence

$$\begin{bmatrix} (T_1^*T_1)^p & 0 \\ 0 & 0 \end{bmatrix} \geq Q(TT^*)^pQ \geq Q(T^*T)^pQ$$
$$\geq \begin{bmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{bmatrix}$$

Hence

$$(TT^*)^p = \begin{bmatrix} |\lambda|^{2p} & A \\ A & B \end{bmatrix}.$$

Let

$$(TT^*)^{\frac{p}{2}} = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}.$$

Then

$$\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} = Q(TT^*)^{\frac{p}{2}}Q \geq Q(TQT^*)^{\frac{p}{2}}Q$$
$$= \begin{bmatrix} |\lambda|^p & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$X \geq |\lambda|^p.$$

Since

$$(TT^*)^p = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$$
$$= \begin{bmatrix} X^2 + YY^* & XY + YZ \\ Y^*X + ZY^* & Y^*Y + Z^2 \end{bmatrix}$$

We have

$$X^2 + YY^* = |\lambda|^{2p}$$

And

$$|\lambda|^p = \sqrt{X^2 + YY^*} \geq X \geq |\lambda|^p.$$

Hence Y=0. Hence

$$(TT^*)^{\frac{p}{2}} = \begin{bmatrix} |\lambda|^p & 0 \\ 0 & Z \end{bmatrix} \text{ and } TT^* = \begin{bmatrix} |\lambda|^2 & 0 \\ 0 & Z^{\frac{p}{2}} \end{bmatrix}.$$

On the other hand we have

$$TT^* = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ T_2^* & T_3^* \end{bmatrix} = \begin{bmatrix} |\lambda|^2 + T_2T_2^* & T_2T_3^* \\ T_3T_2^* & T_3T_3^* \end{bmatrix}$$

Hence $T_2 = 0$. Thus

$$T^*X = \begin{bmatrix} \bar{\lambda} & 0 \\ 0 & T_3 \end{bmatrix} \begin{matrix} x \\ 0 \end{matrix} = \bar{\lambda}X.$$

The first generalization of Fuglede-Putnam Theorem is as follows:

Theorem 3. *If $A \in$ **B**(H) is (p, k)-quasiposinormal operator and $B \in$ **B**(H) is invertible and co-posinormal operator such that $AX = XB$, for some $X \in S_1$, then $A^*X = XB^*$.*

Proof: Let $AX = BX$ for some $X \in S_1$, then

$$M_{A,B^{-1}}(X) = AXB^{-1}$$
$$= XBB^{-1}$$
$$= X.$$

Since B is invertible and co-posinormal operator, that is B^* is invertible and posinorml operator, so $(B^*)^{-1}$ is posinormal by Lemma 1. Also $M_{A,B^{-1}}$ is posinormal operator by Proposition 2. Hence $M_{A,B^{-1}}^* = X$ by Lemma 2, and so

$$A^*X(B^*)^{-1} = X$$

that is

$$A^*X = X B^*.$$

The Supraposinormal Operator Case

Definition 2. *(1)An operator $T \in$ **B**(H) is said to be supraposinormal if there exist two positive operators $U\&V \in$ **B**(H) such that*

$$TVT^* = T^*UT,$$

where at least one of the U and V has dense range in H. *It will sometimes be convenient to refer to the ordered pair (U, V) as an interrupter pair associated with T.*

(2)For a positive integer k and a positive number $0 < p \leq 1$, An operator T is said to be (p, k)-quasisupraposinormal if

$$(T^*)^k((TVT^*)^p = (T^*UT)^p)T^k.$$

The following theorem with proof can be found in [11, Theorem 4.6.7]

Theorem 4. *If A is an invertible positive operator, then its inverse A^{-1} is positive.*

Proposition 3. *If T is an invertible supraposinormal with invertible interrupter (U, V), then it's inverse T^{-1} is also supraposinormal.*

Proof: Since T is supraposinormal
$TV T^* = T^*UT$

$$(T^*)^{-1}TVT^*T^{-1} = (T^*)^{-1}T^*UTT^{-1}$$
$$(T^*)^{-1}TVT^*T^{-1} = U$$

Take inverses

$$T(T^*)^{-1}V^{-1}T^{-1}T^* = U^{-1}$$
$$(T^*)^{-1}V^{-1}T^{-1} = T^{-1}U^{-1}(T^*)^{-1}$$

by Theorem4 U^{-1} and V^{-1} are positive, so T^{-1} is a supraposinormal.

Proposition 4. *If $A \in$* **B**(H) *is (p, k)-quasisupraposinormal operator and $B^* \in$* **B**(H) *is a supraposinormal operators then $M_{A,B}$ is supraposinormal operator.*

Proof : Since A and B are supraposinormal

$$M_{A,B}\, V\, M_{A,B}^* = AVA^*XB^*VB$$
$$A^*UAXBUB^*$$
$$= M_{A,B}^*\, U\, M_{A,B}.$$

Theorem 5. *Let T be a (p, q)-quasisupraposinormal operator. Then*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_1 \end{bmatrix}$$

On $H = \overline{ran(T^k)} + \ker(T^{*k})$, where T_1 *is p-supraposinormal operator and $T_3^k = 0$.*

Proof: Consider the decomposition $H = \overline{ran(T^k)} + \ker(T^{*k})$, since $\overline{ran(T^k)}$ is an invariant subspace of *T*, *T* has the matrix representation

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_1 \end{bmatrix}$$

with respect to $H = \overline{ran(T^k)} + \ker(T^{*k})$ Let *Q* be the orthogonal projection on $\overline{ran(T^k)}$. Then $T_1 = TQ$ and *T** = *QT**, because of

$$(T^*)^k (TVT^*)^p = (T^*UT)^p T^k,$$

Lowner-Heinz's inequality [9, 10], and Hansen's inequality we have

$$\begin{aligned}(T_1VT_1^*)^p &= (TQVQT^*)^p \le Q(TVT^*)^p Q \\ &= Q(T^*UT)^p Q \le (QT^*UTQ)^p \\ &= (T_1^*UT_1)^p \quad .... \quad (1)\end{aligned}$$

$$\begin{aligned}(T_1^*UT_1)^p &= (QT^*UTQ)^p \le Q(T^*UT)^p Q \\ &= Q(TVT^*)^p Q \le (TQVQT^*)^p \\ &= (T_1VT_1^*)^p . \quad ... \quad (2)\end{aligned}$$

From (1) and (2) we see that *T*1 is *p*-supraposinormal on $\overline{ran(T^k)}$.

Let $x = x_1 + x_2$ and $y = y_1 + y_2$ in $H = \overline{ran(T^k)} + \ker(T^{*k})$. Then

$$(T_3^* x_2, y_2) = (T^k (I - Q)x, (I - Q)y) = 0$$

for all *x*, *y* ∈ H. Thus $T_3^k = 0$.

Lemma 3. *Let T be a (p, q)-quasisupraposinormal operator on a Hilbert space H. If* $\lambda \in \mathbb{C}, x \in H$ *and* $T_x = \lambda x$*, then* $T^*x = \bar{\lambda}x$.

Proof of this Lemma is similar to the proof of Lemma 2.

The second generalization of Fuglede-Putnam Theorem is as follows:

Theorem 6. *If* A ∈ ***B***(H) *is (p, k)-quasisupraposinormal operator and* B ∈ ***B***(H) *is invertible with invertible interrupters (U,V) and supraposinormal operator such that AX = XB, for some X* ∈ *S*1*, then A*X = XB*.*

Proof of this Theorem is similar to the proof of Theorem 3.

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تعميم مبرهنة فاكليد– بوتنام الى (p,q) مؤثر شبه طبيعي ايجابي و (p,q) معكوس مؤثر طبيعي ايجابي

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الملخص

في هذا البحث العلمي، عممنا مبرهنة فاكليد – بوتنام الى المؤثرات غير العادية عممناها الى مؤثرات شبه طبيعي ايجابي و معكوس مؤثر طبيعي ايجابي. وكذلك عممنا هذه المبرهنة الى مؤثرات سوبر كلاس شبه طبيعي ايجابي ومعكوس مؤثر سوبر كلاس طبيعي ايجابي.

الكلمات المفتاحية: مبرهنة فاكليد – بوتنام، مؤثر طبيعي، مؤثر طبيعي ايجابي