On the Domination Numbers of Certain Prism Graphs

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ABSTRACT

A dominating set $S$ of a graph $G = (V, E)$, is a subset of the vertex set $V(G)$ such that any vertex not in $S$ is adjacent to at least one vertex in $S$. The domination number of a graph $G$ denoted by $\gamma(G)$ is the minimum size of the dominating sets of $G$. In this paper we introduced the domination numbers of certain prism graphs.

1. Introduction

Throughout this paper we consider simple graphs, finite, undirected and contain no loops or multiple edges. Our terminology and notations will be standard except as indicated. For undefined terms see [1], [2] and [3]. For a graph $G = (V, E)$, $V$ denotes its vertex set while $E$ its edge set. If $D \subseteq V$ then $<D>$ denotes the induced sub graph of $G$ by the vertices of $D$. The cardinality of a set $S$ denoted by $|S|$ is the number of elements of the $S$. A set $D \subseteq V$ is said to be dominating set of $G$ if every vertex in $V - D$ is adjacent to some vertex in $D$. The cardinality of a minimum dominating set $D$ is called the domination number of $G$ and is denoted by $\gamma(G)$ [4]. In other words we defined the domination number $\gamma(G)$ of a graph $G$ as the order of smallest dominating set of $G$. A dominating set $S$ with $|S| = \gamma(G)$ is called a minimum dominating set [5].

As usual we use $\lfloor x \rfloor$ for the smallest integer not greater than $x$.

Definition 1.1: A Prism graph $Y_{m,n}$ is a simple graph given by the Cartesian product graph $Y_{m,n} = C_m \times P_n$ where $C_m$ is a cycle with $m$ vertices and $P_n$ is a path with $n$ vertices. It can therefore be viewed and formed by connecting $n$ concentric cycle graphs $C_m$ along spokes. Also sometime $Y_{m,n}$ calls a circular ladder graph [11].
The following theorems have been used in this paper:

**Theorem 1.1** (see [12]): If $G$ has no isolated vertices, then $(G) \leq \frac{|V(G)|}{2}$.

**Theorem 1.2** (see [8]): A dominating set $D$ of a graph $G$ is minimal if and only if for each vertex $v \in V(D)$ one of the following conditions satisfied

(i) There exist a vertex $u \in V(G)$ such that $N(u) \cap D = \{v\}$

(ii) $v$ is an isolated vertex in $D$.

We consider for $m \geq 3$ the following three cases:

**Case (1):** Suppose $m \equiv 1 (mod \ 4)$, we give the set of vertices $S = S_1 \cup S_2 \cup \{m\}$ is dominating set for $Y_{m,2}$.

Where $S_1 = \{1 + 4k | k = 0,1,2, \ldots, \lfloor \frac{m-2}{4} \rfloor \}$

and $S_2 = \{(3 + m) + 4k | k = 0,1,2, \ldots, \lfloor \frac{m-2}{4} \rfloor \}$.

Therefore $\gamma(Y_{m,2}) \leq |S| = 2 \lfloor \frac{m-2}{4} \rfloor + 3$.

Now, to prove that $\gamma(Y_{m,2}) \geq 2 \lfloor \frac{m-2}{4} \rfloor + 3$ it is sufficient to show that there is no proper subset of $S$ dominating $Y_{m,2}$. So if $v$ is any vertex in $S$, $S' = S - \{v\} \subset S$ be dominating $Y_{m,2}$ and $|S'| \leq 2 \lfloor \frac{m-2}{4} \rfloor + 2$.

For any $v \in S$, implies that $v$ either in $S_1$ or in $S_2$ or $v = \{m\}$. According to the structure of the labeling it is clear that for any choices of $v \in S_1$, we have at least two vertices of the form $(2 + 4k, m + 4k | k = 0,1,2, \ldots, \lfloor \frac{m-2}{4} \rfloor )$ that adjacent to $v$ not dominating with any vertex in $S'$. For $v \in S_2$, $v$ dominates itself and we have three vertices of the form:

$\{3 + 4k, (m + 2) + 4k, (4 + m) + 4k | k = 0,1,2, \ldots, \lfloor \frac{m-2}{4} \rfloor \}$ not dominating with any vertex in $S'$.

Also if $v = \{m\}$ then we have at least two vertices of the form $(m - 1,2m)$ adjacent to $v$ not dominating yet with any vertex in $S'$. So for any choices of $v$ we have at least two vertices not dominating with any vertex in $S'$, this implies that $S'$ is not a dominating set or in other words we show that there is no proper subset $S' \subset S$ dominating $Y_{m,2}$.

Therefore we have a contradiction and $\gamma(Y_{m,2}) \geq 2 \lfloor \frac{m-2}{4} \rfloor + 3$.

This implies that $\gamma(Y_{m,2}) = 2 \lfloor \frac{m-2}{4} \rfloor + 3$.

**Case(2):** For $m \equiv 2 (mod \ 4)$, we give the set of vertices $S = S_1 \cup S_2 \cup \{2m\}$ is dominating set for $Y_{m,2}$.

Where $S_1$ is the same set in case(1) and $S_2 = ((3 + m) + 4k | k = 0,1,2, \ldots, \lfloor \frac{m-2}{4} \rfloor - 1)$.

Therefore $|S| = |S_1| + |S_2| + 1 = 2 \lfloor \frac{m-2}{4} \rfloor + 2$.

Hence $\gamma(Y_{m,2}) \leq |S| = 2 \lfloor \frac{m-2}{4} \rfloor + 2$.

Similar to that in case(1), let $v \in S$, $S' = S - \{v\} \subset S$, $|S'| \leq 2 \lfloor \frac{m-2}{4} \rfloor + 1$ and $S'$ be the dominating set of $Y_{m,2}$. Now if $v \in S_1$ or in $S_2$ the proof is similar to that in the case (1) remained when $v = \{2m\} \in S'$.
this vertex dominates itself which is not dominated with any vertex in $S' = S_1 \cup S_2$. So we have for all choices of $v$ at least one vertex not dominating with any vertex in $S'$.

\[ \therefore S' \text{ not dominating set of } Y_{m,2}. \]

Hence $\gamma(Y_{m,2}) \geq |S| = 2\left[\frac{m-2}{4}\right] + 2$ implies that $\gamma(Y_{m,2}) = 2\left[\frac{m-2}{4}\right] + 2$.

**Case (3):** For $m \equiv 0,3 \pmod{4}$, we give the dominating set for these two cases of $Y_{m,2}$ by: $S = S_1 \cup S_2$ where $S_1$ and $S_2$ are the same sets in case(1).

\[ \therefore |S| = |S_1| + |S_2| = 2\left[\frac{m-2}{4}\right] + 2 \text{ Hence } \gamma(Y_{m,2}) \leq 2\left[\frac{m-2}{4}\right] + 2. \]

The proof of $\gamma(Y_{m,2}) \geq 2\left[\frac{m-2}{4}\right] + 2$ is similar to that in case(1) when $v \in S_1$ or $v \in S_2 \therefore \gamma(Y_{m,2}) = 2\left[\frac{m-2}{4}\right] + 2$.

From the above cases we have:

\[ \gamma(Y_{m,2}) = \begin{cases} 
2\left[\frac{m-2}{4}\right] + 3 & \text{for } m \equiv 1 \pmod{4} \\
2\left[\frac{m-2}{4}\right] + 2 & \text{otherwise}.
\end{cases} \]

**3. On domination numbers of prism graph $Y_{m,3}$:**

**Theorem (2):** For $m \geq 3$,

\[ \gamma(Y_{m,3}) = \begin{cases} 
3\left[\frac{m-1}{2}\right] + 4 & \text{for } m \equiv 1 \pmod{4} \\
3\left[\frac{m-2}{4}\right] + 2 & \text{for } m \equiv 2 \pmod{4} \\
3\left[\frac{m-2}{4}\right] + 3 & \text{for } m \equiv 0,3 \pmod{4}.
\end{cases} \]

**Proof:** Let the vertices of this graph labeled by:

\[ V(Y_{m,3}) = \{1,2,3, \ldots, 3m\} \]

see Fig. 3.

![Fig. 3: $Y_{m,3}$](image)

Similarly in Theorem(1) we consider for $m \geq 3$ the three following cases:

**Case (1):** Suppose $m \equiv 1 \pmod{4}$, first we give the set of dominating set of $Y_{m,3}$ by the set of the vertices $S = S_1 \cup S_2 \cup S_3 \cup \{2m - 1\}$. Where $S_1 = \{1 + 4k | k = 0,1,2, \ldots, \frac{m-2}{4}\}$

\[ S_2 = \{(3 + m) + 4k | k = 0,1,2, \ldots, \frac{m-2}{4}\} \]

and $S_3 = \{(1 + 2m) + 4k | k = 0,1,2, \ldots, \frac{m-2}{4}\}.$

\[ |S| = |S_1| + |S_2| + |S_3| + 1 = 3\left[\frac{m-2}{4}\right] + 4 \]

Therefore $\gamma(Y_{m,3}) \leq |S| = 3\left[\frac{m-2}{4}\right] + 4.$ By the same way as in Theorem (1) we prove $\gamma(Y_{m,3}) \geq 3\left[\frac{m-2}{4}\right] + 4,$ let $v$ any vertex in $S$. So $S' = S - \{v\} \subset S$, $|S'| \leq 3\left[\frac{m-2}{4}\right] + 3$ and $S'$ be the dominating set of $Y_{m,3}$.

For any $v \in S$ implies that $v$ is either in $S_1$ or in $S_2$ or in $S_3$ or $v = \{2m - 1\}$. The proof is similar to that in Theorem(1), case(1) when $v \in S_1$ or $S_2$. So if $v \in S_3$ according to the structure of the labeling it is clear that $v$ is dominated itself in addition there is at least one vertex of the form:

\[ \{2(m + 1) + 4k | k = 0,1,2, \ldots, \frac{m-2}{4}\} \]

that adjacent to $v$ not dominating with any vertex in $S'$, remained when $v = \{2m - 1\}$ we note $v$ dominates itself and there are three vertices of the form $\{2m, m - 1, 3m - 1\}$ that adjacent to this vertex not dominating with any vertex in $S'$. So for any choices of $v$ we have at least two vertices not dominating with any vertex in $S'$.

\[ \therefore S' \text{ is not a dominating set implies } \gamma(Y_{m,3}) \geq 3\left[\frac{m-2}{4}\right] + 4. \]

Therefore $\gamma(Y_{m,3}) = 3\left[\frac{m-2}{4}\right] + 4.$

**Case (2):** For $m \equiv 2 \pmod{4}$.

We have give the set of vertices $S = S_1 \cup S_2 \cup S_3 \cup \{2m-1, 2(m-1)\}$

Where $S_1 = \{1 + 4k | k = 0,1,2, \ldots, \frac{m-2}{4}-1\}$

\[ S_2 = \{(3 + m) + 4k | k = 0,1,2, \ldots, \frac{m-2}{4}-1\} \]

and $S_3 = \{(1 + 2m) + 4k | k = 0,1,2, \ldots, \frac{m-2}{4}-1\}$.

Therefore $|S| = |S_1| + |S_2| + |S_3| + 2 = 3\left[\frac{m-2}{4}\right] + 2.$

Hence $\gamma(Y_{m,3}) \leq |S| = 3\left[\frac{m-2}{4}\right] + 2.$

Now, let $v \in S$. So $S' = S - \{v\} \subset S, |S'| \leq 3\left[\frac{m-2}{4}\right] + 1$ and $S'$ be the dominating set of $Y_{m,3}$.

Therefore $\gamma(Y_{m,3}) \geq 3\left[\frac{m-2}{4}\right] + 1.$

So either $v = \{2m - 1\}$ that dominates itself in addition there are at least three vertices of the form $\{m - 1, 3m - 1, 2m\}$ that adjacent to $v$ not dominating with any vertex in $S '$. So for any choices of $v$ we have at least two vertices not dominating with any vertex in $S'$.
least two vertices not dominating with any vertex in $S'$.
\[ \therefore S' \text{ is not a dominating set.} \]
Hence $\gamma(Y_{m,3}) \geq 3 \left\lceil \frac{m-2}{4} \right\rceil + 2$.
This implies that $\gamma(Y_{m,3}) = 3 \left\lceil \frac{m-2}{4} \right\rceil + 2$.

**Case(3):** For $m \equiv 0, 3 \pmod{4}$ give the set of dominating set of $Y_{m,3}$ by the set of the vertices $S = S_1 \cup S_2 \cup S_3$ where $S_1, S_2$ and $S_3$ are the same sets in case(1).
Therefore $\gamma(Y_{m,3}) \leq |S| = 3 \left\lceil \frac{m-2}{4} \right\rceil + 3$.
Let $v \in S \cdot S' \subset S \cdot |S'| \leq 3 \left\lceil \frac{m-2}{4} \right\rceil + 3$, the proof of $\gamma(Y_{m,3}) \geq 3 \left\lceil \frac{m-2}{4} \right\rceil + 3$ is similar to that in case (1)
when $v \in S_1, v \in S_2$ and $v \in S_3$ hence $S'$ is not a dominating set, implies that $\gamma(Y_{m,3}) \geq 3 \left\lceil \frac{m-2}{4} \right\rceil + 3$.
Hence from the above cases we have:
\[
\begin{align*}
\gamma(Y_{m,3}) &= \\
&= \begin{cases} 
3 \left\lceil \frac{m-2}{4} \right\rceil + 4 & \text{for } m \equiv 1 \pmod{4} \\
3 \left\lceil \frac{m-2}{4} \right\rceil + 2 & \text{for } m \equiv 2 \pmod{4} \\
3 \left\lceil \frac{m-2}{4} \right\rceil + 3 & \text{for } m \equiv 0, 3 \pmod{4}.
\end{cases}
\end{align*}
\]

4. **On domination numbers of prism graph $Y_{m,4}$:**
In this section we first determine the domination number of $Y_{3,4}, Y_{4,4}$ and $Y_{5,4}$ as special cases then we determine the domination numbers of $Y_{m,4}$ in general.

**Lemma(1):** $\gamma(Y_{3,4}) = 4$

**Proof:** Let the vertices of this graph labeled as shown in Fig.4.

![Fig.4: $Y_{3,4}$](image)

We have first give the dominating set of this graph by the set $S = \{3,4,11,12\}$, therefore $\gamma(Y_{3,4}) \leq |S| = 4$.
Let $v \in S \cdot S' \subset S \cdot |S'| \leq 3$.
It is easy to show that there is no proper subset $S' = S - \{v\}$ dominating $Y_{3,4}$.
So if we choose $v \in S$ we have always at least one vertex that adjacent to $v$ not dominating with any vertex in $S'$. So if $v = \{3\} \notin S'$, $S' = \{4, 11, 12\}$ it is clear $\{2, 6\}$ not dominating with any vertex in $S'$.
Now if $v = \{4\} \notin S'$, we have $\{7\}$ not dominating with any vertex in $S'$. If $v = \{11\} \notin S'$, we have $\{8\}$ not dominating with any vertex in $S'$. Also if $v = \{12\} \notin S'$, we have $\{9\}$ not dominating with any vertex in $S'$. So for any choices of $v$ we have always one vertex not dominating with any vertex in $S'$.

\[ \therefore S' \text{ is not dominating set, hence we have a contradiction and } \gamma(Y_{3,4}) \geq 4. \]

\[ \therefore \gamma(Y_{3,4}) = 4. \]

**Lemma(2):** $\gamma(Y_{5,4}) = 6$

**Proof:** Let the vertices of this graph labeled as shown in Fig.5.

![Fig.5: $Y_{5,4}$](image)

We have first give the dominating set of this graph by the set $S = \{3,5,6,14,17,20\}$, therefore $\gamma(Y_{5,4}) \leq |S| = 6$.
Let $v \in S \cdot S' \subset S \cdot |S'| \leq 5$.
It is very easy as in lemma(1) to show that there is no proper subset $S' \subseteq S$ dominating $Y_{5,4}$. So if $v = \{3\}$, we have three vertices $\{2,4,8\}$ not dominating with any vertex in $S'$. If $v = \{5\}$ not dominating with any vertex in $S'$ because it is dominates itself. If $v = \{6\}$, we have two vertices $\{7,11\}$ not dominating with any vertex in $S'$.
If $v = \{14\}$, we have three vertices $\{9,13,15\}$ not dominating with any vertex in $S'$.
If $v = \{17\}$, we have three vertices $\{12,16,18\}$ not dominating with any vertex in $S'$.
If $v = \{20\}$, we have two vertices $\{15,16\}$ not dominating with any vertex in $S'$. So for any choices of $v$ we have always one vertices not dominating with any vertex in $S'$.

\[ \therefore S' \text{ is not a dominating set. Hence } \gamma(Y_{5,4}) \geq 6. \]

\[ \therefore \gamma(Y_{5,4}) = 6. \]

**Lemma(3):** $\gamma(Y_{9,4}) = 10$

**Proof:** Let the vertices of this graph labeled as shown in Fig.6.

![Fig.6: $Y_{9,4}$](image)
We have given the dominating set of this graph by the set $S$ where: $S = \{3, 5, 8, 10, 16, 22, 26, 29, 33, 36\}$ therefore $\gamma(Y_{9, 4}) \leq |S| = 10$. Similarly as in lemma(1) and lemma(2) we have always $v$ dominates itself or there is at least one vertex that adjacent to $v$ not dominating with any vertex in $S'$.

So no proper subset $S' \subseteq S$ dominating $Y_{9, 4}$.

Hence $\gamma(Y_{9, 4}) \geq 10$.

$\therefore \gamma(Y_{9, 4}) = 10$. 

**Theorem (3):** For $m \geq 4$ and $m \neq 5, 9$ then $\gamma(Y_{m, 4}) = m$.

**Proof:** Let the vertices of this graph labeled by: $V(Y_{m, 4}) \{1, 2, 3, \ldots, 4m\}$ see Fig.7.

![Graph Diagram](image)

Fig. 7: $Y_{m, 4}$

Similar to that in Theorem(1) and Theorem(2), we consider for $m \geq 4$ and $m \neq 5, 9$ the four following cases:

**Case (1):** For $m \equiv 0 \pmod{4}$

We give the set of dominating set of $Y_{m, 4}$ by the set of the vertices:

$S = \{(i + m), (i + 3m + 1), (i + 2), (i + 2m + 3)\} | i = 1, 5, 9, \ldots, m - 3\}$

$|S| = 4 \left(\frac{m}{4}\right) = m$.

Therefore $\gamma(Y_{m, 4}) \leq |S| = m$.

For any vertex in $S$ then $S' = S - \{v\} \subseteq S$, $|S'| \leq m - 1$ and $S'$ be the dominating set of $Y_{m, 4}$, according to the structure of the labeling if we take $v \in S$ we have four choices to $v$.

If $v = (i + m)$ then the always three vertices are of the form $i, (i + m + 1), (i + 2m)$. If $v = (i + 3m + 1)$ then, we have three vertices are of the form:

$\{(i + 2m + 1), (i + 3m + 2), (i + 3m)\}$ not dominating with any vertex in $S'$.

If $v = (i + 2)$ we have three vertices of the form $\{(i + 1), (i + 3), (i + m + 2)\}$ not dominating with any vertex in $S'$.

If $v = (i + 2m + 3)$ we have two vertices are of the form:

$\{(i + 3m + 3), (i + 2m + 2)\}$ not dominating with any vertex in $S'$. So for any choices of $v$ we have at least two vertices not dominating with any vertex in $S'$.

Hence $\gamma(Y_{m, 4}) \geq m$.

$\therefore \gamma(Y_{m, 4}) = m$.

**Case (2):** For $m \equiv 1 \pmod{4}$ and note that $m \neq 5, 9$ first we give the set of dominating set of $Y_{m, 4}$ by the set of the vertices $S = S_1 \cup S_2$.

Where:

$S_1 = \{(i + m), (i + 3m + 1), (i + 2), (i + 2m + 3)\} | i = 8, 12, 16, \ldots, m - 5\}$

and $S_2 = \{(5, (m + 1), (2m + 4), (3m + 2), (m + 7), (3m + 6), (m - 1), 4m\}$.

$|S| = |S_1| + |S_2| = m$.

Therefore $\gamma(Y_{m, 4}) \leq |S| = m$.

Now, let $v$ any vertex in $S$, $S' = S - \{v\} \subseteq S$, $|S'| \leq m$ and $S'$ be the dominating set of $Y_{m, 4}$.

Similarly for any chooses of $v \in S$ either in $S_1$ or in $S_2$.

For $v \in S_1$, $v$ is dominates itself and all the vertices that adjacent to $v$ except when $v = (8 + m)$ we have three vertices of the form $\{(5, (8 + 2m), (9 + m)\}$ that adjacent to $v$ not dominating with any vertex in $S'$.

Also, for $v \in S_2$, $v$ is dominates itself and all the vertices that adjacent to $v$ except when $v = (m - 1, m + 7)$ we have two vertices of the form $\{(m, (2m - 1)\} \text{ and } \{(m + 6), (2m + 7)\}$ that adjacent to $v$ not dominating with any vertex in $S'$, similarly for any chooses of $v$ we have at least two vertices not dominating with any vertex in $S'$, so we have a contradiction, hence $S'$ is not a dominating set and $\gamma(Y_{m, 4}) \geq m$.

$\therefore \gamma(Y_{m, 4}) = m$.

**Case (3):** For $m \equiv 2 \pmod{4}$ first we give the set of dominating set of $Y_{m, 4}$ by the set of the vertices $S = S_1 \cup S_2$.

Where:

$S = \{(i + m), (i + 3m + 1), (i + 2), (i + 2m + 3)\} | i = 1, 5, 9, \ldots, m - 3\}$

and $S_2 = \{(m - 1), 4m\}$.

$|S| = |S_1| + |S_2| = m$.

Therefore $\gamma(Y_{m, 4}) \leq |S| = m$.

Now, let $v$ any vertex in $S$, $S' = S - \{v\} \subseteq S$, $|S'| \leq m - 1$ and $S'$ be the dominating set of $Y_{m, 4}$.

For $v \in S_1$ similarly, if $v = (i + m)$ or $v = (i + 2)$ always $v$ dominates itself and all the vertices that adjacent to him.

For $v = (i + 3m + 1)$ dominates itself and two vertices of the form $\{(i + 2m + 1), (i + 3m + 2)\}$ now. If $v = (i + 2m + 3)$ also dominates itself and two vertices of the form $\{(i + 3m + 3), (i + 2m + 2)\}$ not dominating with any vertex in $S'$. Now, for $v \in S_2$ implies that $v$ either $(m - 1)$ or $4m$.

If $v = (m - 1)$ we have two vertices of the form $\{(2m - 1), m\}$ adjacent to $v$ not dominating with any vertex in $S'$. Also, if $v = (4m)$ we have two vertices of the form $\{(4m - 1), 3m\}$ adjacent to $v$ not dominating with any vertex in $S'$. So for any chooses of $v$ we have always two vertices not dominating with any vertex in $S'$.

Therefore $S'$ is not a dominating set.

Hence $\gamma(Y_{m, 4}) \geq m$. 

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This implies that $\gamma(Y_{m,4}) = m$.

**Case (4):** For $m \equiv 3 \pmod{4}$ first we give the set of dominating set of $Y_{m,4}$ by the set of the vertices $S = S_1 \cup S_2$.

Where $S_1 = \{(i + m), (i + 3m + 1), (i + 3m + 2), (i + 2m + 3)\mid i = 1, 5, 9, \ldots, m-6\}$ and $S_2 = \{(m - 2), 2m, (4m - 1)\}$. Therefore $\gamma(Y_{m,4}) \leq |S| = m$. Now, let $v$ any vertex in $S$, $S' = S - \{v\} \subset S$, $|S'| \leq m - 1$ and $S'$ be the dominating set of $Y_{m,4}$. If $v \in S$ implies that $v$ either in $S_1$ or in $S_2$, for $v \in S_1$ according to the structure of the labeling for $v = (i + 2)$ and $v = (i + 3m + 1)$ it is clear that $v$ is dominates itself and all the three vertices that adjacent to him. If $v = (i + m)$ then, we have the three vertices of the form:

$[i, (i + m + 1), (i + 2m)]$ not dominating with any vertex in $S'$. Now if $v = (i + 2m + 3)$, $v$ dominates itself and the vertex of the form $(i + 3m + 3)$.

Also if $v \in S_2$ we have three choices:

If $v = (m - 2)$ then, there are two vertices of the form $[(2m - 2), (m - 1)]$ that adjacent to $v$ not dominating with any vertex in $S'$. For $v = (4m - 1)$ then, we have all the vertices that adjacent to $v$ not dominating with any vertex in $S'$. Now if $v = (2m)$ then, there are three vertices of the form $[(m, 3m), (2m - 1)]$ that adjacent to $v$ not dominating with any vertex in $S'$. So for any choices of $v$ we have at least two vertices not dominating with any vertex in $S'$ so we have a contradiction.

Hence $S'$ is not a dominating set implies $\gamma(Y_{m,4}) \geq m$.

$\therefore \gamma(Y_{m,4}) = m$. ■

5. **On domination numbers of prism graph** $Y_{m,5}$:

As previous section we first determine the domination number of $Y_{3,5}$ and $Y_{6,5}$ as a special cases then we determined the domination numbers of $Y_{m,5}$ in general.

**Lemma (4):** $\gamma(Y_{3,5}) = 4$

**Proof:** Let the vertices of this graph labeled as shown in Fig.8.

First we give the dominating set of this graph by: $S = \{3, 5, 7, 15, 16, 24, 26, 28\}$. Therefore $\gamma(Y_{6,5}) \leq |S| = 8$. Similarly in Lemma(4) for $v \in S$, $S' = S - \{v\} \subset S$, $|S'| \leq 7$ now, let $v$ any vertex in $S$ implies that:

If $v = (3)$ in additional $v$ dominates itself there is the vertex $[2]$ not dominating with any vertex in $S'$.

If $v = (5)$ then, there are two vertices $[6, 11]$ that adjacent to $v$ not dominating with any vertex in $S'$. If $v = (7)$ then, there are four vertices $[1, 8, 12, 13]$ that adjacent to $v$ not dominating with any vertex in $S'$.

If $v = (15)$ then, there are two vertices $[14, 21]$ that adjacent to $v$ not dominating with any vertex in $S'$.

If $v = (16)$ then, there are two vertices $[10, 17]$ that adjacent to $v$ not dominating with any vertex in $S'$.

If $v = (24)$ then, there are four vertices $[18, 19, 23, 30]$ that adjacent to $v$ not dominating with any vertex in $S'$. If $v = (26)$ then, there are three vertices $[20, 25, 27]$ that adjacent to $v$ not dominating with any vertex in $S'$.

If $v = (28)$ then, there is it is easy to show that there is no proper subset $S' \subset S$ dominating $Y_{3,5}$. So if $v = (3)$ then it dominates itself and there are three vertices $\{1, 2, 6\}$ not dominating with any vertex in $S'$. If $v = (7)$ then it dominates itself and the vertex $\{4\}$ that adjacent to $v$ not dominating with any vertex in $S'$. So if $v = (8)$ then it dominates itself and there are $\{5, 9, 11\}$ not dominating with any vertex in $S'$. Now, if $v = (15)$ then it dominates itself and there are $\{13, 14\}$ not dominating with any vertex in $S'$. So for all choices of $v$ we have at least two vertices not dominating with any vertex in $S'$.

We have a contradiction and $\gamma(Y_{3,5}) \geq 4$.

$\therefore \gamma(Y_{3,5}) = 4$.

**Lemma (5):** $\gamma(Y_{6,5}) = 8$

**Proof:** Let the vertices of this graph labeled as in Fig.9.
one vertex \(\{29\}\) that adjacent to \(v\) not dominating with any vertex in \(S'\). Hence \(S'\) is not dominating set since for all chooses of \(v\) we have at least one vertex not dominating with any vertex in \(S'\).

So \(\gamma(Y_{6,5}) \geq 8\) and \(\gamma(Y_{6,5}) = 8\). ■

**Theorem 4:** For \(m \geq 4\) and \(m \neq 6\), \(\gamma(Y_{m,5}) = \begin{cases} \frac{5m^2}{4} & \text{for } m \equiv 0 \pmod{4} \\ \frac{5m+3}{4} & \text{for } m \equiv 1 \pmod{4} \\ \frac{5m-2}{4} & \text{for } m \equiv 2 \pmod{4} \\ \frac{5m+1}{4} & \text{for } m \equiv 3 \pmod{4} \end{cases}\)

**Proof:** Let the vertices of this graph labeled by: \(V(Y_{m,5}) = \{1, 2, 3, ..., 5m\}\) see Fig. 10.

Similarly from above theorems we consider for \(m \geq 4\) and \(m \neq 6\) the four following cases:

**Case (1):** For \(m \equiv 0 \pmod{4}\)

First we give the set of dominating set of \(Y_{m,5}\) by the set of the vertices \(S\) where:

\[S = \{(i + m), (i + 4m + 1), (i + 2), (i + 2m + 1), (i + 3m + 3)\}|i = 1, 5, ..., m - 3}\]

\[\therefore |S| = \frac{5m}{4}, \text{ therefore } \gamma(Y_{m,5}) \leq |S| = \frac{5m}{4}\]

For any vertex \(v\) in \(S\), \(S' = S - \{v\} \subseteq S\), \(|S'| \leq \frac{5m}{4} - 1\) and \(S'\) be the dominating set of \(Y_{m,5}\).

According to the structure of labeling if we take \(v \in S\) then, we have five choices:

For \(v = \{(i + m), (i + 2), (i + 3m + 3)\}\) we note \(v\) dominates itself and all the vertices that adjacent to \(v\) which is not dominated with any vertex in \(S'\).

For \(v = (i + 2m + 1)\) dominate itself and one vertex of the form \((i + 2m + 2)\) which is not dominating with any vertex in \(S'\). Now, for \(v = (i + 4m + 1)\) dominates itself and the two vertices of the form \([(i + 4m + 2), (i + 4m)]\) that adjacent to \(v\) not dominating with any vertex in \(S'\). So for any chooses of \(v\) we have at least two vertices not dominating with any vertex in \(S'\), hence \(S'\) is not dominating set implies \(\gamma(Y_{m,5}) \geq \frac{5m}{4}\).

\[\therefore \gamma(Y_{m,5}) = \frac{5m}{4}\].

**Case (2):** For \(m \equiv 1 \pmod{4}\)

First we give the set of dominating set of \(Y_{m,5}\) by the set of the vertices:

\[S = S_1 \cup \{2m, 5m\}\]

Where:

\[S_1 = \{(i + 2), (i + m), (i + 2m + 3), (i + 3m + 1), (i + 4m + 3)|i = 1, 5, ..., m - 4\}\]

\[|S| = |S_1| + 2 = 5\left(\frac{m^2 - 1}{4}\right) + 2 = \frac{5m + 3}{4}\]

Therefore \(\gamma(Y_{m,5}) \leq |S| = \frac{5m + 3}{4}\). Now, let \(v\) be any vertex in \(S\). \(S' = S - \{v\} \subseteq S\), \(|S'| \leq \frac{5m + 3}{4} - 1\) and \(S'\) be the dominating set of \(Y_{m,5}\).

Similarly in case (1) we have five chooses when \(v \in S_1\).

For \(v = (i + 2), (i + 3m + 1)\) we note \(v\) dominates itself and all the vertices that adjacent to \(v\) which is not dominated with any vertex in \(S'\). For \(v = (i + m)\) we always at least one vertex of the form \((i + m + 1)\) which is not dominating with any vertex in \(S'\).

For \(v = (i + 2m + 3)\) dominates itself and one vertex of the form \((i + 2m + 2)\) which is not dominated any vertex in \(S'\).

Now, for \(v = (i + 4m + 3)\) dominates itself and then, there are two vertexes of the form \([(i + 4m + 1), (i + 4m + 2)]\) that adjacent to \(v\) not dominating with any vertex in \(S'\). To complete the proof remained when \(v = \{5m\}\) and \(v = \{2m\}\) where in two chooses of \(v\) there is at least one vertex of the form \((m)(4m)\) that adjacent to \(v\) not dominating with any vertex in \(S'\). So for any chooses of \(v\) we have always one vertex not dominating with any vertex in \(S'\) therefore \(S'\) is not a dominating set.

Hence \(\gamma(Y_{m,5}) \geq \frac{5m + 3}{4}\).

This implies that \(\gamma(Y_{m,5}) = \frac{5m + 3}{4}\).

**Case (3):** For \(m \equiv 2 \pmod{4}\) and \(m \neq 6\)

We give the set of dominating set of \(Y_{m,5}\) by the set of the vertices:

\[S = S_1 \cup S_2\]

Where:

\[S_1 = \{(i + 2), (i + m), (i + 2m + 3), (i + 3m + 1), (i + 4m + 3)|i = 1, 5, ..., m - 9\}\]

and \(S_1 = \{5m, (m - 1), (3m - 1), (2m - 3), (4m - 4), (m - 5), (5m - 2)\}\).

\[|S| = |S_1| + 7 = 5\left(\frac{m^2 - 1}{4}\right) + 7\]

Therefore \(\gamma(Y_{m,5}) \leq |S| = \frac{5m^2 - 2}{4}\).

Now, let \(v\) any vertex in \(S\). \(S' = S - \{v\} \subseteq S\), \(|S'| \leq \frac{5m^2 - 2}{4} - 1\) and \(S'\) be the dominating set of \(Y_{m,5}\) .

The proof is similarly in the case (2) when \(v \in S_1\) so to complete the proof remained when \(v \in S_2\) hence if \(v = \{5m\}\) which dominates itself and all the vertices that adjacent to \(v\) not dominating with any vertex in \(S'\).

If \(v = \{m - 1\}\) which dominates itself then we have two vertexes of the form \(\{m, m - 2\}\) that adjacent to \(v\) not dominating with any vertex in \(S'\).

If \(v = \{3m - 1\}\) which dominates itself then we have three vertexes of the form \(\{3m, 4m - 1\}\)
1),\((3m - 2)\) that adjacent to \(v\) not dominating with any vertex in \(S'\).
If \(V = \{(m - 5)\}\) then we have two vertices of the form \([(m - 4), (2m - 5)]\) that adjacent to \(v\) not dominating with any vertex in \(S'\).
Finally for the other choices of \(V = \{(5m - 2), (2m - 3), (4m - 4)\}\) which dominate itself and all the vertices that adjacent to \(v\) not dominating with any vertex in \(S'\). Therefore \(S'\) is not a dominating set.
\[ \therefore S' \text{ is not a dominating set, so we have a contradiction and } \gamma(Y_{m,5}) \geq \frac{5m-2}{4}. \]
Imply that \(\gamma(Y_{m,5}) = \frac{5m-2}{4} \).

**Case (4):** For \(m \equiv 3 \pmod{4}\).
We give the set of dominating set of \(Y_{m,5}\) by the set of the vertices \(S = S_1 \cup S_2\), where \(S_1 = \{(i + 2), (i + 3m + 1), (i + m), (i + 2m + 3), (i + 4m + 3)\} \) and \(S_2 = \{5m(m - 1), (3m - 1), (3m - 2)\}\).
\[ |S| = |S_1| + |S_2| = \frac{5m(m - 2)}{4} + 4 \]
Therefore \(\gamma(Y_{m,5}) \leq |S| = \frac{5m+1}{4}. \)
Now, let \(v\) any vertex in \(S'\), \(S' = S - \{v\} \subset S\), \(|S'| \leq \frac{5m+1}{4} - 1\) and \(S'\) be the dominating set of \(Y_{m,5}\).
If \(v \in S\) implies that \(v\) either in \(S_1\) or in \(S_2\).

**Reference**
حول العدد المهيمن لبعض الحالات الخاصة للبيان الموشوري
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الملخص
تعرف المجموعة المهيمنة للبيان {G} بأنها المجموعة الجزئية {S} من مجموعة الرؤوس {V} إذا كان لكل رأس ليس في {S} يجاور في الأقل {S} رأسا واحداً في {S}. ويعرف العدد المهيمن {\gamma(G)} بأنه حجم أصغر مجموعة ضمن المجموعات المهيمنة {Dominating sets}. يركز هذا البحث درسنا العدد المهيمن {\gamma(G)} لبعض الحالات الخاصة للبيان الموشوري {Prism graph}. 

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