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Nearly 2-Absorbing Submodules And Related Concepts

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Abstract

Throughout this note R is commutative ring with identity, and X be a left unitary R-module. A proper submodule K of an R-module X is called nearly prime, if whenever $\boldsymbol{r} \in \boldsymbol{R}$, $\boldsymbol{m} \in \boldsymbol{X}$, $\boldsymbol{rm} \in \boldsymbol{K}$ implies that either $\boldsymbol{m} \in \boldsymbol{K}+\boldsymbol{J}(\boldsymbol{X})$ or $\boldsymbol{r} \in [\boldsymbol{K}+\boldsymbol{J}(\boldsymbol{X}):\boldsymbol{X}]$. This concept us to introduce the concept of nearly prime submodule, where a proper submodule K of an R-module X is called nearly 2-absorbing submodule, if wherever $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$, $\boldsymbol{x} \in \boldsymbol{X}$ and $\boldsymbol{abx} \in \boldsymbol{K}$ then either $\boldsymbol{ax} \in \boldsymbol{K}+\boldsymbol{J}(\boldsymbol{X})$ or $\boldsymbol{bx} \in \boldsymbol{K}+\boldsymbol{J}(\boldsymbol{X})$ or $\boldsymbol{ab} \in [\boldsymbol{K}+\boldsymbol{J}(\boldsymbol{X}):\boldsymbol{X}]$. The aim of this paper is to study this concept, and gives some of its basic properties, characterization and examples. Furthermore, we study the relation of this concept with some kind of submodules .

1. Introduction

The concept of nearly prime submodule, was first introduce in [1]. A proper submodule K of an R-module X is called a prime if wherever $rx \in K, r \in R, x \in X$, implies that either $x \in K$ or $r \in [K:x]$ [2]. Darani and soheilnia in [3] introduced the concept of 2-absorbing submodule as generalization of prime submodule, where a proper submodule K of an R-module X is called 2-absorbing if wherever $a,b \in R, x \in X, abx \in K$ implies that either $ax \in K$ or $bx \in K$ or $ab \in [K:X]$. From this we generalization the concept of nearly prime submodule to nearly 2-absorbing submodule. Every prime submodule is nearly prime[1] and every prime submodule is 2-absorbing [3]. And we prime that Every nearly prime submodule is nearly 2-absorbing but the converse is not true in general. And every 2-absorbing submodule is nearly 2-absorbing and the converse is true under certain condition .

2. Nearly 2-absorbing submodules

We introduce in this part of this work the concept of nearly 2-absorbing submodule as a generalization of nearly prime submodule .

Definition (2.1)

A proper submodule K of anR-module X is called nearly 2-absorbing if wherever $a,b \in R, x \in X$ with $abx \in K$, implies that either $ax \in K+J(X)$ or $ab \in [K+J(X):X]$. An ideal I of a ring R is called nearly 2-absorbing if I is nearly 2-absorbing submodule of the R-module R. Where

$J(X) =$ *the inrersection of all maximal submodules of X*

Remark (2.2)

It is clear that every nearly prime submodule is nearly 2-absorbing submodule. However the converse is not true in general: For example: The submodule $K=(\bar{0})$ of the z-module Z_6 is not nearly prime submodule, since $2 \in Z_1 \bar{3} \in Z_6$with $2.\bar{3} \in K$, but $\bar{3} \in K+J(Z_6)=(\bar{0})+(\bar{0})=(\bar{0})$ and $2 \in [K+J(Z_6):Z_6]=[(\bar{0})+(\bar{0}):Z_6]=6Z$ but$(\bar{0})$ is nearly 2-absorbing submodule of Z_6 since $2.3.\bar{0} \in (\bar{0})$, with $2,3 \in Z, \bar{1} \in Z_6$ implies that $2.\bar{1} \notin (\bar{0})+J(Z_6)3.\bar{1} \notin (\bar{0})+J(Z_6)$ but $2.3 \in [(\bar{0})+J(Z_6):Z_6]=[(\bar{0})+(\bar{0}):Z_6]=6Z$.

Before we introduce the next proposition, we need to recall the following lemma .

Lemma (2.3) [4]

Let L and K are submodules of an R-module M, then

1. If $N \subseteq K$ then $[N:M] \subseteq [K:M]$.
2. If $N \subseteq K$ then $[N:M] \subseteq [N:K]$.

Proposition (2.4)

Let K,L be two submodule of an R-module X and $K \subseteq L$. If K is a nearly 2-absorbing of X and $J(X) \subseteq J(L)$, then K is nearly 2-absorbing submodule of L .

Proof

Let $abx \in K$, where $a,b \in R, x \in L$.Since K is nearly 2-absorbing submodule of X, so either $ax \in K+J(X)$ or $bx \in K+J(X)$ or $ab \in [K+J(X):X]$. But $J(X) \subseteq J(L)$. Hence either $ax \in K+J(L)$ or $bx \in$

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$K + J(L)$ or $ab \in [K + J(L): X] \subseteq [K + J(L): L]$by Lemma(2.3)[2].Therefore K is nearly 2-absorbing submodule of L .

Proposition (2.5)

Let L be a submodule of an R-module X. If $L + J(X)$ is nearly 2-absorbing, then L is nearly 2-absorbing .

Proof

Let $abx \in L$ with $a, b \in R, x \in X$. Hence $abx \in L \subseteq L + J(X)$, implies that, $abx \in L + J(X)$: Since $L + J(X)$ is nearly 2-absorbing submodule of X, then either $ax \in L + J(X) + J(X) = L + J(X)$ or $bx \in L + J(X) + J(X) = L + J(X)$ or $ab \in [L + J(X) + J(X) = Y] = [L + J(X): X]$. Hence L is nearly 2-absorbing in X .

Proposition (2.6)

Every prime submodule of an R-module Y is nearly 2-absorbing submodule of Y.

Proof

Let K be a prime submodule of Y, and $abx \in K$ where $a, b \in R, x \in K$ with $ax \notin K + J(Y)$, and $ab \notin [K + J(Y): Y]$.Since K is a prime submodule of Y, then we get $bx \in K \subseteq K + J(Y)$. Hence K is nearly 2-absorbing in Y .

Proposition (2.7)

If K is a 2-absorbing submodule of an R-module Y, then is nearly 2-absorbing in Y .

Proof

Assume that $aby \in K$, where $a, b \in R, x \in Y$. Since K is 2-absorbing, then either $ay \in K$ or $by \in K$ or $ab \in [K: Y]$, it follows that, either $ay \in K \subseteq K + J(Y)$ or $by \in K \subseteq K + J(Y)$ or $ab \in [K: Y] \subseteq [K + J(Y): Y]$ by Lemma(2.3)(1). Hence either $ax \in K + J(Y)$ or $bx \in K + J(Y)$ or $ab \in [K + J(Y): Y]$.

The following proposition give the converse of Proposition (2.7) .

Proposition (2.8)

Let L be a nearly 2-absorbing submodule of an R-module Y with $J(Y) \subseteq L$. Then L is 2-absorbing submodule of Y .

Proof

Assume that $aby \in L$, where $a, b \in R, y \in Y$, since L is nearly 2-absorbing in Y, then either $ay \in L + J(Y)$ or $by \in L + J(Y)$ or $ab \in [L + J(Y): Y]$. But $J(Y) \subseteq L$, it follows that $L + J(Y) = L$.Hence either $ay \in L$ or $bg \in L$ or $ab \in [L: Y]$ in Y .

Remark (2.9)

Let L and F are two submodule of an R-module X with $L \cong F$. If L is a nearly 2-absorbing in X, then F need not to be nearly 2-absorbing in X. As the following example clear that .

Let $L = 3Z$ and $F = 30Z$ are submodule of the Z-module Z, we have $3Z \cong 30Z$ and $3Z$ is nearly 2-absorbing in Z, but $30Z$ is not nearly 2-absorbing in Z since if we take $5.3.2 \in 30Z$ and $5.3 = 15 \notin [30Z + J(Z): Z] = 30Z$.

Proposition (2.10)

If a submodule L of an R-module X is nearly 2-absorbing with $J(X) \subseteq L$, then $[L: X]$ is nearly 2-absorbing ideal in R.

Proof

Since L is nearly 2-absorbing, with $J(X) \subseteq L$ by Proposition (2.8) L is 2-absorbing in X. Hence by [5, theo.1.1.9] we get $[L: X]$ is 2-absorbing ideal in R. Hence by Remark (2.2) $[L: X]$ is nearly 2-absorbing in R .

Recall that R is a good ring if $J(R).M = J(M)$ where M is an R-module [6].

Remark (2.11) [6]

If R is a good ring, and M is an R-module, and N is a submodule of M, then $J(M) \cap N = J(N)$.

Lemma (2.12) [6, Lemma 2.3.15]

Let M be an R-module, and A,B and C are submodules of M with B proper submodule of C. Then
$[(A + B) \cap C = (A \cap C) + (B \cap C) + (A \cap C) + B$.

Proposition(2.13)

Let X be an R-module over a good ring R, and F, L be two submodules of X such that $L \not\subseteq F$ and $J(X) \subseteq L$. If is a nearly 2-absorbing submodule of X, then $L \cap F$ is a nearly 2-absorbing submodule of L .

Proof

$L \cap F$ is a proper submodule of L because $L \not\subseteq F$. Assume that $rsx \in L \cap F$, where $r, s \in F$. But F is nearly 2-absorbing in X, then either $rx \in F + J(X)$ or $sx \in F + J(X)$ or $rs \in [F + J(X): X]$. That is either $rx \in (F + J(X)) \cap rL \subseteq (F + J(x)) \cap L$, implies that $rx \in (F + J(X)) \cap L = (L \cap F) + (J(X) \cap L)$ or $sx \in (F + J(X)) \cap sL \subseteq (F + J(X)) \cap L = (L \cap F) + (J(X) \cap L)$ or $rs \in (F + J(X))$, implies that $rsL = rsX \cap rsL \subseteq (F + J(X)) \cap rsL \subseteq (F + J(X)) \cap L = (L \cap F) + (J(X) \cap L)$ by Lemma (2.12).

But R is a good ring then we have $J(X) \cap L = J(L)$. Hence either $rx \in L \cap F + J(L)$ or $sx \in L \cap F + J(L)$ or $rsL \subseteq L \cap F + J(L)$, implies that $rs \in [L \cap F + J(L): L]$. Thus $L \cap F$ is nearly 2-absorbing in L .

Corollary(2.14)

Let X be an R-module over good ring, and F,L are submodule of X with $L \not\subseteq F$ and L is maximal submodule of X. If F is nearly 2-absorbing in X, then $L \cap F$ is nearly2-absorbing in L .

Proof

Since L is maximal submodule of X, then $J(X) \subseteq L$. Then by Proposition (2.13) $L \cap F$ is nearly 2-absorbing in L.

Corollary (2.15)

Let X be an R-module over good ring and F,L be are submodule of X with $L \not\subseteq F$ and $J\left(\frac{X}{L}\right) = (0)$. If F is a nearly 2-absorbing submodule of X, then $L \cap F$ is a nearly 2-absorbing submodule of L .

Proof

Since $J\left(\frac{X}{L}\right) = (0)$, then by [7, Rem. and Example (1.2.6)(11)] where $J(X) \subseteq L$. Hence the proof follows by Proposition (2.13) .

Proposition (2.16)

Let X be an R-module, and F and L are submodules of X with $L \not\subseteq F$ and $J(X) = J(L)$. If F is a nearly 2-

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absorbing in X, then $L \cap F$ is a nearly 2-absorbing in L .

Proof

Is similar as in Proposition (2.13) .

Proposition (2.17)

Let X be an R-module, and L,F are submodules of X, with $L \subseteq J(X)$ and $F \subseteq J(X)$. If L and F are nearly 2-absorbing in X, then $L \cap F$ is nearly 2-absorbing in X .

Proof

Assume that $rsx \in L \cap F$, where $r,s \in R$, $x \in X$, then $rsx \in L$ and $rsx \in F$. But both L,F are nearly 2-absorbing in X, so either $rx \in F + J(X)$ or $sx \in F + J(X)$ or $rs \in [F + J(X):X]$ and either $rx \in L + J(X)$ or $sx \in L + J(X)$ or $rs \in [L + J(X):X]$.But $F \subseteq J(X)$ and $L \subseteq J(X)$, so either $rx \in J(X)$ or $sx \in J(X)$ or $rs \in [J(X):X]$. If follows and , we have$L \cap F \subseteq J(X)$. Hence, either $rx \in L \cap F + J(X)$ or $sx \in L \cap F + J(X)$ or $rs \in [L \cap F + J(X):X]$. Thus $L \cap F$ is nearly 2-absorbing in X .

Proposition(2.18)

Let X be an R-module, and F is a submodule of X. If $F = \cap_{i \in \wedge} L_i$ where L_i is 2-absorbing submodule of X for each $i \in \wedge$, then F is a nearly 2-absorbing submodule in X.

Proof

Assume that $rsx \in F$, where $r,s \in R, x \in X$, then $rsx \in \cap_{i\in\wedge} L_i$, implies that $rsx \in L_i$ $\forall\, i \in \wedge$. But L_i is 2-absorbing in X, then either $rx \in L_i$ or $sx \in L_i$ or $rs \in [L_i : X]$, that is $rsx \subseteq L_i$ $\forall\, i \in \wedge$. If follows that $rx \in \cap_{i\in\wedge} L_i$ or $sx \in \cap_{i\in\wedge} L_i$ or $rsx \in \cap_{i\in\wedge} L_i$ for each $i \in \wedge$. Then either $rx \in \cap_{i\in\wedge} L_i \subseteq \cap_{i\in\wedge} L_i + J(X)$ or $sx \in \cap_{i\in\wedge} L_i \subseteq \cap_{i\in\wedge} L_i + J(X)$ or $rsx \in \cap_{i\in\wedge} L_i \subseteq \cap_{i\in\wedge} L_i + J(X)$, that is $rs \in [\cap_{i\in\wedge} L_i + J(X):X]$. Hence either $rx \in \cap_{i\in\wedge} L_i + J(X)$ or $sx \in \cap_{i\in\wedge} L_i + J(X)$ or $rs \in [\cap_{i\in\wedge} L_i + J(X):X]$. Therefore $\cap_{i\in\wedge} L_i$ is nearly 2-absorbingin X .

Proposition (2.19)

Let X be an R-module, and L is a proper nearly 2-absorbing submodule of X, then L_S is a nearly 2-absorbing submodule of R_S-module X_S, where S is a multiplicatively closed subset of R .

Proof

Let $\frac{r_1}{s_1}.\frac{r_2}{s_2}.\frac{x}{s_3} \in L_S$, where $\frac{r_1}{s_1},\frac{r_2}{s_2} \in R_S$ and $\frac{x}{s_3} \in X_S$, $r_1, r_2 \in R_1$, $s_1, s_2, s_3 \in S$, $x \in X$, then $\frac{r_1 r_2 x}{t} \in L_S$ where $t = s_1, s_2, s_3 \in S$, implies that there exists $t_1 \in S$ such that $t_1 r_1 r_2 x \in L$. But L is nearly 2-absorbing in X, Then either $t_1 r_1 x \in L + J(X)$ or $t_1 r_2 x \in L + J(X)$ or $t_1 r_1 r_2 x \in [L + J(X):X]$, it follows that either $\frac{t_1 r_1 x}{t_1 s_1 s_3} \in (L + J(X))_S \subseteq L_S + J(X_S)$ or $\frac{t_1 r_2 x}{t_1 s_2 s_3} \in (L + J(X))_S \subseteq L_S + J(X_S)$ or $\frac{t_1 r_1 x}{t_1 s_1 s_3} \in [L + J(X):X]_S \subseteq \left[(L + J(X))_S : X_S\right] \subseteq [L_S + J(X)_S : X_S]$ by [4]. Thus either $\frac{r_1}{s_1}\frac{x}{s_3} \in L_S + J(X_S)$ or $\frac{r_2}{s_2}\frac{x}{s_3} \in L_S + J(X_S)$ or $\frac{r_1}{s_1}\frac{x}{s_3} \in [L_S + J(X_S):X_S]$. Hence L_S is nearly 2-absorbing submodule of X_S .

Recall a submodule N of an R-module X is small if for any submodule A of X with $X = N + A$, then $A = X$[6]. And recall that an R-epimorphism $f: X \longrightarrow X'$ is called small e pimorphism if kerf small submodule in X where X, X' are R-modules[6] .

Proposition (2.20)

Let X, X' be R-modules and $g: X \longrightarrow X'$ be small epimorphism. If L is a nearly 2-absorbing submodule of X containing Ker g, then $g(L)$ is nearly 2-absorbing submodule of X' .

Proof

It is clear that $g(L)$ is a proper submodule of X' If note $g(L) = X'$ since if $x \in X$ such that $g(x) \in X' = g(L)$, then $\exists \in L$ such that $g(n) = g(x)$. Hence $g(n - x) = 0$, implies that $n - x \in \ker g \subseteq L$, then $x \in L$ hence $L = X$ contradiction. Now, assume that $rsx' \in g(L)$, where $r,s \in R, x' \in X'$. Since g is an epimorphism and $x' \in X'$, then $\exists x \in X$ such that $g(x) = x'$. So $rsx' = rsg(x) = g(rsx) \in g(L)$, then $\exists y \in L$ such that $g(y) = g(rsx)$, it follows that $g(rsx - y) = 0$, then $rsx - y \in \ker g \subseteq L$, it follows that $rsx \in L$, but L is nearly 2-absorbing submodule of X, then either $rx \in L + J(X)$ or $sx \in L + J(X)$ or $rs \in [L + J(X):X]$, and hence, either $rg(x) \in g(J) + g(J(X))$ or $sg(x) \in g(J) + g(J(X))$ or $rsX \subseteq L + J(X)$. It follows that either $rx^1 \in g(L) + g\,(J(X))$ or $sx' \in g(L) + g\,(J(X))$ or $rsg(X) \subseteq g(L) + g\,(J(X))$. But g is small epimorphism, so, either $rx' \in g(L) + J(X')$ or $sx' \in g(L) + J(X')$ $rsX \subseteq g(L) + J(X')$. Thus $rs \in [g(L) + J(X'):X]$. Hence $g(L)$ is a nearly 2-absorbing submodule in X' .

Proposition (2.21)

let $g: X \longrightarrow X'$ be small epimorphism, where X, X' are R-modules. If L is nearly 2-absorbing submodule of X', then $g^{-1}(L)$ is nearly 2-absorbing submodule of X.

Proof

Assume that $rsx \in g^{-1}(L)$, where $r,s \in R, x \in X$. And suppose that $rx \notin g^{-1}(L) + J(X)$ and $sx \notin g^{-1}(L) + J(X)$. To prove that $rs \in [g^{-1}(L) + J(X):X]$. Then $g(rx) = rg(X) \notin g[g^{-1}(L) + J(X)] = L + J(X')$ or $g(sx) = rg(X) \notin g[g^{-1}(L) + J(X)] = L + J(X')$ since g is small epimorphism. But $rsx \in g^{-1}(L)$, implies that $rsg(x) \in L$. But L is nearly 2-absorbing in X' and $rg(x) \notin L + J(X')$ and $sg(x) \notin L + J(X')$, hence $rs \in [L + J(X'):X']$, implies that $rsX' \subseteq L + J(X')$ it follows that $rsg(X) \subseteq L + J(X')$, so $g(rsx) \subseteq L + J(X')$, implies that $rsx \subseteq g^{-1}(L + J(X')) \subseteq g^{-1}(L) + J(X)$, so $rsx \in [g^{-1}(L) + J(X):X]$. Therefore $g^{-1}(L)$ is nearly 2-absorbing in X.

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المقاسات الجزئية المستحوذة من النمط – 2 تقريبا ومفاهيم ذات العلاقة

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الملخص

خلال هذا البحث R حلقة إبداليه بمحايد و X مقاسا احاديا ايسرا . يدعى المقاس الجزئي الفعلي K من المقاس X مقاسا اوليا تقريبا اذا كان $r \in R$ ، $m \in X$ بحيث ان $r\,m \in K$ يؤدي الى اما $m \in K+J(X)$ او $r \in [K+J(x):X]$. هذا المفهوم قادنا لكي نقدم مفهوم المقاس الجزئي المستحوذ من النمط – 2 تقريبا كأعمام للمقاس الجزئي الاولي تقريبا، حيث المقاس الجزئي الفعلي K يدعى مقاسا جزئيا مستحوذا من النمط – 2 تقريبا اذا كان $a, b \in R$ و $x \in X$ و $a\,b\,x \in K$ فانه يؤدي الى اما $a\,x \in K+J(X)$ او $b\,x \in K+J(X)$ او $a\,b \in [K+J(x):X]$. الهدف من هذا البحث هو دراسة هذا المفهوم واعطاءه بعض الصفات الاساسية والتشخيصات مع بعض الامثلة ، بالإضافة لذلك ندرس علاقته ببعض انواع المقاسات الجزيئية الاخرى.