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Partial b-Rectangular Metric Space with Some Results

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ABSTRACT

A new generalization of metric space called partial b-rectangular metric space is introduced. Also, the relation between this generalization and the other generalizations for example a b-rectangular metric space is given. Moreover, we have proved Banach theorem and Kannan theorem of fixed Point in partial b-rectangular metric space. Furthermore, some definitions and results dealing with partial b-rectangular metric space are discussed.

1. Introduction

The definition of metric space was presented by Freshet in 1906 [1]. Many authors after Freshet gave several generalization of metric space. In [2], Shukla S. gave the concept of partial b-metric space. The notion of b-rectangular metric space was given in [3]. Banach in 1922 introduced the famous and important type of fixed point which is Banach contraction principle [4]. In 1969, Kannan gave another type of fixed point which is Kannan fixed point theorem [5]. Many fixed point theorems were stated and proved in generalization metric space [6-14].

Definition 1[2]

Let S be a non-empty set. A function $L{:}\times S \to [0,\infty)$ is a partial b-metric on S if $\forall\, a, b$ and c in S:

(L1): $L(a,a) = L(a,b) = L(b,b)$ iff $a = b$.

(L2): $L(a,a) \le L(a,b)$.

(L3): $L(a,b) = L(b,a)$.

(L4): $L(a,b) \le k[L(a,c) + L(c,b)] - L(c,c)$, for some $k \in \mathbb{R}$, $k \ge 1$.

Then the pair (S, L) is said to be partial b-metric space, where $k \in \mathbb{R}, k \ge 1$ is the coefficient number of (S, L, k).

Definition 2[3]

Let S be a non-empty set. A function $N: S \times S \to [0,\infty)$ is a b-rectangular metric on S if $\forall\, a, b, c$ and d in S:

(N1): $N(a,b) = 0$ iff $a = b$.

(N2): $N(a,b) = N(b,a)$.

(N3): $N(a,b) \le k[N(a,c) + N(c,d) + N(d,b)]$, for some $k \in \mathbb{R}$, $k \ge 1$.

Then the pair (S, N) is said to be b-rectangular metric space, where $k \in \mathbb{R}$, $k \ge 1$ is the coefficient number of (S, N, k).

2. Partial b-Rectangular Metric Space

In this section we generalize the definitions of partial b-metric space and b-rectangular metric space, we named partial b-rectangular metric space.

Definition 3.

Let S be a non-empty set. A function $p_{br}: S \times S \to [0,\infty)$ is a partial b-rectangular metric on S if satisfies the following conditions: for a, b in S and for $c \ne d$ in $S - \{a, b\}$:

(PBR1): $p_{br}(a,a) = p_{br}(a,b) = p_{br}(b,b)$ if and only if $a = b$.

(PBR2): $p_{br}(a,a) \le p_{br}(a,b)$.

(PBR3): $p_{br}(a,b) = p_{br}(b,a)$.

(PBR4): $p_{br}(a,b) \le k[p_{br}(a,c) + p_{br}(c,d) + p_{br}(d,b)] - p_{br}(c,c) - p_{br}(d,d)$ for some $k \in \mathbb{R}, k \ge 1$.

Then the pair (S, p_{br}) is said to be partial b-rectangular metric space, where $k \in \mathbb{R}$, $k \ge 1$ is the coefficient number of (S, p_{br}, k).

We explain that by some examples and remarks.

Example 4.

Let $S = \left\{0, \frac{1}{4}, \frac{1}{2}, 1\right\}$. Define $p_{br}: S \times S \to (0,\infty)$ as $p_{br}(a,b) = \alpha + |a - b|$, α is a real number greater than zero. Then (S, p_{br}) is a partial b-rectangular metric space.

Proof: We see that the properties (PBR1), (PBR2) and (PBR3) are verified for all $a, b \in S$.

To proof (PBR4) consider:

$p_{br}(a,b) = \alpha + |a-b|$
$= \alpha + |a-c+c-d+d-b| - 2\alpha + 2\alpha$
$\leq \alpha + |a-c| + \alpha + |c-d| + \alpha + |d-b| - \alpha - \alpha$
$\leq k[(\alpha+|a-c|) + (\alpha+|c-d|) + (\alpha + |d-b|)] - \alpha - \alpha$ (for some $k \in \mathbb{R}, k \geq 1$)
$= k[p_{br}(a,c) + p_{br}(c,d) + p_{br}(d,b)] - p_{br}(c,c) - p_{br}(d,d)$
$\forall\, a,b \in S$ and for $c \neq d \in S - \{a,b\}$

Hence (S, p_{br}) is a partial b-rectangular metric space.

Example 5.

Let $A = \{0,1,2,3,4,5\}$, $A^2 = S$. Define $p_{br}: S \times S \to [0,\infty)$ as follows:

$p_{br}(a,b) = 2 + (|a_1 - a_2| + |b_1 - b_2|)$, for each $a = (a_1, b_1), b = (a_2, b_2)$ in S. Then (S, p_{br}) is a partial b-rectangular metric space.

Proof: We notice that the properties (PBR1), (PBR2) and (PBR3) are verified for all $a, b \in S$.

To proof (PBR4) consider:

$p_{br}(a,b) = 2 + (|a_1-a_2| + |b_1-b_2|)$
$\leq 2 + (|a_1-a_3| + |b_1-b_3|) + 2 + (|a_3-a_4| + |b_3-b_4|) + 2 + (|a_4-a_2| + |b_4-b_2|) - 2 - 2$
$\leq k\,[2 + (|a_1-a_3| + |b_1-b_3|)$
$+2 + (|a_3-a_4| + |b_3-b_4|)$
$+2 + (|a_4-a_2| + |b_4-b_2|)] - 2 - 2$
$= k[p_{br}(a,c) + p_{br}(c,d) + p_{br}(d,b)] - p_{br}(c,c) - p_{br}(d,d)$
$\forall\, a,b \in S$ and for $c \neq d \in S - \{a,b\}$, for some $k \in \mathbb{R}, k \geq 1$

Hence (S, p_{br}) is a partial b-rectangular metric space.

Remarks 6.

1. If $a, b \in S$ such that $p_{br}(a,b) = 0$ we have $a = b$. The opposite is not necessary true.
2. b-rectangular metric space is a partial b-rectangular metric space with the self-distance between two points equals zero but the convers not necessary true.

Note 7. In (Examples 4, 5) we have $p_{br}(a,a) \neq 0, \forall\, a = b$ in S. So the opposite of remark 1 not true and (S, p_{br}) is not a b-rectangular metric space.

3. Fixed Point in Partial b-Rectangular Metric Space

In this section we prove Banach contraction principle theorem and Kannan fixed point theorem in a partial b-rectangular metric space. In order to obtain that we need to give some definitions and information in a partial b-rectangular metric space.

Definition 8.

Let $< a_n >$ be a sequence in (S, p_{br}), we say that $< a_n >$ converges to $a \in S$, if $\forall\, \varepsilon > 0, \exists\, n_0(\varepsilon) \in \mathbb{N}$ such that $|p_{br}(a_n, a) - p_{br}(a,a)| < \varepsilon, \forall\, n > n_0(\varepsilon)$.

Definition 9.

Let $< a_n >$ be a sequence in (S, p_{br}), we say that $< a_n >$ Cauchy sequence, if $\forall\, \varepsilon > 0, \exists\, n_0(\varepsilon) \in \mathbb{N}$ such that $p_{br}(a_n, a_m) < \varepsilon, \forall\, n,m > n_0(\varepsilon)$, in other words $\lim_{n,m\to\infty} p_{br}(a_n, a_m)$ exists and finite.

Definition 10.

A partial b-rectangular metric space (S, p_{br}) is said to be complete if every Cauchy sequence in S is convergent, in other words $\lim_{n,m\to\infty} p_{br}(a_n, a_m) = \lim_{n\to\infty} p_{br}(a_n, a) = p_{br}(a,a)$.

Example 11.

Let $A = \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\}, B = \{0,1\}$ and $A \cup B = S$, $\beta \geq 0$. Define $p_{br}: S \times S \to [0,\infty)$ as follows:

$$p_{br}(a,b) = \begin{cases} \beta, & \text{if } a = b \text{ or } a,b \in B \\ \beta + \frac{1}{n+1}, & \text{if } a \in A \text{ and } b \in B \text{ or } b \in A \text{ and } a \in B \\ \beta + \frac{1}{2}, & \text{otherwise} \end{cases}$$

We see that the properties (PBR1), (PBR2) and (PBR3) are verified for all $a, b \in S$.

To proof (PBR4) we have:

$p_{br}(a,b) = \beta + \frac{1}{n+1} \leq k\left[\left(\beta + \frac{1}{n+1}\right) + \frac{1}{2} + \frac{1}{n+1}\right] + 2\beta - 2\beta$
$= k\left[\left(\beta + \frac{1}{n+1}\right) + \left(\beta + \frac{1}{2}\right) + \left(\beta + \frac{1}{n+1}\right)\right] - \beta - \beta$
$= k[p_{br}(a,c) + p_{br}(c,d) + p_{br}(d,b)] - p_{br}(c,c) - p_{br}(d,d)$
$\forall\, a,b \in S$ and for $c \neq d \in S - \{a,b\}$, for some $k \in \mathbb{R}, k \geq 1$.

So, (S, p_{br}) is a partial b-rectangular metric space.

In (S, p_{br}) the convergent sequence may have more than one convergence point and may not be Cauchy sequence.

Consider the following sequence

$$a_n = \begin{cases} \frac{1}{n+1} & , n \text{ is odd} \\ 1 & , n \text{ is even} \end{cases} \quad \text{in } S$$

$$\lim_{n\to\infty} p_{br}(a_n, 0) = \begin{cases} \lim_{n\to\infty} p_{br}\left(\frac{1}{n+1}, 0\right) & , n \text{ is odd} \\ \lim_{n\to\infty} p_{br}(1,0) & , n \text{ is even} \end{cases} = \begin{cases} \beta \\ \beta \end{cases}$$

$$\lim_{n\to\infty} p_{br}(a_n, 1) = \begin{cases} \lim_{n\to\infty} p_{br}\left(\frac{1}{n+1}, 1\right) & , n \text{ is odd} \\ \lim_{n\to\infty} p_{br}(1,1) & , n \text{ is even} \end{cases} = \begin{cases} \beta \\ \beta \end{cases}$$

So, $\langle a_n \rangle$ convergence to two different points 0 and 1.

Also, $\lim_{n\to\infty} p_{br}(a_{2n}, a_{2n+2}) = \beta + \frac{1}{2}$ and $\lim_{n\to\infty} p_{br}(a_{2n}, 1) = \beta$

It means the limit does not exist, so $\langle a_n \rangle$ is not Cauchy sequence.

Definition 12.

Let $\mathrm{T}: (S, p_{br}) \to (S, p_{br})$ be a mapping, we say that $a_0 \in S$ is a fixed point of T if $\mathrm{T}(a_0) = a_0, \forall a_0 \in S$.

Definition 13.

Let $\mathrm{T}: (S, p_{br}) \to (S, p_{br})$ be a mapping, then T is called a contraction mapping if $p_{br}(T(a), T(b)) \leq \theta p_{br}(a,b)$, where $0 \leq \theta < 1, \forall\, a,b \in S$.

In the following theorem we prove Banach contraction principle (Banach type of fixed Point) in partial b-rectangular metric space

Theorem 14. (Banach Type of fixed Point)

Let (S, p_{br}) be a complete partial b-rectangular metric space with a real number $k > 1$, let

$T:(S,p_{br}) \to (S,p_{br})$ be a mapping such that $p_{br}(T(a),T(b)) \leq \theta p_{br}(a,b)$, where $0 \leq \theta < \frac{1}{k}$, $\forall\, a,b \in S$, then there exists a unique fixed point $a_0 \in S$ of T and the self-distance between two points equals zero.

Proof: Firstly to prove if T has a fixed point $a_0 \in S$, then a_0 is unique. We shall prove that if $a_0 \in S$ is a fixed point of T, i.e. $T(a_0) = a_0$, then the self-distance between two points equals zero.

Suppose that $a_0 \in S$ is a fixed point of T, i.e. $T(a_0) = a_0$

Consider $p_{br}(a_0, a_0) = p_{br}(T(a_0), T(a_0)) \leq \theta p_{br}(a_0, a_0) < \frac{1}{k} p_{br}(a_0, a_0) < p_{br}(a_0, a_0)$

Which is a contradiction. Thus, we must have the self-distance between two points equals zero.

Now, to prove a_0 is a unique, suppose there exist two fixed points of T in S say a_0 and b_0 such that $a_0 \neq b_0$, i.e. $T(a_0) = a_0 \neq b_0 = T(b_0)$

Consider $p_{br}(a_0, b_0) = p_{br}(T(a_0), T(b_0)) \leq \theta\, p_{br}(a_0, b_0) < \frac{1}{k} p_{br}(a_0, b_0) < p_{br}(a_0, b_0)$

Which is a contradiction. Thus, we must have the self-distance between two points equals zero, i.e. $a_0 = b_0$.

Next, to prove the existence of fixed point, let $< a_n >$ be a sequence in S such that $T(a_n) = a_{n+1}$, $n = 0,1,2\cdots$, if $a_{n+1} = a_n$ then T has a fixed point which is a_n.

Assume that $a_{n+1} \neq a_n, n = 0,1,2,\cdots$, put $p_{brn} = p_{br}(a_n, a_{n+1})$

Consider

$p_{brn} = p_{br}(a_n, a_{n+1}) = p_{br}(T(a_{n-1}), T(a_n)) \leq \theta p_{br}(a_{n-1}, a_n) = \theta p_{brn-1}$

And

$\theta p_{brn-1} = \theta p_{br}(a_{n-1}, a_n) = \theta p_{br}(T(a_{n-2}), T(a_{n-1})) \leq \theta^2 p_{br}(a_{n-2}, a_{n-1}) = \theta^2 p_{brn-2}$

After repetition of this process we get:

$p_{brn} \leq \theta p_{brn-1} \leq \theta^2 p_{brn-2} \leq \cdots \leq \theta^{n} p_{br0}$ i.e. $p_{brn} \leq \theta^{n} p_{br0}$

Again put $p^*_{brn} = p_{br}(a_n, a_{n+2})$

Consider

$p^*_{brn} = p_{br}(a_n, a_{n+2}) = p_{br}(T(a_{n-1}), T(a_{n+1})) \leq \theta p_{br}(a_{n-1}, a_{n+1}) = \theta p^*_{brn-1}$

After repetition of this process we get:

$p^*_{brn} \leq \theta p^*_{brn-1} \leq \theta^2 p^*_{brn-2} \leq \cdots \leq \theta^{n} p^*_{br0}$ i.e. $p^*_{brn} \leq \theta^{n} p^*_{br0}$

To prove that $< a_n >$ is a Cauchy sequence considered $p_{br}(a_n, a_m), m = n + q, q > 0$, we have two cases:

Case 1: If $q = 2p + 1$ (odd) we have:

$p_{br}(a_n, a_{n+2p+1}) \leq k[p_{br}(a_n, a_{n+1}) + p_{br}(a_{n+1}, a_{n+2}) + p_{br}(a_{n+2}, a_{n+2p+1})] - p_{br}(a_{n+1}, a_{n+1})$
$- p_{br}(a_{n+2}, a_{n+2})$
$\leq k p_{br}(a_n, a_{n+1}) + k p_{br}(a_{n+1}, a_{n+2}) + k^2 p_{br}(a_{n+2}, a_{n+3}) + k^2 p_{br}(a_{n+3}, a_{n+4})$
$+ k^3 p_{br}(a_{n+4}, a_{n+5}) + k^3 p_{br}(a_{n+5}, a_{n+6}) + \cdots + k^p p_{br}(a_{n+2p}, a_{n+2p+1})$
$- p_{br}(a_{n+1}, a_{n+1}) - p_{br}(a_{n+2}, a_{n+2}) - p_{br}(a_{n+3}, a_{n+3}) - p_{br}(a_{n+4}, a_{n+4})$
$- p_{br}(a_{n+5}, a_{n+5}) - p_{br}(a_{n+6}, a_{n+6}) - \cdots - p_{br}(a_{n+2p}, a_{n+2p})$
$\leq k p_{br}(a_n, a_{n+1}) + k p_{br}(a_{n+1}, a_{n+2}) + k^2 p_{br}(a_{n+2}, a_{n+3}) + k^2 p_{br}(a_{n+3}, a_{n+4})$
$+ k^3 p_{br}(a_{n+4}, a_{n+5}) + k^3 p_{br}(a_{n+5}, a_{n+6}) + \cdots + k^p p_{br}(a_{n+2p}, a_{n+2p+1})$

Since $p^*_{brn} = p_{br}(a_n, a_{n+2})$, we have:

$p_{br}(a_n, a_{n+2p+1}) = k p_{brn} + k p_{brn+1} + k^2 p_{brn+2} + k^2 p_{brn+3} + k^3 p_{brn+4} + k^3 p_{brn+5} + \cdots + k^p p_{brn+2p}$

Since $p_{brn} \leq \theta^{n} p_{br0}$, we have:

$p_{br}(a_n, a_{n+2p+1}) \leq k\theta^{n} p_{br0} + k\theta^{n+1} p_{br0} + k^2\theta^{n+2} p_{br0} + k^2\theta^{n+3} p_{br0} + k^3\theta^{n+4} p_{br0} + k^3\theta^{n+5} p_{br0}$
$+ \cdots + k^p\theta^{n+2p-1} p_{br0} + k^p\theta^{n+2p} p_{br0}$
$= [(1 + k\theta^2 + k^2\theta^4 + \cdots + k^{p-1}\theta^{2p})$
$+ \theta(1 + k\theta^2 + k^2\theta^4 + \cdots$
$+ k^{p-1}\theta^{2p-2})]k\theta^{n} p_{br0}$
$\leq \frac{k\theta^{n} p_{br0}(1+\theta)}{1-k\theta^2}$ (since $k > 1$ and $0 \leq \theta < \frac{1}{k}$ so $k\theta^2 < 1$)

Case 2: If $q = 2p$ (even), by using the same way as the proof of the first case with $\theta^{n} p^*_{br0} \geq p^*_{brn}$ we get:

$p_{br}(a_n, a_{n+2p+1}) \leq \frac{k\theta^{n} p_{br0}(1+\theta) + \theta^{n-2} p^*_{br0}}{1-k\theta^2}$

From case 1 and case 2 we have $p_{br}(a_n, a_m) = 0$, as $n, m \to \infty, m = n + q, q > 0$

Hence $< a_n >$ is a Cauchy sequence in (S, p_{br}).

From completeness property of (S, p_{br}) we have $a_0 \in S$ such that $a_n \to a_0$ as $n \to \infty$

To prove $a_0 \in S$ is a fixed point of T

Consider, T satisfies $p_{br}(T(a), T(b)) \leq \theta p_{br}(a, b)$ we have:

$p_{br}(a_0, T(a_0)) \leq k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + p_{br}(a_{n+1}, T(a_0))] - p_{br}(a_n, a_n) - p_{br}(a_{n+1}, a_{n+1})$
$\leq$
$k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + p_{br}(T(a_n), T(a_0))]$
$\leq k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + \theta p_{br}(a_n, a_0)]$

Since $p_{br}(a_n, a_m) = 0$, $< a_n >$ converges to $a_0 \in S$ we obtain:

$p_{br}(a_0, T(a_0)) = 0 \implies a_0 = T(a_0)$

Hence T has a fixed point which is $a_0 \in S$.

In the following theorem we prove Kannan fixed point theorem in partial b-rectangular metric space

Theorem 15. (Kannan Type of fixed Point)

Let (S, p_{br}) be a complete partial b-rectangular metric space with a real number $k \geq 1$, let $T:(S,p_{br}) \to (S,p_{br})$ be a mapping such that $p_{br}(T(a),T(b)) \leq \theta[p_{br}(a,T(a)) + p_{br}(b,T(b))]$ where $0 \leq \theta < \frac{1}{2k}$, $\forall\, a,b \in S$, then there exists a unique fixed point $a_0 \in S$ of T and the self-distance between two points equals zero.

Proof: Firstly to prove if T has a fixed point $a_0 \in S$, then a_0 is unique. We shall prove that if $a_0 \in S$ is a fixed point of T, i.e. $T(a_0) = a_0$, then the self-distance between two points equals zero.

Suppose that $a_0 \in S$ is a fixed point of T, i.e. $T(a_0) = a_0$

Consider $p_{br}(a_0, a_0) = p_{br}(T(a_0), T(a_0))$
$\leq \theta[p_{br}(a_0, T(a_0)) + p_{br}(a_0, T(a_0))]$
$= 2\theta p_{br}(a_0, a_0) < \frac{1}{k} p_{br}(a_0, a_0) < p_{br}(a_0, a_0)$
Which is a contradiction. Thus, we must have the self-distance between two points equals zero.

Now, to prove a_0 is a unique, suppose there exist two fixed points of T in S say a_0 and b_0 such that $a_0 \neq b_0$, i.e. $T(a_0) = a_0 \neq b_0 = T(b_0)$, then we have the self-distance between two points equals zero.

Consider $p_{br}(a_0, b_0) = p_{br}(T(a_0), T(b_0))$
$\leq \theta[p_{br}(a_0, T(a_0)) + p_{br}(b_0, T(b_0))]$
$= \theta[p_{br}(a_0, a_0) + p_{br}(b_0, b_0)] = 0$
Which is contradiction. So, we must have the self-distance between two points equals zero, i.e. $a_0 = b_0$.

Next, to prove the existence of fixed point, let $< a_n >$ be a sequence in S such that $T(a_n) = a_{n+1}$, $n = 0,1,2 \cdots$, if $a_{n+1} = a_n$ then T has a fixed point which is a_n.

Assume that $a_{n+1} \neq a_n, n = 0,1,2,\cdots$, put $p_{brn} = p_{br}(a_n, a_{n+1})$
Consider $p_{brn} = p_{br}(a_n, a_{n+1})$
$= p_{br}(T(a_{n-1}), T(a_n))$
$\leq \theta[p_{br}(a_{n-1}, T(a_{n-1})) + p_{br}(a_n, T(a_n))]$
$= \theta[p_{br}(a_{n-1}, a_n) + p_{br}(a_n, a_{n+1})]$
$= \theta[p_{brn-1} + p_{brn}]$
After repetition of this process we get:
$p_{brn} \leq \left(\frac{\theta k}{1-\theta k}\right)^n p_{br0}$ (sice $k \geq 1$ and $0 \leq \theta < \frac{1}{2k}$ so $\theta k < 1$)
Thus, $\lim_{n\to\infty} p_{brn} = \lim_{n\to\infty} p_{br}(a_n, a_{n+1}) = 0$, it means for any $\varepsilon > 0$ we can find $n_0(\varepsilon) \in \mathbb{N}$ such that $p_{brn} < \frac{\varepsilon}{2}, \forall n > n_0(\varepsilon)$.

To prove that $< a_n >$ is a Cauchy sequence in S
Consider $p_{br}(a_n, a_m) = p_{br}(T(a_{n-1}), T(a_{m-1}))$
$\leq \theta[p_{br}(a_{n-1}, T(a_{n-1})) + p_{br}(a_{m-1}, T(a_{m-1}))]$
$= \theta[p_{br}(a_{n-1}, a_n) + p_{br}(a_{m-1}, a_m)]$
$= \theta[p_{brn-1} + p_{brm-1}]$
Since $p_{brn} = p_{br}(a_n, a_{n+1}) < \frac{\varepsilon}{2}, \forall n > n_0(\varepsilon)$, so $p_{brm} = p_{br}(a_m, a_{m+1}) < \frac{\varepsilon}{2}, \forall m > n_0(\varepsilon)$
Therefore, $p_{br}(a_n, a_m) < \varepsilon, \forall n, m > n_0(\varepsilon)$
Hence $< a_n >$ is a Cauchy sequence in (S, p_{br}) and $\lim_{n,m\to\infty} p_{br}(a_n, a_m) = 0$
From completeness property of (S, p_{br}) we have $a_0 \in S$ such that $a_n \to a_0 \; as \; n \to \infty$
$0 = \lim_{n,m\to\infty} p_{br}(a_n, a_m) = \lim_{n\to\infty} p_{br}(a_n, a_0) = p_{br}(a_0, a_0)$.
To prove $a_0 \in S$ is a fixed point of T
$p_{br}(a_0, T(a_0)) \leq k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + p_{br}(a_{n+1}, T(a_0))] - p_{br}(a_n, a_n) - p_{br}(a_{n+1}, a_{n+1})$
$\leq$
$k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + p_{br}(T(a_n), T(a_0))]$
$\leq$
$k[[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + \theta\,[p_{br}(a_n, T(a_n)) + p_{br}(a_0, T(a_0))]]$
$=$
$k[[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + \theta\,[p_{br}(a_n, a_{n+1}) + p_{br}(a_0, T(a_0))]]$
Since $\lim_{n\to\infty} p_{br}(a_n, a_0) = 0$ and $\lim_{n\to\infty} p_{brn} = \lim_{n\to\infty} p_{br}(a_n, a_{n+1}) = 0$, we obtain
$p_{br}(a_0, T(a_0)) = 0 \Longrightarrow a_0 = T(a_0)$
Hence T has a fixed point which is $a_0 \in S$.

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الفضاء المتري الجزئي من النمط b المستطيل مع بعض النتائج

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الملخص

قدمنا في هذا البحث تعميماً جديداً للفضاء المتري يدعى الفضاء المتري الجزئي من النمط b المستطيل. كذلك وضحنا العلاقة بين هذا التعميم والتعميمات الأخرى مثل الفضاء المتري من النمط b المستطيل. علاوةً على ذلك قمنا ببرهان مبدأ بناخ الانكماشي للنقطة الصامدة ومبرهنة كانان للنقطة الصامدة في الفضاء المتري الجزئي من النمط b المستطيل. بالإضافة الى ذلك ناقشنا بعض التعاريف والنتائج المتعلقة بالفضاء المتري الجزئي من النمط b المستطيل.