The Moment for Some quotient Stochastic Differential Equation with Application

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ABSTRACT
In this article we use the product stochastic differential equations in order to study the solution for some quotient stochastic differential equation by using iô’s formula, Then we find their moments (mean, variance and the k.th moments). Also we gave some examples to explain the method.

1. Introduction
The main definition of the Stochastic differential equations (simply SDE’s) is that differential equations in which one or more of its terms are stochastic (random) processes, for which their solutions may be stochastic process, [Arnold, 1974. The Stochastic differential equations (SDEs) used in many field of science such as biology, chemistry, climatology, mechanics, physics, economics and finance. Many researchers have given their contribution in these field (Akinbo B.J., et. al. (2015)), Guangqiang LAN et. al. (2014) derive the new sufficient conditions of existence, moment of the solution of stochastic differential equation, J.C. Jimenez (2015) uses the explicit formulas for the mean and variance of the solutions of linear stochastic differential equation, Platens [5] study the strong and weak approximation methods for the numerical methods to get the solution of stochastic differential equations, Nayak and Chakraverty [6] worked on numerical solution of fuzzy stochastic differential equation. Christios H.skiadas, [7] Study the exact solution of stochastic differential equation (Gomertz, Generalized logistic and revised exponential. Akinbo B.J. et al [2] study numerical solution of stochastic differential equation, and so on.

In this paper we study some form of stochastic differential equation as a quotient stochastic differential equation, then we explain how to apply iô’s integral formula to find the solution of those equations and we find the moments of their solutions.

2. Preliminaries and method
Definition 1 : (random variable) [5]
A random variable is a mapping or a function from the sample space Ω onto the real line R, (i.e. X: Ω →R)
Definition 2 : (Expectation) or (mean) of a random variable:[5]
Let X is a random variable defined on the probability space (Ω, F, P), then the expected values or the mean of X is:
E(X) = µ =∑ixi P(xi).
That is the average of X over the entire probability space.
For a random variable continuous over R:
E(X) = ∫∞-∞xf(x)dx
Definition 3: (Variance):[5]
The Variance is a measure of the spread of data about the mean µ Var(X) = E((X − µ)²) = E(X²) − (E(X))²
Definition 4: The kth -order moment:[5]
The kth -order moment of a continuous random variable is defined by:
E(Xk) = ∫∞-∞xkf(x)dx
Where \( f(x) \) is the probability density function

Or \( E(x^k) = \sum x^k_i p(x_i) \); (For discrete time and \( p(x) \) is probability mass function)

**Definition 5:** (stochastic process) [1]

A stochastic process is a family of random variables denoted by \( x(t), t \in T \) where \( t \) is time parameter and \( T \in R \).

**Definition 6:** (Wiener process) [1]

A wiener process (Brownian motion) over \([0, T]\) denoted by \( \{w(t)\} \) is a continuous-time stochastic process satisfying:

1: \( W(0) = 0 \)
2: For all \( t, s \geq 0 \), \( W(t) - W(s) \) is normally distributed with mean zero and variance \((t - s)\).
3: The increments’ \( W(t) - W(s) \) and \( W(v) - W(u) \) are independent.

**Definition 7:** (Itô -formula)[1]

Let \( X(.) \) be a real-valued stochastic process which satisfy

\[
\begin{align*}
E\left( \int_0^t |x(s)| \, ds \right) &< \infty \\
E\left( \int_0^t |y(s)| \, ds \right) &< \infty
\end{align*}
\]

If \( u : R \times [0, T] \rightarrow R \) is continuous and their first and second derivative for \( t \) exist and are continuous.

If we take \( Y(t) = u(x(t), t) \), then we have the following Itô formula:

\[
dY = \left( \frac{\partial u}{\partial x} F_t + \frac{\partial u}{\partial t} G_t + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (G_t)^2 \right) dt + \frac{\partial u}{\partial x} G_t dw
\]

**Theorem 1:** [1]

Let \( u(x) = x^m, \; m = 0, 1, 2, \ldots \) then \( d(x^m) = mx^{m-1}dx + \frac{1}{2}m(m-1)x^{m-2}G^2 dt \) ...(3)

**see [1]***

**Lemma 1:** [2]

Let \( w_t \) is a Brownian motion then, by Itô’s Formula, we have:

\( (dw_t)^2 = dt \), \( dt dw_t = 0 \) and \( (dt)^2 = 0 \). ...(4)

**Lemma 2:** [4]

Suppose \( \{w_t\} \) is a Brownian motion then, by using Itô's Formula, we get:

\( dw_t^2 = 2w_t dw_t + dt \), \( (dw_t)^2 \rightarrow dt, \) \( dt dw_t = 0 \) and \( dw_t^2 = 3w_t dw_t + 3w_t dt \) ...(5)

**Theorem 2:** (Itô product rule) [1]

Let \( dx_t = \alpha_t dt + \beta_t dw_t(t) \); (i = 1, 2)

\((0 \leq t \leq T); \) \( \alpha_t(t) L^1(0, T), \) \( \beta_t(t) L^2(0, T). \) Then

\[
d(x_1x_2) = X_1(t)dx_2(t) + X_2(t)dx_1(t) + \beta_1(t)\beta_2(t)dt.
\]

Let \( \alpha_t = \alpha; \) \( \beta_t = \beta \) independent of \( t \), where \( i = 1, 2 \)

Therefore \( d(x_1x_2)(t) = X_1(t)dx_2(t) + X_2(t)dx_1(t) + \beta_1(t) \beta_2(t)dt \) ...(6)

**3: Propositions:**

**a:** The quotient stochastic differential equation of the first order:

Let \( x_1 \) and \( x_2 \) are two stochastic processes and time independent. Then by using Itô Formula we have:

\[
d\left( \frac{x_1}{x_2} \right) = \left( \frac{x_1F_2 + \frac{F_1}{x_2} + G_1G_2}{x_1G_2 + \frac{G_1}{x_2}} \right) dw + \frac{1}{x_2} \left( \frac{x_1F_2 + \frac{F_1}{x_2} + G_1G_2}{x_1G_2 + \frac{G_1}{x_2}} \right) dt
\]

**Proof:** from equation (6) then we have

\[
\frac{dx_1}{x_2} = dx_1 \frac{1}{x_2} + \frac{1}{x_2} dx_1 + G_1G_2 dt \ldots (8)
\]

\[
dx_1 = F_1 dt + G_1 dw; \; dz = F_2 dt + G_2 dw
\]

**Remark:** [3]

\( L^1[0, T], L^2[0, T] \) denotes the space of all real-valued, adaptive processes \( \{x_t\} \), \( \{y_t\} \) respectively, such that

\[
E\left( \int_0^t |x| \, dt \right) < \infty
\]

\( E\left( \int_0^t |y| \, dt \right) < \infty
\]

**b:** The quotient stochastic differential equation of degree two (i.e. \( d\left( \frac{x_1}{x_2} \right)^2 \))

\[
d\left( \frac{x_1}{x_2} \right)^2 = \left( \frac{2x_1F_1}{x_2} + \frac{2x_2F_2}{x_1} + 4x_1G_1G_2 \right) dt + \left( \frac{2x_1G_1}{x_2^2} \right) dw \ldots (10)
\]

**proof:**

\[
\frac{dx_1}{x_2} = \frac{1}{x_2} \; dz \quad \text{and} \quad dz^2 = \frac{1}{x_2^2} \; dt.
\]

From (2) then we have

\[
\frac{dx_1}{x_2} = \frac{1}{x_2} \; dw + \frac{1}{x_2} \; dx_1 + G_1G_2 dt
\]

\[
d\left( \frac{x_1}{x_2} \right)^2 = 2x_1Z_1 dx_1 + 2x_1^2Z_2 dx_1 + 2x_1Z_3 dt + dx_1^2
\]

By using theorem(1), we have

\[
t^2 = 2x_1dx_1 + G_1G_2 dt; \; dx_z^2 = 2x_2 dx_1 + G_2 dt
\]

Then,

\[
\frac{dx_1}{x_2} = \frac{1}{x_2} \; dw + \frac{1}{x_2} \; dx_1 + G_1G_2 dt
\]

\[
dx_1^2 = 2x_1Z_1 dx_1 + 2x_1^2Z_2 dx_1 + 2x_1Z_3 dt + dx_1^2
\]

\[
dx_1Z_2 = 2x_1Z_2 dx_1 + 2x_1^2Z_3 dt + dx_1^2
\]

\[
dx_1^2 = 2x_1Z_1 dx_1 + 2x_1^2Z_2 dx_1 + 2x_1Z_3 dt + dx_1^2
\]

\[
dx_1^2 = 2x_1Z_1 dx_1 + 2x_1^2Z_2 dx_1 + 2x_1Z_3 dt + dx_1^2
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\]

\[
dx_1^2 = 2x_1Z_1 dx_1 + 2x_1^2Z_2 dx_1 + 2x_1Z_3 dt + dx_1^2
\]
4. The moment

In this paragraph we find the moments to the solution of the Quotient stochastic differential equation (Mean, Variance and the k-moment) by using the above proposition:

Let we have

\[ d \left( \frac{x_1}{x_2} \right) = \left( \frac{F_1}{x_2} + \frac{F_2}{x_2} + G_1 + G_2 \right) dt + \left( x_1 G_2 + \frac{F_1}{x_2} + G_2 \right) dw \]

Then the mean of \( \left( \frac{x_1}{x_2} \right) \) is

\[ E \left( \frac{x_1}{x_2} \right) = E \left( \frac{x_1(0)}{x_2(0)} \right) + E \left( \int_0^t \left( \frac{x_1 F_2}{x_2} + \frac{F_1}{x_2} + G_1 + G_2 \right) ds \right) + E \left( \int_0^t x_1 G_2 + \frac{F_1}{x_2} + G_2 \right) dw_s \]

Then \( \text{Var}[X] = E[X^2] - (E[X])^2 \), where \( X = \frac{x_1}{x_2} \)

The moment for general form (equation (13)) or the k\(^{th}\) moment is:

\[ E \left( \frac{x_1^k}{x_2^k} \right) = E \left( \frac{x_1^k(0)}{x_2^k(0)} \right) + E \left( \int_0^t \left( \frac{mx_1^{m-1}F_1}{x_2^k} + \frac{mx_1^{m-1}F_2}{x_2} + \frac{mx_1^{m-1}G_1}{x_2} \right) ds \right) + E \left( \int_0^t \left( \frac{mx_1^{m-1}F_1}{x_2^k} + \frac{mx_1^{m-1}G_1}{x_2} \right) dw_s \right) \]

Example: (1)

Suppose \( d \left( \frac{x_1}{x_2} \right) = \left( \frac{1}{x^2} \right) dw \) or we can write it as \( dx_1 = s dw \) and \( dx_2 = rdw \)

s and r are constants, also let \( x_1 = t^2 + 1 \), \( x_2 = t \), where t is a scalar (time) . By using Itô’s formula find \( E \left( \frac{x_1}{x_2} \right) \) and \( \text{Var} \left( \frac{x_1}{x_2} \right) \).

Solution: To find the mean of \( \frac{x_1(t)}{x_2(t)} \):

Let \( z = \frac{1}{x_2} \) then \( dz = d \left( \frac{1}{x_2} \right) \), by equation (2) we get

\[ dx_1 = F_1 dt + G_1 dw ; \quad dz = F_2 dt + G_2 dw \]

Then

\[ d(x_1, x_2) = (x_1 F_2 + z F_1 + G_1 + G_2) dt + (x_1 G_2 + G_2) dw \]

So

\[ \int_0^t d(x_1, x_2) = \int_0^t srdw + \int_0^t (sz + rx_1)dw \]

Then the expected value (mean) of \( \frac{x_1(t)}{x_2(t)} \) is:

\[ E \left( \frac{x_1(t)}{x_2(t)} \right) = E \left( \frac{x_1(0)}{x_2(0)} + \int_0^t \left( s \frac{dz}{x_2} + rz \right) dt \right) \]

The variance: First we need to find \( E \left( \frac{x_1(t)}{x_2(t)} \right)^2 \).
Let $z = \frac{1}{x_2}$ , $z^2 = \frac{1}{x_2^2}$ and $dz^2 = d\left(\frac{1}{x_2}\right)^2$

\(d\left(x_2^2z^2\right) = (2x_2^2z^2F_1 + 2x_2^2zF_2 + 4x_2zG_1G_2)dt + (2x_2zG_1 + 2x_2^2zG_2)dw\)

\(\int_{0}^{t} d\left(x_2^2z^2\right) = \int_{0}^{t} d\left(2x_2^2z^2F_1 + 2x_2^2zF_2 + 4x_2zG_1G_2\right)ds + \int_{0}^{t} d\left(2x_2zG_1 + 2x_2^2zG_2\right)dw_s\)

\(x^2_1(t)z^2(t) = x^2_1(0)z^2(0) + \int_{0}^{t} 4sz_1zds + \int_{0}^{t} 2sx_2z + r^2x_1z^2 dz\)

Where $z = \frac{1}{x_2}$ and $z^2 = \frac{1}{x_2^2}$ then

\(E \left(x^2_1(t)/x^2_2(0)\right) = E \left(x^2_1(0)/x^2_2(0)\right) + \int_{0}^{t} 4sz_1zds = \frac{x^2_1(0)}{x^2_2(0)} + 4sr\int_{0}^{t} x_1^2 ds = \frac{x^2_1(0)}{x^2_2(0)} + 4sz^2_1z^2 d\)

\(\int_{0}^{t} E \left(z^2_1\right) ds = \frac{x^2_1(0)}{x^2_2(0)} + 4sz^2_1z^2 + 4sz_1z^2 = \frac{x^2_1(0)}{x^2_2(0)} + 4sz^2_1z^2 + 4sz_1z^2 + 2s^2r^2t^2\)

\(\text{var} \left(z^2_1\right) = E \left(z^4_1\right) - E \left(z^2_1\right)^2 = \left(\frac{x^2_1(0)}{x^2_2(0)} + 4sz^2_1z^2 + 4sz_1z^2 + 2s^2r^2t^2\right) - \left(\frac{x^2_1(0)}{x^2_2(0)} + 4sz^2_1z^2 + 4sz_1z^2\right)^2\)

\(\text{In the same way we can find the higher moment.}\)

Example (2): Suppose \(dx_i = F_i dt + G_i dw\), \(i=1,2\) , \(x_i = \frac{x_i}{x_2}\) and let \(dx_1 = t^2 dt + 2tdw\) and \(dx_2 = t^2 dt + 4tdw\) . Then by using Itô's formula find \(E \left(z^2_1\right), \text{var} \left(z^2_1\right)\), where \(x_1(0) = 0, x_2(0) \neq 0\).

\(\text{Solution: we have }\)

\(d\left(\frac{1}{x_2}\right) = \left(\frac{x_2F_1 + 2x_2F_2 + 4x_1G_1G_2}{x_2^2} dt + \frac{G_1}{x_2^2} dw\right) + \left(\frac{x_2G_1}{x_2} + \frac{2x_2G_2}{x_2} + 4x_1G_1G_2\right)\frac{dt}{d\left(\frac{1}{x_2}\right)}\)

\(\int_{0}^{t} d\left(\frac{1}{x_2}\right) = \int_{0}^{t} d\left(\frac{x_2F_1 + 2x_2F_2 + 4x_1G_1G_2}{x_2^2} dt + \frac{G_1}{x_2^2} dw\right) + \int_{0}^{t} d\left(\frac{x_2G_1}{x_2} + \frac{2x_2G_2}{x_2} + 4x_1G_1G_2\right)\frac{dt}{d\left(\frac{1}{x_2}\right)}\)

\(= \int_{0}^{t} \left(\left(\frac{x_2F_1 + 2x_2F_2 + 4x_1G_1G_2}{x_2^2} dt + \frac{G_1}{x_2^2} dw\right) + \left(\frac{x_2G_1}{x_2} + \frac{2x_2G_2}{x_2} + 4x_1G_1G_2\right)\frac{dt}{d\left(\frac{1}{x_2}\right)}\right)\)

\(\text{To find the variance of } \left(z^2_1\right) \text{ we need } E \left(\frac{z^2_1}{x_2^2}\right)^2\)

\(E \left(\frac{z^2_1}{x_2^2}\right)^2 = \left(\frac{x_2F_1 + 2x_2F_2 + 4x_1G_1G_2}{x_2^2} dt + \frac{G_1}{x_2^2} dw\right) + \left(\frac{x_2G_1}{x_2} + \frac{2x_2G_2}{x_2} + 4x_1G_1G_2\right)\frac{dt}{d\left(\frac{1}{x_2}\right)}\)
So that
\[
\text{Var}(\frac{x_1}{x_2}) = \text{E}(\frac{2x_1}{x_2})^2 - (\text{E} \left( \frac{x_1}{x_2} \right))^2 = \frac{1}{2} t - \left( \frac{1}{2} \ln(t) \right)^2 = \frac{1}{2} t - \frac{1}{2} \ln(t)
\]
For the higher moment we can use the same way.

5. Conclusion
In this paper, we showed using Itô’s formula that the quotient stochastic differential equations can be found by the same method for product stochastic differential equations with some attention when we using their theorem’s. (That is, Itô’s formula is valid for rational form of the functions \(u(x(t), t)\) of the variables \(x_1\) and \(x_2\)). Also we find the moments to the solution of the Quotient stochastic differential equation by using the above proposition with some examples.

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