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Dynamical Behavior of some families of cubic functions in complex plane

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ABSTRACT

The current study deals with the dynamical behavior of three cubic functions in the complex plane. Critical and fixed points of all of them were studied . Properties of every point were studied and the nature of them was determined if it is either attracting or repelling. First function $f(z,\alpha)=\alpha z(1-z^2)$ such that $\alpha, z \in \mathbb{C}$, have two critical points $z_{1,2}=\mp\frac{1}{\sqrt{3}}$ and three fixed points $z_1=0, z_{2,3}=\mp\sqrt{\frac{\alpha-1}{\alpha}}$ such that $z_1=0$ is attracting when $|\alpha|<1$ is origin point As shown in figure (2).And $z_{2,3}=\mp\sqrt{\frac{\alpha-1}{\alpha}}$ are attracting when $|3-2\alpha|<1$ is the region specified by open disc $|3-2\alpha|<1$ shown in figure (1.(c)).Second function $f(z,\alpha)=\alpha z^3+(1-\alpha)z$ such that $\alpha, z \in \mathbb{C}$, have two critical points $z_{1,2}=\mp\sqrt{\frac{\alpha-1}{3\alpha}}$ and three fixed points $z_1=0, z_{2,3}=\mp 1$ such that $z_1=0$ is attracting when $|1-\alpha|<1$ and that its path is to the origin point as shown in figure (4).And $z_{2,3}=\mp 1$ are attractive when $|2\alpha+1|<1$ represents the open disc shown in the figure (3.(c)).Third function $f(z,\alpha)=\alpha z^3$ such that $\alpha, z \in \mathbb{C}$, have one critical point $z=0$ and three fixed points $z_1=0$ is attracting that is path is the origin point and$z_{2,3}=\frac{1}{\sqrt{\alpha}}$ are repelling as shown in figure (5). And all 2-cycles of $f(z,\alpha)=\alpha z^3$ are repelling and unstable .

1. Introduction

Nonlinear systems have played an important role in the study of natural phenomena because Nonlinear dynamics is concerned with these systems whose time evolution equations or differential equations such as the dynamical variables describing the system properties appear in the equation in a nonlinear form [1] and many engineering, social, and biological phenomena are dynamic, and these Phenomena can be modeled as dynamical systems [2]. Most dynamical systems are presented by difference equations or ordinary differential equations. Generally, these systems are nonlinear and contain various parameters. Little modification on their Parameters values may give big effects on the system behaviors. The main problem is determining a manner for analyzing such a dynamical system. A large number of proposed methods have been found to analyze nonlinear dynamical systems. Analytically, the solution for most nonlinear dynamical systems cannot be acquired. When the system parameter changes slightly, the properties of the solution to the Dynamical system may change. The phenomenon is named bifurcation, as a Chaotic state or from a stable equilibrium state to oscillating motion. The analysis system is a manner for gaining deep insights into the essential properties of dynamical systems. Additionally, bifurcation analysis enables us to determine the total behavior of the solution in the large, the range of a parameter over which a system behaves stable, and the transition mechanisms of the dynamic responses [3]. If the parameter of the system is changed, then, a bifurcation occurs as a system which is a qualitative change in the dynamics. A bifurcation diagram presents the potential long-term values a system variable can gain a function of a parameter of the logistic map chaotic dynamical discrete systems classes of dynamical systems. In the last two decades, the study of fixed points was studied and their properties were studied and nature was either attracting or repelling. A lot of research has been presented in this area where fixed points, their characteristic and schemes have been functions in the real plane, Dynamical systems has acquired much attention. The utilize of the concept "chaos" was firstly presented into dynamical systems by Li and Yorke (Li and Yorke, [4] for a Map. Zhou presents a chaos definition for a topological dynamical system on a general metric space (Zhou [5]) based on the definition of Li and Yorke. Another explicit Definition of chaos belongs to Devaney (Devaney[6]). Lots of manners quantitative measurement of complicated or chaotic nature of the dynamic are found for more specifics , see [1][7]. In this paper we were presented a set of cubic functions in the complex plane and studied their fixed points and their properties.

2. Preliminaries

This section consists of the basic definitions and theorems without proof that which we have used in this paper.

Definition 2.1 (Discrete Dynamical System) [8]

A discrete dynamical system is a study of how things change over Time. A discrete dynamical system on x is just a map $f: x \to x$. This map describe the deterministic evolution of some physical system: if the system is in state x at time t. then it shall be $f(x)$ is in the time $t+1$. Study of a discrete dynamical system is concerned with iterates of the Map : the sequence $x, f(x), f^{[2]}(x), \ldots$ Is called the orbit of x. The definition of the orbit , will be give later on

Definition 2.2 (Orbit)[4]

Let $f(x)$ a defined function in a domain. The function's orbit, referred to as $f^n(x)$, is created for point x by repeating a function starting from that point to obtain a list of numbers. The front orbit of x is a set of points into the sequence $x, f(x), f(f(x))$, $f(f(f(x)))$, more concise written $x, f(x), f^2(x), f^3(x), \ldots$ and can be referred to as $f^n(x)$ for $n \in N$.

Definition 2.3(Fixed point) [12] [11]

A point x^* is said to be fixed point of the map f or an equilibrium point if $f(x^*) = x^*$.

Remark 2.1

Geometrically , the fixed points of $f(x)$ are the intersection points of the Graphs of the two curves $y = f(x)$ and $y = x$.

Theorem 2.1 [12]

Suppose that f is continuous at c , and let x be in the domain of f .If $f^{[n]}(x) \to c$ as $n \to \infty$, then c is a fixed point of f .

Definition 2.4 (Hyperbolic)[11]

A fixed point c of f is hyperbolic if $|f'(c)| \neq 1$. If $|f'(c)| = 1$ it is non-hyperbolic. The reasons for these names becomes apparent when one looks at the fixed points of maps $f: R^2 \to R^2$ Thus if c is anon- hyperbolic fixed point, then $f'(c) = 1$ or $f'(c) = -1$, the graph of $f(x)$ eithers meets the line $y = x$ tangentially, or at 90°.

Definition 2.5(Asymptotically) [11]

Let p be fixed point of f :

a- point p is an attracting fixed point of f provided in which an interval $(p\text{-}\epsilon, p+\epsilon)$ containing p such that if x is in the domain of f and in $(p-\epsilon, p+\epsilon)$, then $f^{[n]}(x) \to p$ as n increases without bound .(Such a point is also called asymptotically stable in the literature).

b- point p is a repelling (unstable) fixed point of f provided that there is an interval $(p-\epsilon, p+\epsilon)$ containing p such that if x is in these domain and in $(p-\epsilon, p+\epsilon)$ but $x \neq p$, then $|f(x) - p| > |x - p|$

c- The point p is an asymptotically stable fixed point if it is both stable and attracting .

Theorem 2.1 [6] [7]

Suppose that f is differentiable at fixed point x .

- If $|f'(x)| < 1$, then x is attracting (stable).
- If $|f'(x)| > 1$, then x is repelling (unstable).
- If $|f'(x)| = 1$, then x can be attracting , repelling or nether .

.**Definition 2.6(Periodic orbit) [6]**

Let x_0 be in the domain of f . Then x_0 has period n or (is a period-n point) if $f^{[n]}(x_0) = x_0$ and if in addition $x_0, f(x_0), f^2(x_0), \ldots, f^{[n-1]}(x_0$)are distinct. If x_0 has period n , then the orbit of x_0 , which is $\{x_0, f(x_0), f^2(x_0), \ldots, f^{[n-1]}(x_0)\}$ is **a periodic orbit and is called an n-cycle.** A point x is eventually periodic of period n if x is not periodic but there exist$m > 0$ such that $f^{n+i}(x) = f^i(x)$ for all $i > m$,that is $f^i(x)$ is a periodic for $i \geq m$. .

3. Cubic Functions in the Complex Plane

This section deals with the study of the dynamical behavior of three cubic functions in the complex plane.

1- $f(z, \alpha) = \alpha z(1 - z^2)$, $\alpha \in \mathbb{C}, z \in \mathbb{C}$

2- $f(z, \alpha) = \alpha z^3 + (1 - \alpha)z$, $\alpha \in \mathbb{C}, z \in \mathbb{C}$

3-$f(z, \alpha) = \alpha z^3$, $\alpha \in \mathbb{C}, z \in \mathbb{C}$

Critical points and fixed points were studied and their properties were studied and their nature was either attracting or repelling.

3-1 Fixed points of cubic functions.

In this sub section, fixed points and their properties of each function have been discussed and the diagrams for different value of α have been drawn .

Example 3.1

Consider the function $f(z, \alpha) = \alpha z(1 - z^2)$, $z, \alpha \in \mathbb{C}$ (1)

The function $f(z, \alpha)$ is analytic function, the roots of $f'(z, \alpha) = 0$ its critical points $f'(z, \alpha) = \alpha - 3\alpha z^2 = 0$

$3\alpha z^2 = \alpha$

$z^2 = \frac{1}{3}$

$z = \mp \frac{1}{\sqrt{3}}$, the critical points of $f(z, \alpha)$,The fixed points for the function $f(z,\alpha)$ is $f(z, \alpha) = z$

$\alpha z - \alpha z^3 = z$
$\alpha z - \alpha z^3 - z = 0$
$z(\alpha - \alpha z^2 - 1) = 0$
$z = 0$ or $\alpha - \alpha z^2 - 1 = 0$ then $z^2 = \frac{\alpha-1}{\alpha}$
$\Rightarrow z = \mp\sqrt{\frac{\alpha-1}{\alpha}}$ so the fixed points for the function $f(z,\alpha)$
$z_1 = 0$
$z_2 = \sqrt{\frac{\alpha-1}{\alpha}}$
$z_3 = -\sqrt{\frac{\alpha-1}{\alpha}}$
And prove it attractive should be achieved below
$|f'(0,\alpha)| < 1$ (2)
$\left|f'\left(\sqrt{\frac{\alpha-1}{\alpha}},\alpha\right)\right| < 1$ (3)
$\left|f'\left(-\sqrt{\frac{\alpha-1}{\alpha}},\alpha\right)\right| < 1$ (4)
from the equation (1) $f'(z,\alpha) = \alpha - 3\alpha z^2$ (5)
we make up fixed points in the equation (5), we get
$f'(0,\alpha) = \alpha$
$f'\left(\mp\sqrt{\frac{\alpha-1}{\alpha}},\alpha\right) = 3 - 2\alpha$
And applying equations (2), (3) and (4), we get
$|\alpha| < 1$
$|3 - 2\alpha| < 1$

Theorem 3.2

The set of all points α such that , The function $f(z,\alpha)$ defined by the equation(1)has fixed points , attractive and non-zero is the region specified by the open disc$|3 - 2\alpha| < 1$ shown in figure(1.(c)).

proof: Let $f(z,\alpha)$ be the function which defined in the equation (1) , The non- zero fixed points of the function $f(z,\alpha)$ are $z = \mp\sqrt{\frac{\alpha-1}{\alpha}}$,And be attractive if
$\left|f'\left(\mp\sqrt{\frac{\alpha-1}{\alpha}},\alpha\right)\right| = |3 - 2\alpha| < 1 \Rightarrow |1 - (2\alpha - 2)| < 1$,Let w complex number, and suppose that $w = 2\alpha - 2$. $|1 - w| < 1$,In complex plan $|1 - w|$ represents a circle with radius 1 and center (1,0) . It is expressed by polar coordinates as $r = 2\cos\theta$, And shown in figure (1.(a)). But $w = 2\alpha - 2 \Rightarrow \alpha = \frac{w}{2} + 1$
This mean values α , Represents the circumference of the circle centered $\left(\frac{3}{2},0\right)$ and radius $\frac{1}{2}$, is shown in figure (1.(b)).Finally, $|3 - 2\alpha| < 1$ in complex plan represent all values of α in $\mathbb{C}$ Which represents the open disc shown in the Figure (1.(c)) .

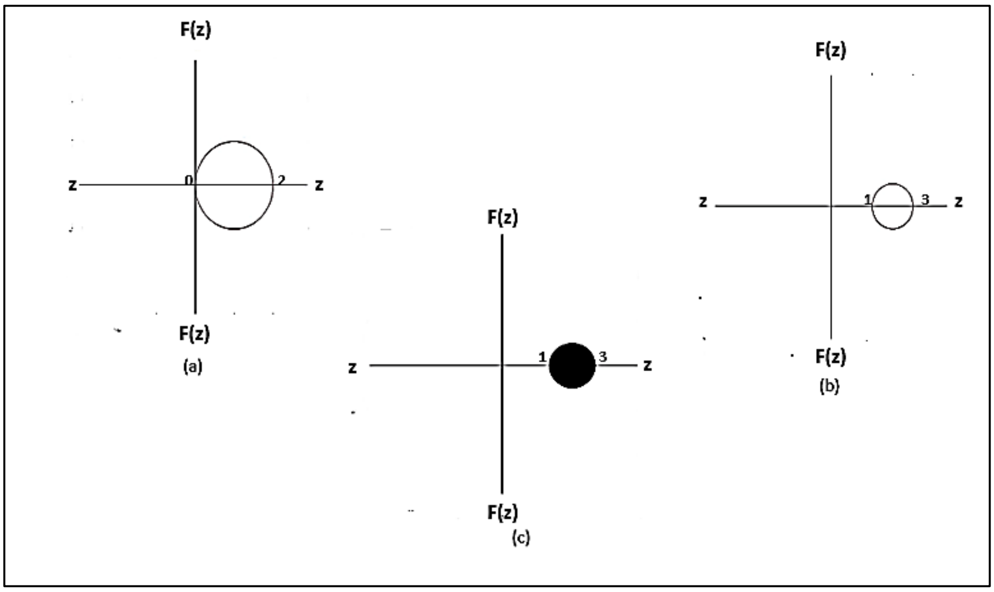

Fig. 1: A set of points α in complex plan such that $f(z,\alpha)$ has non-zero fixed points , attractive.

Remark 3.1

The fixed point $z_1 = 0$ is an attractive point , and that its path is to the origin point, As shown in Figure (2).

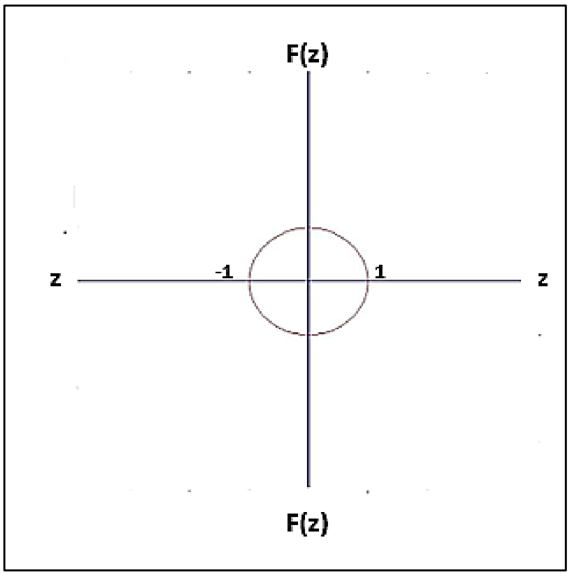

Fig. 2: A set of points α in complex plan , $z_1 = 0$ fixed point and attractive.

Example 3.2

Consider the function $f(z,\alpha) = \alpha z^3 + (1-\alpha)z$, such that $z,\alpha \in \mathbb{C}$ (6).

The function $f(z,\alpha)$ is analytic and differentiable , the roots of $f'(z,\alpha) = 0$ its critical points $f'(z,\alpha) = 3\alpha z^2 + (1-\alpha) = 0$

$$3\alpha z^2 + (1-\alpha) = 0$$

$$z^2 = \frac{\alpha-1}{3\alpha}$$

$z = \mp\sqrt{\frac{\alpha-1}{3\alpha}}$, the critical points of $f(z,\alpha)$

The fixed points for the function $f(z,\alpha)$ is $f(z,\alpha) = z$

$$\alpha z^3 + (1-\alpha)z = z$$

$$\alpha z^3 + z - \alpha z - z = 0$$

$$\alpha z^3 - \alpha z = 0$$

$$\alpha z(z^2 - 1) = 0$$

either $z = 0$, $\alpha \neq 0$

$$z = \mp 1$$

So the fixed points for the function $f(z,\alpha)$

$$z_1 = 0$$

$$z_2 = 1$$

$$z_3 = -1$$

And prove it attracting should be achieved below

$$|f'(0,\alpha)| < 1 \quad (7)$$

$$|f'(1,\alpha)| < 1 \quad (8)$$

$$|f'(-1,\alpha)| < 1 \quad (9)$$

From the equation (6) get $f'(z,\alpha) = 3\alpha z^2 + (1-\alpha)$ (10)

we make up fixed points in the equation (10), we get $f'(0,\alpha) = 1-\alpha$ and $f'(\mp 1,\alpha) = 2\alpha + 1$ And applying equations (7),(8) and (9), we get $|1-\alpha| < 1$ and $|2\alpha + 1| < 1$ and by theorem (2-1), we get $|f'(\mp 1,\alpha)| < 1$ and $|2\alpha + 1| < 1$ then$|1-(-2\alpha)| < 1$ Let w be a complex number , and suppose that $w = -2\alpha$ $|1-w| < 1$ in complex plan $|1-w|$, represents a circle with radius 1 and center (1,0). It is expressed by polar coordinates as $r = 2\cos\theta$ which $|1-w| = 1$ if and only if $r = 2\cos\theta$, And shown in figure (3.(a)), But $w = -2\alpha \implies \alpha = \frac{-w}{2}$ This means values α , Represents the circumference of the circle central $(\frac{-1}{2}, 0)$ and radius $\frac{1}{2}$, is shown in figure (3.(b)) Finally, $|2\alpha + 1| < 1$ in complex plan represent all values of α in $\mathbb{C}$ which represents the open disk shown in the Figure $(3.(c))$.

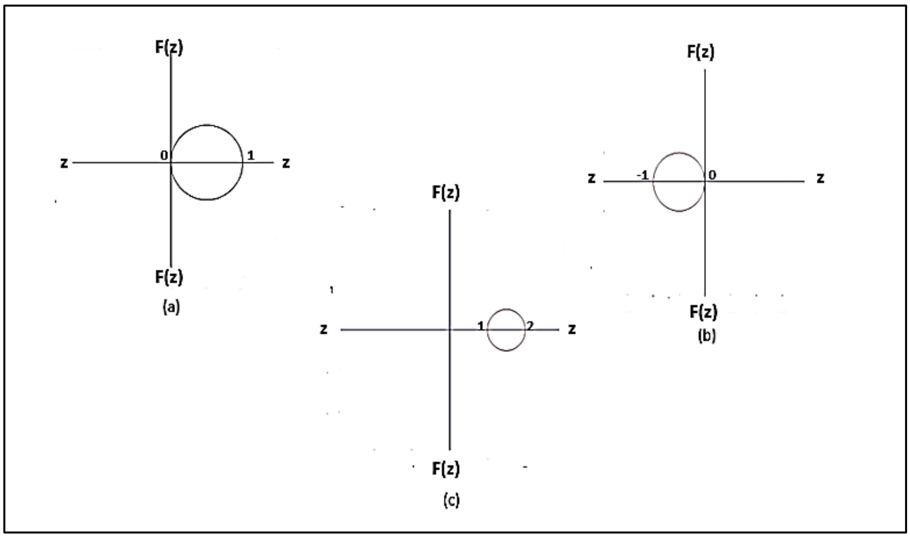

Fig. 3: A set of points α in complex plan such that $f(z,\alpha)$ has non-zero fixed points, attractive .

Remark 3.2

The fixed point $z_1 = 0$ is an attracting point, and that its path is to the origin point, as shown in Figure (4).

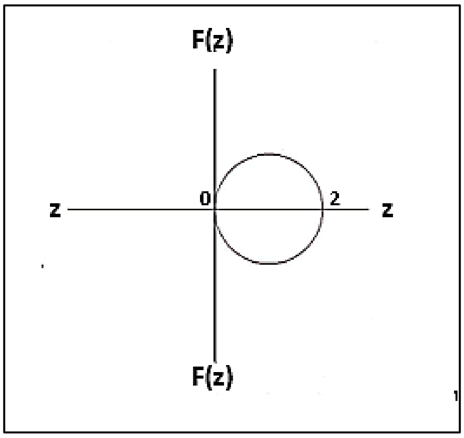

Fig. 4: A set of points (1-α) in complex plan, $z_1 = 0$ fixed point and attractive.

Example 3.3

Consider the function $f(z,\alpha) = \alpha z^3$,such that $\alpha, z \in \mathbb{C}$ (11)

The function $f(z,\alpha)$ is analytic function, the roots of $f'(z,\alpha) = 0$, its critical points. $f'(z,\alpha) = 3\alpha z^2 = 0 \Longrightarrow z^2 = 0 \Longrightarrow z = 0$, the critical points of $f(z,\alpha)$ and The fixed points for the function $f(z,\alpha)$

$$f(z,\alpha) = z$$
$$\alpha z^3 = z$$
$$\alpha z^3 - z = 0$$
$$z(\alpha z^2 - 1) = 0$$

either $z = 0$ *or* $z = \mp\frac{1}{\sqrt{\alpha}}$, So the fixed points for the function $f(z,\alpha)$

$$z_1 = 0$$
$$z_2 = \frac{1}{\sqrt{\alpha}}$$
$$z_3 = -\frac{1}{\sqrt{\alpha}}$$

And prove it attractive should be achieved below

$$|f'(0,\alpha)| < 1 \quad (12)$$
$$|f'\left(\frac{1}{\sqrt{\alpha}},\alpha\right)| < 1 \quad (13)$$
$$|f'\left(\frac{-1}{\sqrt{\alpha}},\alpha\right)| < 1 \quad (14)$$

from the equation (11) $f(z,\alpha) = 3\alpha z^2$ (15)

we make up fixed points in the equation (15) we get

$$f'(0,\alpha) = 0$$
$$f'\left(\mp\frac{1}{\sqrt{\alpha}},\alpha\right) = 3$$

And applying equations (12), (13) and (14), we get $|f'(0,\alpha)| = 0 < 1$ and $\left|f'\left(\mp\frac{1}{\sqrt{\alpha}},\alpha\right)\right| = 3 > 1$,The fixed point $z_1 = 0$ is an attractive point, and that its path is to the origin point.

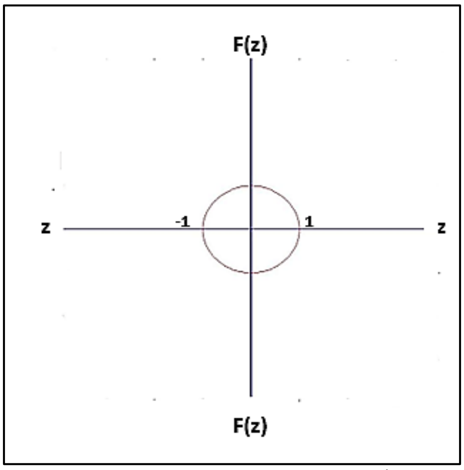

Fig. 5: And the fixed point $z_{2,3} = \mp\frac{1}{\sqrt{\alpha}}$ are an repelling fixed points.

Example 3.4

Find the 2-cycles of $f(z,\alpha) = \alpha z^3$, such that $\alpha, z \in \mathbb{C}$, and determine their stability.

Solution: To find the 2-cycles , we solve the equation $f^2(z,\alpha) = z$

Hence $f^2(z,\alpha) = f(f(z,\alpha))$

$$\alpha^4 z^9 - z = 0$$
$$z(\alpha^4 z^8 - 1) = 0$$
$$z(\alpha^2 z^4 - 1)(\alpha^2 z^4 + 1) = 0$$
$$z(\alpha z^2 - 1)(\alpha z^2 + 1)(\alpha^2 z^4 + 1) = 0 \quad (16)$$

And divide all the boundaries of the equation (16) on $z(\alpha z^2 - 1)$, to eliminate the effect of fixed points of the function $f(z,\alpha)$ for values $z = 0$ and $z = \mp\frac{1}{\sqrt{\alpha}}$, we get then the equation

$$(\alpha z^2 + 1)(\alpha^2 z^4 + 1) = 0$$
$$\alpha z^2 + 1 = 0$$
$$z^2 = \frac{-1}{\alpha} \qquad (17)$$

The two roots of equation (17) are z_1, z_2

$$z_1 = \frac{i}{\sqrt{\alpha}} \ , \ z_2 = -\frac{i}{\sqrt{\alpha}}$$
$$\alpha^2 z^4 + 1 = 0$$
$$z^4 = \frac{-1}{\alpha^2} \qquad (18)$$

The four roots of equation (18) are z_3, z_4, z_5 and z_6

$$z_3 = \sqrt{\frac{i}{\alpha}} \quad , \quad z_4 = -\sqrt{\frac{i}{\alpha}}$$
$$z_5 = \frac{i^{3/2}}{\sqrt{\alpha}} \quad , \quad z_6 = -\frac{i^{3/2}}{\sqrt{\alpha}}$$

Hence , the 2-cycles are $\{z_1, z_2\}$, $\{z_3, z_4\}$ and

$$|f'(z_1,\alpha).f'(z_2,\alpha)| = \left|3\alpha\frac{i^2}{\alpha} \ .3\alpha\frac{i^2}{\alpha}\right| = 9 > 1$$
$$|f'(z_3,\alpha).f'(z_4,\alpha)| = \left|3\alpha\frac{i}{\alpha} \ .3\alpha\frac{i}{\alpha}\right| = 9 > 1$$
$$|f'(z_5,\alpha).f'(z_6,\alpha)| = \left|3\alpha\frac{i^3}{\alpha} \ .3\alpha\frac{i^3}{\alpha}\right| = 9 > 1$$

Then all 2-cycle are unstable.

Remark 3.3

The fixed points for the function $f^2(z,\alpha)$ are repelling.

conclusion

1) $f(z,\alpha) = \alpha z(1 - z^2)$ such that $\alpha, z \in \mathbb{C}$, have two critical points $z_{1,2} = \mp\frac{1}{\sqrt{3}}$ and three fixed points $z_1 = 0, z_{2,3} = \mp\sqrt{\frac{\alpha-1}{\alpha}}$ such that $z_1 = 0$ is attracting when $|\alpha| < 1$ is origin point As shown in figure (2).And $z_{2,3} = \mp\sqrt{\frac{\alpha-1}{\alpha}}$ are attracting when $|3 - 2\alpha| < 1$ is the region specified by open disc $|3 - 2\alpha| < 1$ shown in figure (1.(c)) **.**

2) $f(z,\alpha) = \alpha z^3 + (1-\alpha)z$ such that $\alpha, z \in \mathbb{C}$, have two critical points $z_{1,2} = \mp\sqrt{\frac{\alpha-1}{3\alpha}}$ and three fixed points $z_1 = 0, z_{2,3} = \mp 1$ such that $z_1 = 0$ is attracting when $|1-\alpha| < 1$ and that its path is to the origin point as shown in figure (4).And $z_{2,3} = \mp 1$ are attractive when $|2\alpha + 1| < 1$ represents the open disc shown in the figure (3.(c)).

3) $f(z,\alpha) = \alpha z^3$ such that $\alpha, z \in \mathbb{C}$, have one critical point $z = 0$ and three fixed points $z_1 = 0$ is attracting that is path is the origin point and $z_{2,3} = \frac{1}{\sqrt{\alpha}}$ are repelling as shown in figure (5). And all 2-cycles of $f(z,\alpha) = \alpha z^3$ are repelling and unstable .

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TJPS

السلوك الديناميكي لبعض عوائل الدوال التكعيبية في المستوي العقدي

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الملخص

عني هذا البحث بدراسة السلوك الديناميكي لثلاث من عوائل الدوال التكعيبية ، حيث تم تحديد النقاط الحرجة لها وايجاد النقاط الثابتة لكل منها، وقد تم دراسة ووصف صفة كل من تلك النقاط الجاذبة من حيث الجذب والطرد.

حيث وجد أن الدالة $f(z,\alpha) = \alpha z(1 - z^2)$ عندما $\alpha , z \in \mathbb{C}$, تمتلك نقطتين حرجتين هما $z_{1,2} = \mp\frac{1}{\sqrt{3}}$ والنقاط الثابتة $z_1 = 0, z_{2,3} = \mp\sqrt{\frac{\alpha-1}{\alpha}}$ بحيث أن $z_1 = 0$ جاذبة عند قيم $|\alpha| < 1$ وأن النقطتين الأخريين تكونان جاذبتين عند قيم $|3 - 2\alpha| < 1$ كما وجد أن الدالة الثانية $f(z,\alpha) = \alpha z^3 + (1-\alpha)z$ تمتلك نقطتين حرجتين هما $z_{1,2} = \mp\sqrt{\frac{\alpha-1}{3\alpha}}$ وثلاثة نقاط ثابتة هي: $z_1 = 0, z_{2,3} = \mp 1$ حيث أن $z_1 = 0$ تكون جاذبة عند $|1-\alpha| < 1$ وأن $z_{2,3} = \mp 1$ تكون جاذبة عند القيم $|2\alpha + 1| < 1$ ، أما الدالة الثالثة $f(z,\alpha) = \alpha z^3$ فانها تمتلك نقطة حرجة واحدة هي $z = 0$ وثلاثة نقاط ثابتة، حيث تكون $z_1 = 0$ جاذبة فيما تكون $z_{2,3} = \frac{1}{\sqrt{\alpha}}$ طاردتين . كما تبين أن جميع الحلقات – 2 لعائلة الدوال $f(z,\alpha) = \alpha z^3$ طاردة وغير مستقرة.