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The Implementations of the Embedded Diagonally Implicit Type Runge-Kutta Method (EDITRKM) For Special Third Order of the Ordinary Differential Equations

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ABSTRACT

The derivation of the Embedded pair Diagonally Implicit Type Runge-Kutta Method (EDITRKM) for solving 3rd special order ordinary differential equations (ODEs) is introduced in the current study. The EDITRKM techniques are the name of the approach. This approach in the present study has two types: EDITRKM 4(3) for order 4 and 3 of the first pair and EDITRKM 5(4) for orders 5 and 4 of the second pair. To investigate the current study, a variety of tests for five various initial value problems (IVPs) with different step sizes h were implemented. Then, a comparation of the present study between the EDITRKM 4(3) and EDITRKM 5(4) for five different problems are made. The numerical techniques elucidated as the qualification regarding the efficiency and decimal logarithm for highest the time curve against logarithm of number of the function call estimate.

1. Introduction

Third-order ODEs are used in neural network engineering and applied sciences, the dynamics of fluid flow, the ship's motion, and electric circuits, among other fields [1-6]. The starting value problem for third-order ODEs where the second derivative does not appear implicitly is addressed as

$y'''(x) = f\big(x, y(x)\big)$ with $y(x_0) = \alpha\,,\ y'(x_0) = \beta$ and $y'(x_0) = \gamma$ ....(1)

The implicit methods are important because they can reach high orders of accuracy at the equivalent number of stages, which can be represented as an advantage that leads to the more accurate than the explicit approaches. This manufactures it easier to exist the solution to the difficulties of the problems. So, the implicit RK techniques play an important role for denomination the physical and mathematical problems, like a differential algebraic equation. The diagonal implicit RK (DIRK) techniques are also pointed to as semi-implicit approaches or semi explicit RK techniques since they obtained at minimum one value does not zero for the lowest of the triangular diagonal matrices. Therefore, to solve Eq. (1), two general strategies can be employed. The elementary way is to transfer the Eq. (1) into a problem with first-order then apply any pattern of the RK approach to it. Calvo M. et al. (1996) proposed novel of embedded pairs RK approaches particularly modified to the approximate computations of 1st order sets of differential equations that supposed to get oscillating approximations are found [7]. The dispersion and dissipation orders besides the validation of accuracy, approximation of local error and analysis of the stability are studied according to Van der Houwen and Sommeijer (1989) [8]. The dispersion and dissipation of three nine stage embedded pairs of Runge-Kutta methods of algebraic 7,5 and higher-orders that have various free parameters are examined [7]. Moreover, [9-10] developed a solution of the special third order for the ODEs directly by RK technique. Finally, Senu [11] and Fawzi et al. [12] constructed the embedded RK technique to solve third order for the ODEs. The explicit embedded pair Runge-Kutta (RK) method that known as TFRKF6 (5) is improved to compute the numerical solution of the initial value problems of first-order for oscillatory approximations. The suggested approach has been studied a 1st order IVPs via first decreasing higher order of IVPs to the identical system of 1st order. The embedded techniques have algebraic 6 and 5-order according to Fawzi, F. A. et al. (2016) [13]. Therefore, Senu et al. [14] structted a novel embedded explicit RK method to solve special third order problems. Ismail, F. and et al. (2008) are purposed the Singly Embedded Diagonally Implicit Runge-Kutta (SDIRK) methods to combine Delay Differential Equations (DDEs) and the computational results are compared. The singly known as all the eigenvalues of the coefficient matrix A are equivalent and all the diagonal elements are same. He mentioned will use the expression loosely for the first diagonal element that equal to zero [15]. The set of test problems are studied using the singly diagonally implicit RK-Nystróm general (SDIRKNG) approach of 3rd -order embedded in 4th -order for the integration second-order IVPs according work of Ismail, F. et al. (2007) [16]. In this work, the special third order of the ordinary differential equations (ODE) of the form $y'''(x)=f(x,y(x))$ will be study. The first and the second order are not occurred as a perfect third order of the ODEs in the formula $y'''(x)=f(x,y(x),y'(x),y''(x))$. Results for special third order of ODE are implemented via the implicit embedded of DITRKM.

Section 2 demonstrates the basic idea of construction and derivation of the DITRK system for addressing Initial Value Problems (IVPs). The DITRK technique's order conditions are outlined in Section 3. Section 4 describes Derivation Embedded DITRK Methods. In Section 5, the Test of Problems are presented. In Section 6, validation of the EDITRK approach with five IVPs are computed. The Discussion and Conclusion are given in Section 7.

2. The Methodology of DITRK Techniques

For solving IVPs in eq. (1), the prevalent formula of the implicit RK approach for the m stage digit can be expressed as follows:[18]

$$y_{n+1}=y_n+h\,y'_n+\frac{h^2}{2}y''_n+h^3\sum_{i=1}^{m}d_i\,k_i \quad \text{....(2)}$$

$$y'_{n+1}=y'_n+h\,y''_n+h^2\sum_{i=1}^{m}b_i\,k_i \quad \text{...(3)}$$

$$y''_{n+1}=y''_n+h\sum_{i=1}^{m}g_i\,k_i \quad \text{.....(4)}$$

$$k_1=f(x_n,\ y_n) \quad \text{... (5)}$$

$$k_i=f(x_n+c_ih,y_n+h\,c_iy'_n+\frac{h^2}{2}c_i^2\,y''_n+h^3\sum_{j=1}^{i-1}a_{ij}\,k_j) \quad \text{...(6)}$$

where $i=2,3,\dots,m$. The parameters of diagonal implicit RK type (DITRK) methods are presumed as $c_i, a_{ij}, d_i, b_i, g_i$ where $i,j=1,2,3\dots,s$ are real numbers. This scheme is known as diagonal implicit when $a_{ij}\neq 0$ for $j>i$. The last denomination includes the single DITRK techniques that A indicate that the lower the triangular diagonal matric of A have same values with $a_{ij}\neq 0$ where $i=j$ at the diagonal. The DITRK approach proposed from the work of Butcher, as illustrated in Table 1 [17].

Table 1: Butcher form DITRK method.

c_1	$a_{1,1}$		
c_2	$a_{2,1}$	$a_{2,2}$	
c_2	a_{31}	$a_{3,2}$	$a_{3,3}$
d_i	d_1	d_2	d_3
b_i	b_1	b_2	b_3
g_i	g_1	g_2	g_3

3. Order Conditions of the DITRK Technique

According to Mechee et al. [18], the orders of algebraic criteria for RKD approached over order 6 are as follow:

Order conditions of y:

order 3 $\sum d_i=\frac{1}{6}$ .... (7)

order 4 $\sum d_i\,c_i=\frac{1}{24}$ .....(8)

order 5 $\sum d_i\,c_i^2=\frac{1}{60}$ .... (9)

order 6 $\sum d_i\,c_i^3=\frac{1}{120}$ and $\sum d_i\,a_{i,j}=1/720$. ....(10)

Order conditions of y':

order 2 $\sum b_i=\frac{1}{2}$ ....(11)

order 3 $\sum b_i\,c_i=\frac{1}{6}$ ....(12)

order 4 $\sum b_i\,c_i^2=\frac{1}{12}$ .....(13)

order 5 $\sum b_i\,c_i^3=\frac{1}{20}$ and $\sum b_i\,a_{i,j}=\frac{1}{120}$ ....(14)

order 6 $\sum b_i\,c_i^4=\frac{1}{30}$, $\sum b_i\,a_{i,j}c_j=\frac{1}{720}$ and $\sum b_i\,c_i\,a_{i,j}=\frac{1}{180}$ ....(15)

Order conditions of y'':

order 1 $\sum g_i=1$ .... (16)

order 2 $\sum g_i\,c_i=\frac{1}{2}$ ....(17)

order 3 $\sum g_i\,c_i^2=\frac{1}{3}$ ....(18)

order 4 $\sum g_i\,c_i^3=\frac{1}{4}$ and $\sum g_i\,a_{i,j}=\frac{1}{24}$.....(19)

order 5 $\sum g_i\,c_i^4=\frac{1}{5}$, $\sum g_i\,a_{i,j}c_j=\frac{1}{120}$ and $\sum g_i\,c_i\,a_{i,j}=\frac{1}{30}$ ...(20)

order 6 $\sum g_i\,c_i^2a_{i,j}=\frac{1}{36}$, $\sum g_i\,a_{i,j}\,c_j^2+\sum g_i\,c_ia_{i,j}c_j=\frac{7}{720}$,

$\sum g_i\,c_i^5=\frac{1}{6}$, $\sum g_i\,a_{i,j}\,c_j^2=\frac{1}{360}$, $\sum g_i\,c_ia_{i,j}c_j=\frac{1}{144}$ &

$\frac{1}{2}\sum g_i\,a_{i,j}\,c_j^2+\sum g_i\,c_ia_{i,j}c_j=\frac{1}{120}$ (21)

4. Derivation Embedded DITRK Methods

The general form of DITRK technique with m-stage for numerical solution of eq. (1) is provided. Then, there is the creation for embedded pair RK approach, which is active study topic that is always improving existing codes. The derivation of p(q) pairs of implicit DITRK techniques are employed in values of step size codes to give a minimum error estimation. They based on the order p method (C, A, d, b, g) and order q method (C, A, d', b', g') In Butcher Tabular, the embedded pair can be started as follows:

Table 2: Butcher Tabular of the embedded pair DITRK Method.

c_1	a_{11}			
c_1	a_{21}	a_{22}		
$\vdots$	$\vdots$	$\vdots$		
c_m	a_{m1}	$\cdots$		a_{mm}
	d_1	d_2	$\cdots$	d_m
	b_1	b_2	$\cdots$	b_m
	g_1	g_2	$\cdots$	g_m
	d'_1	d'_2	$\cdots$	d'_m
	b'_1	b'_2	$\cdots$	b'_m
	g'_1	g'_2	$\cdots$	g'_m

The primary proposes for constructing the embedded pair of implicit DITRK techniques is to get a lone cost error estimation for use in values of step size approach. The techniques are represented by improving the significant pairs and estimations of the local error that is employed via bounded the step size h as follow

$$h_{n+1} = 0.9hn\left(\frac{Tol}{LTE}\right)^{\frac{1}{q+1}} \ldots (22)$$

where 0.9 is the achieve factor, local error estimation (LTE) computed at each step, and Tol refers to requirement of the accuracy. So, if LTE $\leq$ Tol that mean the step will accept and the technique of executing local extrapolation which refers to more accurate computations will be employed to progress the integration and h can be improved utilizing in eq. (22) If LTE $>$ Tol, the step will be refused and the step size h, will be reduced by half. The EDITRKM technique has been developed as an embedded RK type approach for solving third-order ODEs. Order 4 and 3 are found in the first pair, while orders 5 and 4 are found the second these approaches are developed using factions that ensured the higher- order method were extremely accurate while the lower methods provided the most accurate error estimations. So, the step size h effect to the obtain accurate results by doubling it. For this study, we have two derivations for Embedded DITRK 4(3) Method and Embedded DITRK 5(4) Method as illustrated in Table 3.

In EDITRKM 4(3), the A and C values is computed from the 4$^{\text{th}}$ -order solution then derived the three-stage 3$^{\text{rd}}$-order embedded equation. The solving of the eqs. (7), (11), (12), (16-18) simultaneously then the solution for d'_i and b'_i while g'_i have the same values as the 4$^{\text{th}}$ -order. The solutions are obtained as

$b_1 = -10 + 25b_3 - 15b_3\sqrt{3} + \frac{35\sqrt{3}}{6}, b_2 = -26b_3 + 15b_3\sqrt{3} - \frac{35}{6}\sqrt{3} + \frac{21}{6}, b_3 = b_3, d_1 = \frac{1}{6} - d_2 - d_3, d_2 = d_2, d_3 = d_3, g_1 = 0, g_2 = \frac{1}{2}, g_3 = \frac{1}{2}$

According to [19], the free parameters can be computed via minimizing the LTE, from the minimize commend in Maple then obtained the values of the d2= 0.137801357104202, d3=0.931263538815273 and b3= 0.105662432884725. For optimized value in fractional form then $d_3 = \frac{9}{10}$ $, d_2 = \frac{1}{10}$ and $b_3 = \frac{1}{10}$ are choose.

Table 3: Table of EDITRKM 4(3) and EDITRKM 5(4).

$\frac{4}{5}$	$\frac{1}{600}$		
$\frac{1}{2} - \frac{\sqrt{3}}{6}$	0	$\frac{1}{600}$	
$\frac{1}{2} + \frac{\sqrt{3}}{6}$	0	$\frac{2}{25}$	$\frac{1}{600}$
	0	$\frac{1}{12} + \frac{\sqrt{3}}{24}$	$\frac{1}{2} - \frac{\sqrt{3}}{24}$
	0	$\frac{1}{4} + \frac{\sqrt{3}}{12}$	$\frac{1}{4} - \frac{\sqrt{3}}{12}$
	0	$\frac{1}{2}$	$\frac{1}{2}$
	$\frac{-5}{6}$	$\frac{1}{10}$	$\frac{9}{10}$
	$\frac{-15}{2} + \frac{13\sqrt{3}}{3}$	$\frac{79}{10} - \frac{13\sqrt{3}}{3}$	$\frac{1}{10}$
	0	$\frac{1}{2}$	$\frac{1}{2}$

EDITRKM 4(3)

$\frac{1}{2} - \frac{\sqrt{15}}{10}$	$\frac{1}{24} - \frac{\sqrt{15}}{120}$		
$\frac{1}{2}$	0	$\frac{1}{24} - \frac{\sqrt{15}}{120}$	
$\frac{1}{2} + \frac{\sqrt{15}}{10}$	$\frac{3\sqrt{15}}{100}$	0	$\frac{1}{24} - \frac{\sqrt{15}}{120}$
	$\frac{1}{18} + \frac{\sqrt{15}}{72}$	$\frac{1}{18}$	$\frac{1}{18} - \frac{\sqrt{15}}{72}$
	$\frac{5}{36} + \frac{\sqrt{15}}{36}$	$\frac{2}{9}$	$\frac{5}{36} - \frac{\sqrt{15}}{36}$
	$\frac{5}{18}$	$\frac{4}{9}$	$\frac{5}{18}$
	$\frac{17}{10000} + \frac{\sqrt{15}}{36}$	$\frac{2449}{15000} + \frac{\sqrt{15}}{36}$	$\frac{17}{10000}$
	$\frac{5}{36} + \frac{\sqrt{15}}{36}$	$\frac{2}{9}$	$\frac{5}{36} - \frac{\sqrt{15}}{36}$
	$\frac{5}{18}$	$\frac{4}{9}$	$\frac{5}{18}$

EDITRKM 5(4)

The values of A and C is computed from the 5$^{\text{th}}$ -order solution then derive a three-stage order four embedded formula. solving the eqs. (8), (13), (20-21) simultaneously then the solution for d'_i and b'_i while g'_i have the same as of 5$^{\text{th}}$ -order. The solutions are obtained as

$b_1=\frac{5}{36}+\frac{\sqrt{15}}{36}, b_2=\frac{2}{9}, b_3=-\frac{\sqrt{15}}{36}+\frac{5}{36}, d_1=d_3+\frac{\sqrt{15}}{36}, d_2=-2d_3+\frac{1}{6}-\frac{\sqrt{15}}{36}, d_3=d_3, g_1=\frac{15}{8},$
$g_2=\frac{4}{9}, g_3=\frac{5}{18}$

According to [19], the free parameters can be computed via minimizing the LTE from commend of minimize in Maple then the value of d3= 0.00176412019109205 obtained. For optimized value in fractional formula then $d_3=\frac{17}{10000}$ is choose. Finally, the coefficients of 3rd -stage embedded EDITRK 5(4) technique can be written (see Table 3)

5. Test of Problems

The approaches that demonstrated in section 4 tested with 5 various problems in this part. The numerical experiments were conducted using the following methods:

Problem (1): Consider a nonhomogeneous linear ODE given in [20]

$y'''(x)=y(x)+\cos(x)$, with $y(0)=0,\ y'(0)=0,\ y''(0)=1$ where $x\in[0,1]$,

and analytic solution $y(x)=\frac{(e^x-\cos(x)-\sin(x))}{2}$.

Problem (2): Consider the nonhomogeneous nonlinear ODE

$y'''(x)=(y(x))^2+\cos^2(x)-\cos(x)-1$, with $y(0)=0,\ y'(0)=1,\ y''(0)=0$ where $0\le x\le 2$, and the exact solution $y(x)=\sin(x)$.

Problem (3): Nonhomogeneous nonlinear ODE, reads as

$y'''(x)=8\left(\frac{y^2(x)}{e^{2x}}\right)$ with $y(0)=1,\ y'(0)=2,\ y''(0)=4$ where $0\le x\le 1$,

and analytic solution $y(x)=e^{2x}$.

Problem (4): The homogeneous nonlinear ODEs is considered as

$y_1'''(x)=y_2(x)$, with $y_1(0)=1,\ y_1'(0)=0, y_1''(0)=1$

$y_2'''(x)=-y_1(x)-2\,y_2(x)+2\,y_3(x)$ with $y_2(0)=0,\ y_2'(0)=1, y_2''(0)=0$

$y_3'''(x)=y_1(x)+y_2(x)$ with $y_3(0)=1,\ y_3'(0)=1, y_3''(0)=1$ and

analytic solution $y_1(x)=\cosh(x)$, $y_2(x)=\sinh(x)$ and $y_3(x)=e^x$ where $0\le x\le 1$.

Problem (5): linear system of the ODEs is presented as

$y_1'''(x)=y_2(x)$, with $y_1(0)=1,\ y_1'(0)=0, y_1''(0)=1$,

$y_2'''(x)=y_1(x)$ with $y_2(0)=0,\ y_2'(0)=1, y_2''(0)=0$,

$y_3'''(x)=y_1(x)+y_2(x)-\sinh(x)$ with $y_3(0)=1,\ y_3'(0)=0, y_3''(0)=1$,

and exact solutions $y_1(x)=\cosh(x)$, $y_2(x)=\sinh(x)$ and

$y_3(x)=e^x+1-cosh(x)+\frac{x^2}{2}-x$ where $0\le x\le 1$.

6. Numerical Results

The approximation result that are illustrated in the tables below for solving problems (2.7). Following abbreviations will be used in tables

- **Tol**: Tolerance.
- **Method**: method employed step sizes between two points or positions.
- **F. N**: number of the function call.
- **STEP**: The number of successful steps.
- **FSTEP**: The number of failed steps.
- **Time**: execution time.
- **EDITRKM4(3)**: The novel 4(3) pair derived in this study.
- **EDITRKM5(4)**: The new 5(4) pair embedded derived in current work.

Table 4: Comparisons of number of function call and Time of EDITRKM 4(3) and EDITRKM 5(4) with $h=10^{-2},10^{-4},10^{-6}$ for the problem 1.

TOL(h)	Method	No. of Function Call	Time	Step	FSTEP
10^{-2}	EDITRKM 4(3)	43	0.076	13	2
	EDITRKM 5(4)	15	0.032	5	0
10^{-4}	EDITRKM 4(3)	124	0.092	40	2
	EDITRKM 5(4)	48	0.041	16	0
10^{-6}	EDITRKM 4(3)	388	0.121	128	2
	EDITRKM 5(4)	218	0.054	72	1

Table 5: Comparisons of number of function call and Time of EDITRKM 4(3) and EDITRKM 5(4) with $h=10^{-2},10^{-4},10^{-6}$ for the problem 2.

TOL(h)	Method	No. of Function Call	Time	Step	FSTEP
10^{-2}	EDITRKM 4(3)	21	0.077	7	0
	EDITRKM 5(4)	9	0.051	3	0
10^{-4}	EDITRKM 4(3)	66	0.091	22	0
	EDITRKM 5(4)	24	0.072	8	0
10^{-6}	EDITRKM 4(3)	207	0.122	69	0
	EDITRKM 5(4)	89	0.094	29	1

Table 6: Comparisons of number of function call and Time of EDITRKM 4(3) and EDITRKM 5(4) with $h = 10^{-2}, 10^{-4}, 10^{-6}$ for the problem 3.

TOL(h)	Method	No. of Function Call	Time	Step	FSTEP
10^{-2}	EDITRKM 4(3)	293	0.062	97	1
	EDITRKM 5(4)	87	0.038	29	0
10^{-4}	EDITRKM 4(3)	947	0.075	315	1
	EDITRKM 5(4)	458	0.047	152	1
10^{-6}	EDITRKM 4(3)	3782	0.115	1260	1
	EDITRKM 5(4)	2179	0.071	725	2

Table 7: Comparisons of number of function call and Time of EDITRKM 4(3) and EDITRKM 5(4) with $h = 10^{-2}, 10^{-4}, 10^{-6}$ for the problem 4.

TOL(h)	Method	No. of Function Call	Time	Step	FSTEP
10^{-2}	EDITRKM 4(3)	218	0.085	72	1
	EDITRKM 5(4)	66	0.058	22	0
10^{-4}	EDITRKM 4(3)	701	0.097	233	1
	EDITRKM 5(4)	321	0.066	107	0
10^{-6}	EDITRKM 4(3)	2282	0.121	760	1
	EDITRKM 5(4)	1514	0.078	504	1

Table 8: Comparisons of number of function call and Time of EDITRKM 4(3) and EDITRKM 5(4) with $h = 10^{-2}, 10^{-4}, 10^{-6}$ for the problem 5.

TOL(h)	Method	No. of Function Call	Time	Step	FSTEP
10^{-2}	EDITRKM 4(3)	137	0.065	45	1
	EDITRKM 5(4)	36	0.049	12	0
10^{-4}	EDITRKM 4(3)	437	0.093	145	1
	EDITRKM 5(4)	174	0.059	58	0
10^{-6}	EDITRKM 4(3)	1382	0.112	460	1
	EDITRKM 5(4)	830	0.087	276	1

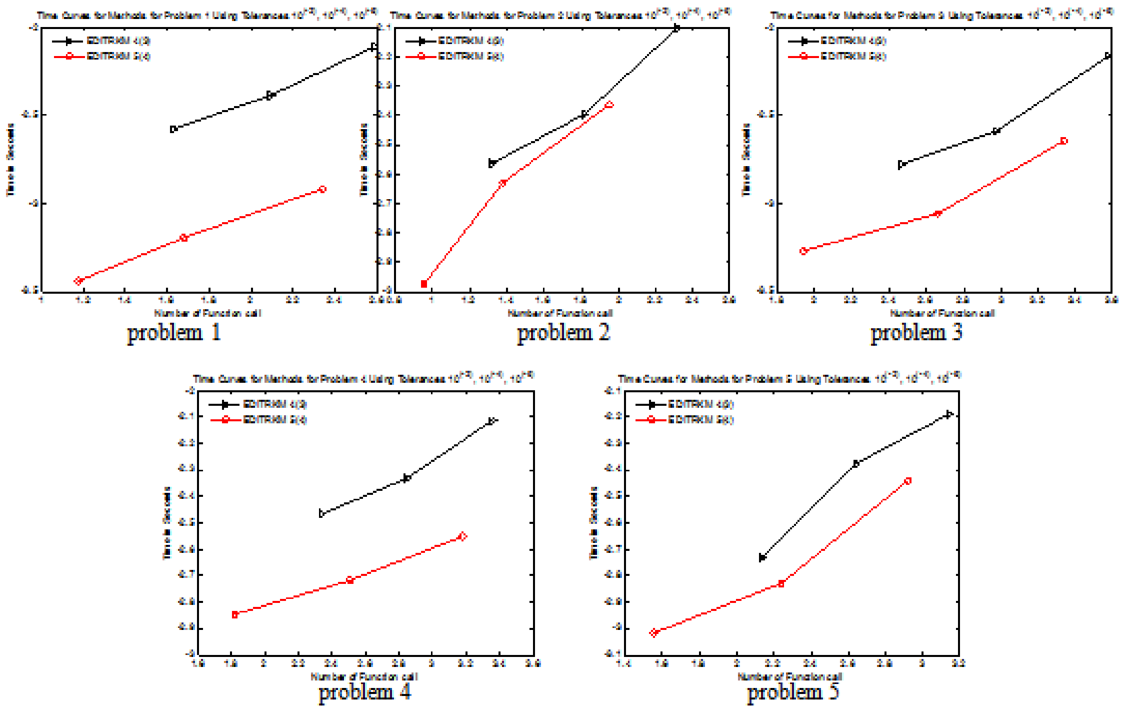

Fig. 1: Accuracy curve for EDITRKM 4(3) and EDITRKM 5(4) with $= 10^{-2}, 10^{-4}, 10^{-6}$.

7. Discussion and Conclusion

Figure (1) show the improvement of the Embedded pair Diagonally Implicit Type Runge-Kutta Method (EDITRKM) created by charting of decimal logarithm for highest the time curve against logarithm of number of the function call estimate which are obtained from the Tables (4-8). The comparative of the present study for the EDITRKM 4(3) and EDITRKM 5(4) with five different problems as mentioned in section 5. In this project, the logarithm of time curve is computed with different Tol $h = 10^{-2}, 10^{-4}, 10^{-6}$ which is known in some literatures as the "Tol" (the given tolerance) for the five test problems. The numerical results that obtained from the Table (4-8) is used to create Figure (1), respectively. As well as, calculations of the numbers of the successful steps (Step) and the failed steps (FSTEP) as illustrated in Table (4-8). In the current study, the numerical results between the EDITRKM 4(3) and EDITRKM 5(4) have a good comparison as shown in Figure (1). The current study was based on Runge-Kutta Method that has been analyzed earlier by [12, 13, 18], however, the research in hand expanded and improved the method from explicit to implicit and from directly to diagonally.

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طريقة رنج-كوتا المضمنة من النوع الضمني قطريا (EDITRKM) للترتيب الثالث الخاص للمعادلات التفاضلية العادية

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الملخص

تم تقديم اشتقاق الزوج المضمنة من النوع الضمني قطريًا طريقة رونج- كوتا (EDITRKM) لحل المعادلات التفاضلية العادية ذات الترتيب الثالث الخاص (ODEs) في الدراسة الحالية. تقنيات EDITRKM هي اسم الطريقة. هذا الطريقة في الدراسة الحالية له نوعان: EDITRKM 4(3) للرتبة 4 و 3 بالنسبة للزوج الأول و 5(4) EDITRKM للرتب 5 و 4 بالنسبة للزوج الثاني. للتحقيق في الدراسة الحالية ، تم تنفيذ مجموعة متنوعة من الاختبارات لخمس مسائل قيمة أبتدائية مختلفة (IVPs) بأحجام خطوات مختلفة (h). تم اجراء مقارنة في الدراسة الحالية بين لـ 4(3) EDITRKM و 5(4) EDITRKM لخمس مسائل متنوعة. تم توضيح التقنيات العددية كمؤهل فيما يتعلق بالكفاءة واللوغاريتم العشري لأعلى منحنى الوقت مقابل لوغاريتم عدد تقدير استدعاء الدالة.