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Approximaitly Primary Submodules

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ABSTRACT

The study deals with the notion of an approximaitly primary submodules of unitary left R-module M over a commutative ring R with identity as a generalization of a primary submodules and approximaitly prime submodules, where a proper submodule N of an R-module M is called an approximaitly primary submodule of M, if whenever $ay \in N$, for $a \in R$, $y \in M$, implies that either $y \in N + soc(M)$ or $a^k M \subseteq N + soc(M)$ for some positive integer k of Z. Several characterizations, basic properties of this concept are given. On the other hand the relationships of this concept with some classes of modules are studied. Furthermore, the behavior of approximaitly primary submodule under R-homomorphism are discussed.

1. Introduction

The study deals with the concept of the prime submodule, which is one on the common concepts. The first to study and submit was by Dauns in 1978, "where a proper submodule N of an R-module M is called prime if whenever $ay \in N$, for $a \in R$, $y \in M$, implies that either $y \in M$ or $r \in [N:_R M]$" [1]. "Primary submodule, was introduced and studied by Lu in 1989 as generalization of prime submodules, where a proper submodule N of an R-module M is called primary if whenever $ay \in N$, for $a \in R$, $y \in M$, implies that either $y \in N$ or $a^k M \subseteq N$ for some positive integer K of Z" [2]. Recently several generalizations of primary submodules were introduced for example "Weakly primary submodules, Quasi-primary submodules, Nearly primary submodules,Ψ-primary submodules, 2-absorbing primary submodules and pseudo primary-2-absorbing submodules" see [3-8]. The study also focused on other generalization of primary submodule, which we called it an approximaitly primary submodule, this concept is also generalization of approximaitly prime submodules and approximaitly quasi-prime submodules see [9,10]. Several basic properties, examples and characterizations of approximaitly primary submodules are given. In this part of the paper we will recall some basic definitions, that we used them in the sequel. Recall that the socle of a module M denoted by $soc(M)$ is the intersection of all essential submodules of M [11], where a non-zero submodule N of an R-module M is called essential if $N \cap K \neq (0)$ for each non-zero submodule K of M [11]. Recall that a proper submodule N of an R-module M is called an approximaitly prime if whenever $ay \in N$ for $a \in R$, y$\in M$, implies that either $y \in N + soc(M)$ or $a \in [N + soc(M):_R M]$ [9], and a proper submodule N of an R-module M is called an approximaitly quasi-prime if whenever $aby \in N$ for $a, b \in R$, y$\in M$, implies that either $ay \in N + soc(M)$ or $by \in N + soc(M)$ [10]. If K is a submodule of an R-module M, and I is an ideal of R, then $[K:_M I] = \{x \in M: xI \subseteq K\}$ is a submodule of M containing K [12] and $[K:_M R] = K$ [13]. Recall that an R-module M is non-singular if $M = Z(M) = \{y \in M: yJ = (0)$ for some essential J of $R\}$ [11]. It is well-know that if M is non-singular then $soc(M) = soc(R)M$ [11, Coro. 1.26]. recall that an R-module M is multiplication if every submodule K of M is of the form $K = IM$ for some ideal I of R. In particular $K = [K:_R M]M$ [14]. Recall that an R-module M is faithful if $ann_R(M) = (0)$. It is well-know if M is faithful multiplication then $soc(M) = soc(R)M$ [14, Coro. 2.14].

2. Approximaitly Primary Submodules

This section intrudes the definition of the notion of approximaitly primary submodule and discuss some of it is basic properties, and some characterizations of this are given.

Definition 2.1 : A proper submodule N of an R-module M is called an approximaitly primary (for short app-primary) submodule of M, if whenever $ay \in L$, where $a \in R$, $y \in M$, implies that either $y \in N + soc(M)$ or $a^n M \subseteq N + soc(M)$ for some positive integer n of Z. And an ideal J of a ring R is called an app-primary ideal of R if J is an app-primary submodule of an R-module R.

Remarks and Examples 2.2 :

1) It is clear that every primary submodule of an R-module M is an app-primary submodule, but not conversely. The following example explains that:

Consider the Z-module Z_{12} , the submodule $N = \langle \bar{6} \rangle$ is not primary submodule of Z_{12} , Since $2.\bar{3} \in N$ for $2 \in Z, \bar{3} \in Z_{12}$, but $\bar{3} \notin N$ and $2 \notin \sqrt{[\langle \bar{6} \rangle :_Z Z_{12}]} = \sqrt{6Z} = 6Z$. While $N = \langle \bar{6} \rangle$ is an app-primary submodule of Z_{12} since $soc(Z_{12}) = \langle \bar{2} \rangle$ and for all $a \in Z$, $y \in Z_{12}$ such that $ay \in \langle \bar{6} \rangle$ implies that either $y \in \langle \bar{6} \rangle + soc(Z_{12}) = \langle \bar{6} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle$ or $a \in \sqrt{[\langle \bar{6} \rangle + soc(Z_{12}) :_Z Z_{12}]} = \sqrt{[\langle \bar{6} \rangle + \langle \bar{2} \rangle :_Z Z_{12}]} = \sqrt{[\langle \bar{2} \rangle :_Z Z_{12}]} = \sqrt{2Z} = 2Z$. That is if $2.\bar{3} \in \langle \bar{6} \rangle$, implies that $2 \in \sqrt{[\langle \bar{6} \rangle + soc(Z_{12}) :_Z Z_{12}]} = 2Z$.

2) It is clear that every approximaitly prime submodule of an R-module M is an app-primary submodule, but the convers is not true in general. The following example shows that.

Consider the Z-module Z, the submodule $N = \langle \bar{8} \rangle$ is not approximaitly-prime submodule of Z, since $2.4 \in \langle \bar{8} \rangle$ but $2 \notin [\langle \bar{8} \rangle + soc(Z) :_Z Z] = [\langle \bar{8} \rangle + (0) :_Z Z] = \langle \bar{8} \rangle$ and $4 \notin \langle \bar{8} \rangle + soc(Z) = \langle \bar{8} \rangle$, while $N = \langle \bar{8} \rangle$ is an app-primary of the Z-module Z, since for all $a \in Z$, $y \in M$ such that $ay \in N = \langle \bar{8} \rangle$, implies that either $y \in N + soc(Z) = \langle \bar{8} \rangle + (0) = \langle \bar{8} \rangle$ or $a \in \sqrt{[N + soc(Z) :_Z Z]} = \sqrt{[\langle \bar{8} \rangle + (0) :_Z Z]} = \sqrt{\langle \bar{8} \rangle} = \langle \bar{2} \rangle$. That is if $2.\bar{4} \in \langle \bar{8} \rangle$, implies that $2 \in \sqrt{[\langle \bar{8} \rangle + soc(Z) :_Z Z]} = \langle \bar{2} \rangle$.

3) It is clear that every approximaitly-quasi-prime submodule of an R-module M is an app-primary submodule of M, but the convers is not true in general for the convers consider the following example.

Let $M = Z$, $R = Z$, the submodule $N = \langle \bar{4} \rangle$ is not approximaitly-quasi-prime submodule of M, since $2.2.\bar{1} = 4 \in \langle \bar{4} \rangle$, but $2.\bar{1} \notin \langle \bar{4} \rangle + soc(Z) = \langle \bar{4} \rangle$. But $N = \langle \bar{4} \rangle$ is an app-primary submodule of M, since for all $a \in Z$, and $y \in Z$ such that $ay \in \langle \bar{4} \rangle$, implies that either $y \in \langle \bar{4} \rangle + soc(Z) = \langle \bar{4} \rangle$ or $a \in \sqrt{[\langle \bar{4} \rangle + soc(Z) :_Z Z]} = \sqrt{[\langle \bar{4} \rangle :_Z Z]} = \sqrt{\langle \bar{4} \rangle} = \langle \bar{2} \rangle$. That is if $2.2 \in \langle \bar{4} \rangle$, then $2 \in \sqrt{[\langle \bar{4} \rangle + soc(Z) :_Z Z]} = \langle \bar{2} \rangle$.

4) It is clear that every prime submodule of an R-module M is an app-primary submodule, but not conversely. Consider the following example for the converse:

Let $M = Z_4$, $R = Z$, the submodule $N = \langle \bar{0} \rangle$ is not prime submodule of Z_4, since $2.\bar{2} = \bar{0} \in N$, for $2 \in Z, \bar{2} \in Z_4$, but $\bar{2} \notin \langle \bar{0} \rangle$ and $2 \notin [\langle \bar{0} \rangle :_Z Z_4] = \langle \bar{4} \rangle$. But $N = \langle \bar{0} \rangle$ is an app-primary submodule of Z_4, since $soc(Z_4) = \langle \bar{2} \rangle$ and for all $a \in Z$, $y \in Z_4$ such that $ay \in \langle \bar{0} \rangle$, implies that either $y \in \langle \bar{0} \rangle + soc(Z_4) = \langle \bar{2} \rangle$ or $a \in \sqrt{[\langle \bar{0} \rangle + soc(Z_4) :_Z Z_4]} = \sqrt{[\langle \bar{2} \rangle :_Z Z_4]} = \sqrt{\langle 2 \rangle} = \langle \bar{2} \rangle$. That is if $2.\bar{2} = \langle \bar{0} \rangle$, implies that either $\bar{2} \in \langle \bar{0} \rangle + soc(Z_4) = \langle \bar{2} \rangle$ or $2 \in \sqrt{[\langle \bar{0} \rangle + soc(Z_4) :_Z Z_4]} = \langle \bar{2} \rangle$.

The following results are characterizations of app-primary submodules.

Proposition 2.3 : Let K be a proper submodule of an R-module M. Then K is an app-primary submodule of M if and only if whenever $JL \subseteq K$, for L is a submodule of M, J is an ideal of R, implies that either $L \subseteq K + soc(M)$ or $J \subseteq \sqrt{[K + soc(M) :_R M]}$.

Proof :

($\Longrightarrow$) Assume that K is an app-primary submodule of an R-module M and $JL \subseteq K$, where J is an ideal of R, L is a submodule of M, with $L \nsubseteq K + soc(M)$, then there exists $l \in L$ such that $l \notin K + soc(M)$. Now we have $JL \subseteq K$, then for any $b \in J$ $bl \in K$. But K is an app-primary submodule of M, and $l \notin K + soc(M)$, it follows that $b^n \in [K + soc(M) :_R M]$ for some $n \in Z^+$, that is $J^n \subseteq [K + soc(M) :_R M]$ for some $n \in Z^+$. Hence $J \subseteq \sqrt{[K + soc(M) :_R M]}$.

($\Longleftarrow$) Assume that $ay \in K$, for $a \in R$, $y \in M$, then $ay = \langle a \rangle \langle y \rangle$, that is $JL \subseteq K$ where $J = \langle a \rangle$, $L = \langle y \rangle$, then by hypothesis either $L \subseteq K + soc(M)$ or $J \subseteq \sqrt{[K + soc(M) :_R M]}$, that is either $a \in \sqrt{[K + soc(M) :_R M]}$ or $y \in K + soc(M)$. Thus K is an app-primary submodule of an R-module M.

As a direct consequence of proposition (2.3) we get the following corollary.

Corollary 2.4 : Let K be a proper submodule of an R-module M. Then K is an app-primary submodule of M if and only if whenever $aL \subseteq K$, for $a \in R$, L is a submodule of M, implies that either $L \subseteq K + soc(M)$ or $a^n \in [K + soc(M) :_R M]$.

Proposition 2.5 : A zero submodule of a non-zero R-module M is an app-primary submodule of M if and only if $ann_R(L) \subseteq \sqrt{[soc(M) :_R M]}$ for all non-zero submodule L of M, with $L \nsubseteq soc(M)$.

Proof :

($\Longrightarrow$) Let L be a non-zero submodule of M, such that $L \nsubseteq soc(M)$, and let $a \in ann_R(L)$, that is $aL = (0)$ but (0) is an app-primary submodule of M and $L \nsubseteq soc(M) = (0) + soc(M)$, it follows by corollary (2.4) that $a \in \sqrt{[(0) + soc(M) :_R M]} = \sqrt{[soc(M) :_R M]}$. That is $ann_R(L) \subseteq \sqrt{[soc(M) :_R M]}$.

($\Longleftarrow$) Suppose that $aL \subseteq (0)$, for $a \in R$ and L is a non-zero submodule of M, with $L \nsubseteq soc(M)$. It follows that $a \in ann_R(L)$, by hypothesis $a \in \sqrt{[soc(M) :_R M]}$, that is $a \in \sqrt{[(0) + soc(M) :_R M]}$. Hence a zero submodule of an R-module M is an app-primary submodule of M.

Proposition 2.6 : Let K be a proper submodule of an R-module M. Then K is an app-primary submodule of M if and only if for every $y \in M$ $[K:_R y] \subseteq \sqrt{[K + soc(M):_R M]}$ with $y \notin K + soc(M)$.

Proof :

($\Longrightarrow$) Suppose that K is an app-primary submodule of M, and $a \in [K:_R y]$, implies that $ay \in K$. Since K is an app-primary submodule of M. and $y \notin K + soc(M)$, then $a \in \sqrt{[K + soc(M):_R M]}$. Thus $[K:_R y] \subseteq \sqrt{[K + soc(M):_R M]}$.

($\Longleftarrow$) Let $ay \in K$, for $a \in R$, $y \in M$, and suppose that $y \notin K + soc(M)$. It follows that $a \in [K:_R y]$ by hypothesis $a \in \sqrt{[K + soc(M):_R M]}$. Thus K is an app-primary submodule of M.

Proposition 2.7 : Let K be a proper submodule of an R-module M with $soc(M) \subseteq K$. Then K is an app-primary submodule of M if and only if $[K:_R I]$ is an app-primary submodule of M for each ideal I of R.

Proof :

($\Longrightarrow$) Suppose that K is an app-primary submodule of M, and $aL \in [K:_R I]$, for $a \in R$, L is a submodule of M, it follows that $aIL \subseteq K$, but K is an app-primary submodule of M, then by corollary (2.4) either $IL \subseteq K + soc(M)$ or $a \in \sqrt{[K + soc(M):_R M]}$. Since $soc(M) \subseteq K$, then $K + soc(M) = K$, it follows that $IL \subseteq K$ or $a \in \sqrt{[K:_R M]}$, hence $L \subseteq [K:_R I]$ or $a^n M \subseteq K$ for some $n \in Z^+$. Thus either $L \subseteq [K:_R I] \subseteq [K:_R I] + soc(M)$ or $a^n M \subseteq K \subseteq [K:_R I] \subseteq [K:_R I] + soc(M)$ for some $n \in Z^+$. Hence either $L \subseteq [K:_R I] + soc(M)$ or $a \in \sqrt{[[K:_R I] + soc(M):_R M]}$. That is $[K:_R I]$ is an app-primary submodule of M.

($\Longleftarrow$) Follows by taking $I = R$, and uasing the fact $[K:_R R] = K$.

Proposition 2.8 : Let K be a proper submodule of an R-module M. Then K is an app-primary submodule of M if and only if $[K:_R a] \subseteq [K + soc(M):_R a^n]$ for $a \in R, n \in Z^+$.

Proof :

($\Longrightarrow$) Suppose that K is an app-primary submodule of M, and let $y \in [K:_R a]$, such that $y \notin K + soc(M)$. Since $y \in [K:_R a]$ it follows that $ay \in K$. But K is an app-primary submodule of M. and $y \notin K + soc(M)$, then $a^n \in [K + soc(M):_R M]$ for some $n \in Z^+$. That is $a^n M \subseteq K + soc(M)$, hence $a^n y \in K + soc(M)$ for all $y \in M$, it follows that $y \in [K + soc(M):_R a^n]$. Thus $[K:_R a] \subseteq [K + soc(M):_R a^n]$.

($\Longleftarrow$) Let $ay \in K$, for $a \in R$, $y \in M$, and suppose that $y \notin K + soc(M)$. Since $ay \in K$ it follows that $y \in [K:_R a] \subseteq [K + soc(M):_R a^n]$, implies that $y \in [K + soc(M):_R a^n]$, that is $a^n y \in K + soc(M)$ for all $y \in M$, hence $a^n M \subseteq K + soc(M)$. That is $a^n \in [K + soc(M):_R M]$. Therefor K is an app-primary submodule of M.

Remark 2.9 : If K is an app-primary submodule of an R-module M, then $[K:_R M]$ need not to be an app-primary ideal of R. The following example explain that:

Let $M = Z_{12}$, $R = Z$, the submodule $K = \langle \bar{0} \rangle$ is an app-primary submodule of Z_{12}, since $soc(Z_{12}) = \langle \bar{2} \rangle$, hence for all $a \in Z$ and $y \in Z_{12}$ such that $ay \in K = \langle \bar{0} \rangle$, implies that either $y \in K + soc(M) = \langle \bar{0} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle$ or $a \in \sqrt{[K + soc(M):_Z M]} = \sqrt{[\langle \bar{0} \rangle + \langle \bar{2} \rangle:_Z Z_{12}]} = \sqrt{[\langle \bar{2} \rangle:_Z Z_{12}]} = \sqrt{2Z} = 2Z$.

That is if $2.\bar{6} \in \langle \bar{0} \rangle$, for $2 \in Z$, $\bar{6} \in Z_{12}$, implies that either $\bar{6} \in \langle \bar{0} \rangle + soc(Z_{12}) = \langle \bar{2} \rangle$ or $2 \in \sqrt{[\langle \bar{0} \rangle + soc(Z_{12}):_Z M]} = 2Z$. But $[\langle \bar{0} \rangle: Z_{12}] = 12Z$ is not app-primary ideal of Z because $4.3 \in 12Z$, for $4,3 \in Z$, but $3 \notin 12Z + soc(Z) = 12Z + (0) = 12Z$ and $4 \notin \sqrt{[12Z + soc(Z):_Z Z]} = \sqrt{[12Z:_Z Z]} = \sqrt{12Z} = 6Z$.

The following proposition shows that under certain condition the resudul of an app-primary submodule is an app-primary ideal.

Proposition 2.10 : Let K be an app-primary submodule of an R-module M with $soc(M) \subseteq K$. Then $[K:_R M]$ is an app-primary ideal of R.

Proof : Let $aI \in [K:_R M]$, for $a \in R$, I is an ideal of R, implies that $aIM \subseteq K$, but K is an app-primary submodule of M, then by corollary (2.4) either $IM \subseteq K + soc(M)$ or $a^n \in [K + soc(M):_R M]$ for some $n \in Z^+$, that is $a^n M \subseteq K + soc(M)$. But $soc(M) \subseteq K$, then $K + soc(M) = K$, it follows that either $IM \subseteq K$ or $a^n M \subseteq K$, so either $I \subseteq [K:_R M] \subseteq [K:_R M] + soc(R)$ or $a^n \in [K:_R M] \subseteq [K:_R M] + soc(R) = \big[[K:_R M] + soc(R): R\big]$. Thus $[K:_R M]$ is an app-primary ideal of R.

Remark 2.11 : Let K be a proper submodule of an R-module M. If $[K:_R M]$ is an app-primary ideal of R, then K need not to be an app-primary submodule of M. The following example shows that:

Let $M = Z \oplus Z$, $R = Z$, and $K = \langle \bar{0} \rangle \oplus 2Z$, then $[K:_R M] = (0)$ which is a prime ideal of Z hence it is an app-primary submodule of Z by remarks and examples (2.2)(4). But K is not app-primary submodule of M, since $2(0,3) = (0,6) \in K$, for $2 \in Z$, $(0,3) \in Z \oplus Z$, but $(0,3) \notin K + soc(Z \oplus Z) = \langle \bar{0} \rangle \oplus 2Z + (0) = \langle \bar{0} \rangle \oplus 2Z$ and $2 \notin \sqrt{[(\langle \bar{0} \rangle \oplus 2Z) + soc(Z \oplus Z):_Z Z \oplus Z]} = \sqrt{[\langle \bar{0} \rangle \oplus 2Z:_Z Z \oplus Z]} = \sqrt{(0)} = (0)$.

Proposition 2.12 : Let K be a proper submodule of faithful multiplication R-module M. Then K is an app-primary submodule of M if and only if $[K:_R M]$ is an app-primary ideal of R.

Proof :

($\Longrightarrow$) Suppose that K is an app-primary submodule of M, and Let $aI \subseteq [K:_R M]$, for $a \in R$, I is an ideal of R, it follows that $aIM \subseteq K$. But K is an app-primary submodule of M, then by corollary (2.4) we have either $IM \subseteq K + soc(M)$ or $a^n M \subseteq K + soc(M)$ for some $n \in Z^+$. Since M is a faithful multiplication, then $soc(M) = soc(R)M$ [14, $Coro.$ 2.14]. Hence either $IM \subseteq [K:_R M]M + soc(R)M$ or $a^n M \subseteq [K:_R M]M + soc(R)M$, it follows that either $I \subseteq [K:_R M] + soc(R)$ or $a^n \in [K:_R M] + soc(R) = [[K:_R M] + soc(R):_R R]$. Hence $[K:_R M]$ is an app-primary ideal of R.

($\Longleftarrow$) Suppose that $[K:_R M]$ is an app-primary ideal of R, and let $aL \subseteq K$, for $a \in R$, L is a submodule of M. Since M is a multiplication then $L = JM$ for some ideal J of R, that is $aJM \subseteq K$, implies that $aJ \subseteq [K:_R M]$. But $[K:_R M]$ is an app-primary ideal of R, then by corollary (2.4) either $J \subseteq [K:_R M] + soc(R)$ or $a^n \in [[K:_R M] + soc(R):_R R] = [K:_R M] + soc(R)$, hence either $JM \subseteq [K:_R M]M + soc(R)M$ or $a^n M \subseteq [K:_R M]M + soc(R)M$. Since M is a faithful multiplication, then by [14, $Coro.$ 2.14] $soc(M) = soc(R)M$. Thus either $JM \subseteq K + soc(M)$ or $a^n M \subseteq K + soc(M)$. That is either $L \subseteq K + soc(M)$ or $a^n \in [K + soc(M):_R M]$. Hence K is an app-primary submodule of M.

Proposition 2.13 : Let K be a proper submodule of a non-singular multiplication R-module M. Then K is an app-primary submodule of M if and only if $[K:_R M]$ is an app-primary ideal of R.

Proof :

($\Longrightarrow$) Suppose that K is an app-primary submodule of M, and Let $as \in [K:_R M]$, for $a, s \in R$, it follows that $asM \subseteq K$. But K is an app-primary submodule of M, then by corollary (2.4) either $sM \subseteq K + soc(M)$ or $a^n M \subseteq K + soc(M)$. But M is non-singular then by [11, $Coro.$ 1.26] $soc(M) = soc(R)M$, and since M is multiplication then $K = [K:_R M]M$. Hence either $sM \subseteq [K:_R M]M + soc(R)M$ or $a^n M \subseteq [K:_R M]M + soc(R)M$, it follows that either $s \in [K:_R M] + soc(R)$ or $a^n \in [K:_R M] + soc(R) = [[K:_R M] + soc(R):_R R]$. Hence $[K:_R M]$ is an app-primary ideal of R.

($\Longleftarrow$) Suppose that $[K:_R M]$ is an app-primary ideal of R, and let $JL \subseteq K$, for J is an ideal of R and L is a submodule of M. Since M is a multiplication then $L = IM$ for some ideal I of R, that is $JIM \subseteq K$, implies that $JI \subseteq [K:_R M]$. But $[K:_R M]$ is an app-primary ideal of R, then by proposition (2.3) either $I \subseteq [K:_R M] + soc(R)$ or $J^n \subseteq [[K:_R M] + soc(R):_R R] = [K:_R M] + soc(R)$, for some $n \in Z^+$, it follows that either $IM \subseteq [K:_R M]M + soc(R)M$ or $J^n M \subseteq [K:_R M]M + soc(R)M$. Since M is non-singular, then by [11, $Coro.$ 1.26] $soc(M) = soc(R)M$, and since M is multiplication then $K = [K:_R M]M$. Hence either $IM \subseteq K + soc(M)$ or $J^n M \subseteq K + soc(M)$. That is either $L \subseteq K + soc(M)$ or $J^n \in [K + soc(M):_R M]$. Hence K is an app-primary submodule of M.

Proposition 2.14 : Let M be a faithful finitely generated multiplication R-module. If A be an app-primary ideal of R. Then AM is an app-primary submodule of M.

Proof : Let $aL \subseteq AM$, for $a \in R$, L be a submodule of M. Since M is a multiplication, then $L = IM$ for some ideal I of R. That is $aIM \subseteq AM$. But M is a finitely generated multiplication R-module, then by [15, Corollary Of Theo. 9], we have $aI \subseteq A + ann_R(M)$, but M is faithful, then $ann_R(M) = (0)$, it follows that $aI \subseteq A$. Now, by hypothesis A is an app-primary ideal of R, then by corollary (2.4) either $I \subseteq A + soc(R)$ or $a^n \in [A + soc(R):_R R] = A + soc(R)$. That is either $IM \subseteq AM + soc(R)M$ or $a^n M \subseteq AM + soc(R)M$, for some $n \in Z^+$. But M is faithful multiplication R-module then by [14, Coro. 2.14] $soc(R)M = soc(M)$. Hence either $L \subseteq AM + soc(M)$ or $a^n M \subseteq AM + soc(M)$. Thus AM is an app-primary submodule of M.

Proposition 2.15 : Let M be a finitely generated multiplication non-singular R-module and B is an app-primary ideal of R with $ann_R(M) \subseteq B$. Then BM is an app-primary submodule of M.

Proof : Let $JK \subseteq BM$, for J is an ideal of R and K be a submodule of M. Since M is a multiplication, then $K = IM$ for some ideal I of R. That is $JIM \subseteq BM$. But M is a finitely generated multiplication, then by [15, Corollary of Theorem. 9] $JI \subseteq B + ann_R(M)$. But $ann_R(M) \subseteq B$, then $B + ann_R(M) = B$, it follows that $JI \subseteq B$. Since B an app-primary ideal of R, then by proposition (2.3) either $I \subseteq B + soc(R)$ or $J^n \subseteq [B + soc(R):_R R] = B + soc(R)$ for some $n \in Z^+$. Thus either $IM \subseteq BM + soc(R)M$ or $J^n M \subseteq BM + soc(R)M$. But M is non-singular then by [11, Corollary (1.26)] $soc(R)M = soc(M)$. Hence either $K \subseteq BM + soc(M)$ or $a^n M \subseteq BM + soc(M)$. Thus BM is an app-primary submodule of M.

Proposition 2.16 : Let K be a proper submodule of faithful finitely generated multiplication R-module M. Then the following statements are equivalent.

1) K is an app-primary submodule of M.
2) $[K:_R M]$ is an app-primary ideal of R.
3) $K = AM$ for some app-primary ideal A of R.

Proof :

(1) $\Longleftrightarrow$ (2) It follows by proposition (2.12).

(2) $\Longrightarrow$ (3) Suppose that $[K:_R M]$ is an app-primary ideal of R, and since M is multiplication $K = [K:M]M = AM$ implies that $A = [K:_R M]$ is an app-primary ideal of R.

(3) $\Longrightarrow$ (2) Suppose that $K = AM$ for some app-primary ideal A of R. Since M is a multiplication, then $K = [K:M]M = AM$. But M is faithful finitely generated multiplication, implies that $A = [K:_R M]$, hence $[K:_R M]$ is an app-primary ideal of R.

Proposition 2.17 : Let H be a proper submodule of non-singular finitely generated multiplication R-module M. Then the following statements are equivalent:

1) H is an app-primary submodule of M.
2) $[H:_R M]$ is an app-primary ideal of R.
3) $H = BM$ for some app-primary ideal B of R with $ann_R(M) \subseteq B$.

Proof :

(1) $\Longleftrightarrow$ (2) It follows by proposition (2.13).

(2) $\Longrightarrow$ (3) Suppose that $[H:_R M]$ is an app-primary ideal of R, and $H = [H:_R M]M$ for M is a multiplication, then $H = BM$ and $B = [H:_R M]$ is an app-primary ideal of R such that $ann_R(M) = [(0):_R M] \subseteq [H:_R M]$.

(3) $\Longrightarrow$ (2) Suppose that $H = BM$ for some app-primary ideal B of R such that $ann_R(M) \subseteq B$. But M is a multiplication, $H = [H:_R M]M$, since M is finitely generated multiplication with $ann_R(M) \subseteq B$ and $[H:_R M]M = BM$, implies that $[H:_R M] = B + ann_R(M) = B$ because $ann_R(M) \subseteq B$, implies that $B + ann_R(M) = B$. Hence $[H:_R M]$ is an app-primary ideal of R.

Remark 2.18 : The intersection of two app-primary submodules of an R-module M need not to be app-primary submodule of M. The following example shows that:

Let $M = Z$, $R = Z$, and $K = 2Z$, $L = 3Z$ are app-primary submodules of M, but $K \cap L = 2Z \cap 3Z = 6Z$ is not app-primary submodule of M, since $2.3 \in 6Z$, but $3 \notin 6Z + soc(Z) = 6Z + (0) = 6Z$ and $2 \notin \sqrt{[6Z + soc(Z):Z]} = \sqrt{[6Z:Z]} = \sqrt{6Z} = 6Z$.

Proposition 2.19 : Let K and L be two app-primary submodule of an R-module M such that $soc(M) \subseteq L$ or $soc(M) \subseteq K$. Then $K \cap L$ is an app-primary submodule of M.

Proof : Since $K \cap L \subseteq L$ and L is a propoer submodule of M, then $K \cap L$ is a proper submodule of M. Now, let $ay \in K \cap L$, for $a \in R$, $y \in M$, and suppose that $a^n \notin [K \cap L + soc(M):M]$ for some $n \in Z^+$, that is $a^n M \nsubseteq K \cap L + soc(M)$, it follows that $a^n M \nsubseteq K + soc(M)$ and $a^n M \nsubseteq L + soc(M)$. Since $ay \in K \cap L$ implies that $ay \in K$ and $ay \in L$. But K and L be two app-primary submodule of an R-module M and $a^n M \nsubseteq K + soc(M)$ and $a^n M \nsubseteq L + soc(M)$, it follows that $y \in K + soc(M)$ and $y \in L + soc(M)$, implies that $y \in (K + soc(M)) \cap (L + soc(M))$. If $soc(M) \subseteq L$ then $L + soc(M) = L$, that is $y \in (K + soc(M)) \cap L$, again since $soc(M) \subseteq L$, then by modular law we have $y \in (K \cap L) + soc(M)$. Similarly if $soc(M) \subseteq K$ we get $y \in (K \cap L) + soc(M)$. Hence $K \cap L$ is app-primary submodule of M.

The following propositions gives the behavior of app-primary submodules under R-homomorphism.

Proposition 2.20 : Let $f: M \longrightarrow M'$ be an R-epimorphism and K be an app-primary submodule of M with $Ker\, f \subseteq K$. Then $f(K)$ is an app-primary submodule of M'.

Proof : $f(K)$ is a proper submodule of M'. If not, we have $f(K) = M'$, that is $f(m) \in M' = f(K)$ for some $m \in M$, it follows that there exists $x \in K$ such that $f(x) = f(m)$, that is $f(x - m) = 0$, so $x - m \in Ker\, f \subseteq K$, it follows that $m \in K$, hence $M = K$ (since K is a propoer submodule of M), contradiction. Now let $ay' \in f(K)$, for $a \in R$, $y' \in M'$. Since f is an epimorphism, then there exists $y \in M$ such that $f(y) = y'$. That is $f(ay) = af(y) \in f(K)$, implies that $f(ay) = f(x)$ for some $x \in K$, so $f(ay - x) = 0$, it follows that $ay - x \in Ker\, f \subseteq K$, hence $ay \in K$. But K is an app-primary submodule of M, then either $y \in K + soc(M)$ or $a^n M \subseteq K + soc(M)$ for some $n \in Z^+$. It follows that $y' = f(y) \in f(K) + f(soc(M)) = f(K) + soc(f(M)) = f(K) + soc(M')$[since f is an epimorphism], or $a^n f(M) \subseteq f(K) + soc(M')$. Hence $f(K)$ is an app-primary submodule of M'.

Proposition 2.21 : Let $f: M \longrightarrow M'$ be an R-epimorphism and K is an app-primary submodule of M'. Then $f^{-1}(K)$ is an app-primary submodule of M.

Proof : It is clear that $f^{-1}(K)$ is a proper submodule of M. Now, let $ay \in f^{-1}(K)$, for $a \in R$, $y \in M$, it follows that $f(ay) = af(y) \in K$. But K is an app-primary submodule of M', then either $f(y) \subseteq K + soc(M')$ or $a^n M' = a^n f(M) \subseteq K + soc(M')$ for some $n \in Z^+$. It follows that either $y \in f^{-1}(K) + f^{-1}(soc(M')) \subseteq f^{-1}(K) + soc(M)$ or $a^n M \subseteq f^{-1}(K) + f^{-1}(soc(M')) \subseteq f^{-1}(K) + soc(M)$. Hence $f^{-1}(K)$ is an app-primary submodule of M.

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المقاسات الجزئية الابتدائية تقريباً

علي شبل عجيل ، هيبة كريم محمدعلي
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الملخص

اكدت الدراسة الحاليه على مفهوم المقاسات الجزئية الابتدائية تقريباً للمقاس الاحادي الايسر M على الحلقة الابداليه بمحايد R كأعمام للمقاسات الجزئية الابتدائية والمقاسات الجزئية الاولية تقريباً, حيث يدعى المقاس الجزئي الفعلي N من المقاس M مقاس جزئي ابتدائي من M اذا كان $ay \in N$ حيث $a \in R$ ، $y \in M$ يؤدي الى اما $y \in N + soc(M)$ او $a^k M \subseteq N + soc(M)$ لبعض k عدد صحيح موجب في Z. اعطينا العديد من المكافئات والخصائص الاساسية لهذا المفهوم. من ناحيه اخرى درسنا العلاقات الشكلية لهذا المفهوم مع بعض اصناف المقاسات الاخرى. أكثر من هذا سلوك المقاسات الجزئية الابتدائية تقريباً تحت تأثير التشاكلات نوقشت.