IASJ المجلات الاكاديمية العراقية

Tikrit Journal of Pure Science
ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

α–almost similar operators

Amjad H. Abdul Majeed , Laith K. Shaakir
Department of Mathematics , College of Computer Science and Mathematics , Tikrit University , Tikrit , Iraq
https://doi.org/10.25130/tjps.v24i5.421

ARTICLE INFO.

Article history:
-Received: 17 / 4 / 2019
-Accepted: 25 / 6 / 2019
-Available online: / / 2019

Keywords: α–almost similar, almost similar, unitarily equivalent, self-adjoint operator.

Corresponding Author:
Name: Amjad H. Abdul Majeed
E-mail:
amjadmath20980@st.tu.edu.iq
Tel:

ABSTRACT

The study focuses on α–almost similar operator which is a new concept of the operator theory and also some basic concepts related to the concept α–almost similar.
The study also defines a new concept called $\beta-operator$ which is an expansion of the concept $\theta-operator$ and the relationship of this concept with the α–almost similar.
At the end of this research, we study some important relationships among similar, unitarily equivalent, and almost similar on the one hand and α–almost similar on the other.

Introduction

We denote $B(\mathcal{H}_1,\mathcal{H}_2)$ to the set of all bounded linear operators from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$. if $\mathcal{H}=\mathcal{H}_1=\mathcal{H}_2$ then we denote $B(\mathcal{H})$ instead of $B(\mathcal{H}_1,\mathcal{H}_2)$. The operator $T\in B(\mathcal{H})$ is called self- adjoint if $T=T^*$ where T^* is the adjoint of T[1]. An operator $A\in B(\mathcal{H})$ is said to be isometric if $A^*A=I$[2]. If $A^*A=AA^*$ then A is called normal operator. And if $A^*A=AA^*=I$ then A is said to be unitary [3]. If $A^*=A$ and $A^2=A$ then A is said to be projection. If $AA^*A=A$ then A is said to be partially isometric, equivalently A^*A is projection (i.e. $(A^*A)^2=A^*A$) [4]. Clearly every unitary operator is isometric and normal.
Two operators $A\in B(\mathcal{H})\,and\, B\in B(\mathcal{H})$ are said to be similar and denoted by $A\sim B$, if there exists an invertible operator X such that $X A = B X$ $(equvalently\; A = X^{-1}BX)$. If $A\sim B$, then A and B have the same: spectrum, point spectrum and approximate point spectrum [5].
Similarly, two operators $A,B\in B(\mathcal{H})$ are said to be unitarily equivalent and denoted by $A\cong B$, if there exists a unitary operator U such that $U A = B U$ $(equvalently\; A = U^*B\,U)$[4]. If A,B are similar normal then they are unitarily equivalent by fugled-Putnam theorem [6].
Let A,B are two bounded linear operators on $B(\mathcal{H})$. Then A,B are said to be almost similar and denoted by A $\overset{a.s}{\approx} B$ if there exists an invertible operator X such that:
$A^*A = X^{-1}B^*B\,X$ and, $A^*+A = X^{-1}(B^*+B)\,X$. The class of almost similar was first introduced by Jibril [7]. we have extended this concept to α–almost similar and demonstrated some different results.
An operator $A\in B(\mathcal{H})$ is said to be $\theta-operator$ if A^*A commutes with A^*+A. The class of all $\theta-operator$ in B $(\mathcal{H})$ is denoted by θ. The class of $\theta-operator$ be which has been widely studied by Campbell [8]. We have extended the concept of $\theta-operator$ to another concept we called it $\beta-operator$, the class of $\beta-operator$ in B $(\mathcal{H})$ is denoted by β.
Let $T\in B(\mathcal{H})$ then the set of all complex number λ for which $T-\lambda I$ is not invertible is called the spectrum of T and denoted by $\sigma(T)$ that is, $\sigma(T)=\{\lambda\in\mathbb{C}:(T-\lambda I)$ is not invertible$\}$. The complement of the spectrum of T is called resolvent set of T. The spectrum of T can be split into many disjoint sets [9].
The point spectrum of the operator T is denoted by$\sigma_p(T)$ is the set of all those λ for which $T-\lambda I$ is not injective, that is $\sigma_p(T)=\{\lambda\in\mathbb{C}: ker(T-\lambda I)\neq\{0\}\}$.
A scalar λ is said to be the approximate point spectrum for the operator T and denoted by $\sigma_{ap}(T)$, if there exists a sequence of unit vector $\{x_n\}$ such that $\|(T-\lambda I)x_n\|\to 0$ [9]. Let T be a linear transformation from a normed space X into a normed space Y $(i.e. T: X\longrightarrow Y)$. Then T is said to be compact if $\overline{T(B)}$ is compact for every bounded

subset $\mathcal{B}$ of X. that is, $\overline{T(\mathcal{B})}$ is relatively compact for every bounded subset $\mathcal{B}$ of X [9].

1. Basic concept on α–almost similarity

Definition 1.1: Let α be a real number, two bounded linear operators $A, B \in B(\mathcal{H})$ are said to be α–almost similar and, denoted by $A \overset{\propto}{\approx} B$. If there exist an invertible operator X such that:
$A^*A = X^{-1}B^*B\ X$ ……….(1) and, $A^* + \alpha A = X^{-1}(B^* + \alpha B)\, X$………… (2).

Example 1.2: Let $A= \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$ be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$, and define the invertible operator on $\mathbb{C}^2$ as follows: $X=X^{-1}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$,take $\alpha = 2$, then $A \overset{2}{\approx} B$. To show that

$A^*A = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} (\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix})\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = X^{-1}B^*B\ X$

$A^* + 2A$

$= \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} + 2\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}3 & 2\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}(\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} + 2\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix})\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$= X^{-1}(B^* + 2B)\, X$

Remark 1.3: Every 1– almost similar operators are almost similar and the converse are true.

The following example show almost similar and α-almost similar are independent when α≠1.

Example 1.4: Let $A= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$ and $B=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$, and define the invertible operator on $\mathbb{C}^2$ as follows: $X=\begin{bmatrix}\frac{1}{2} & 0\\0 & 2\end{bmatrix}$,take $\alpha = -1$. Then $A \overset{-1}{\approx} B$. But A is not almost similar to B Since $A^* + A \neq X^{-1}(B^* + B)\, X$, indeed $A^* + A = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = 2I$, $B^* + B = \begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}$. $B^* + B \neq X(A^* + A)X^{-1} = 2X\, I\, X^{-1} = 2I$ for every invertible operator X.

Theorem 1.5: let α∈ ℝ, the relation $\overset{\propto}{\approx}$ on $B(\mathcal{H})$ is equivalence relation.

Proof: (i)Reflexivity, let $A \in B(\mathcal{H})$ take $X = I$. $A^*A = X^{-1}A^*A\ X$ and, $A^*+\alpha\ A = X^{-1}(A^*+\alpha\ A)\ X$. Then $A \overset{\propto}{\approx} A$.

(ii)Symmetry, suppose that $A, B \in$ B $(\mathcal{H})$ and, $A \overset{\propto}{\approx} B$. Then there exists an invertible operator X such that.
$A^*A = X^{-1}B^*B\ X$ ……… (1), and, $A^* + \alpha A = X^{-1}(B^*+\alpha B)\, X$…... (2).

Now, pre-multiplying and post-multiplying (1) and (2) by X and X^{-1}, respectively yields. $XA^*A\,X^{-1} = B^*B$…………....(3), and, $X(A^*+\alpha A)\ \ X^{-1} = B^*+\alpha B$……….. (4).

Take $Y = X^{-1}$, which is an invertible operator, since $X^{-1}$is an invertible operator.

Substituting X and X^{-1} in (3) and (4) by $Y^{-1}$and Y respectively, we get $B \overset{\propto}{\approx} A$.

(iii) Transitivity, suppose that A, B and $C \in$ B $(\mathcal{H})$. And $A \overset{\propto}{\approx} B$, $B \overset{\propto}{\approx} C$, to show that $A \overset{\propto}{\approx} C$.

Since $A \overset{\propto}{\approx} B$, then there exists an invertible operator X such that.
$A^*A = X^{-1}B^*B\,X$………(1), and $A^*+\alpha A = X^{-1}(B^*+\alpha B)\ X$ …. (2).

Also, since $B \overset{\propto}{\approx} C$, then there exists an invertible operator $Y \in B(\mathcal{H})$ such that
$B^*B = Y^{-1}\ C^*C\ Y$ … (3) and, $B^*+\alpha B = Y^{-1}(C^*+\alpha C)\ Y$ …… (4).

Substituting (3) and (4) in (1) and (2) as follows:
$A^*A = X^{-1}[Y^{-1}C^*C\ Y]\ X = X^{-1}Y^{-1}[C^*C]\,YX = (YX)^{-1}\ C^*C\ (YX)$……… (5)

Also, $A^*+\alpha A = X^{-1}[Y^{-1}\,(C^* + \alpha C)Y]X$. Which implies that $A^*+\alpha A = (YX)^{-1}[C^* + \alpha C]\ (YX)$….. (6).

Then from (5) and (6) we get $A \overset{\propto}{\approx} C$.

Proposition 1.6: Let $A \in B\ (\mathcal{H})$, such that $A \overset{\propto}{\approx} 0$, then $A = 0$.

Proof: Since $A \overset{\propto}{\approx} 0$ then there exists an invertible operator X such that.
$A^*A = X^{-1}0^*0\ X = 0$ ……. (1), and $A^* + \alpha A = X^{-1}(0^*+\alpha 0)\ X = 0$ ……. (2).

Then $A^*A = 0$ and $A^* + \alpha A = 0$. Now, $\|Ax\|^2 = \langle Ax|Ax\rangle = \langle A^*Ax|x\rangle = \langle 0|x\rangle = 0$

Therefore $Ax = 0$ for all $x \in \mathcal{H}$. Thus $A = 0$.

Remark 1.7: suppose that $A, B \in B\ (\mathcal{H})$ such that $A \overset{\propto}{\approx} B$, then clearly by using mathematical induction we can prove:

(i) $(A^*A)^n = X^{-1}\ (B^*B)^n X$,

(ii) $(A^* + \alpha A)^n = X^{-1}(B^* + \alpha B)^n X$. For all-natural number n.

Proposition 1. 8: Let $A, B \in B\ (\mathcal{H})$ such that $A \overset{\propto}{\approx} B$. Then A is isometric if and only if B is isometric.

Proof: Suppose that A is isometric. Since $A \overset{\propto}{\approx} B$ this means that there exists an invertible operator X such that $A^*A = X^{-1}(B^*B)\ X$ ……. (1), and, $A^* + \alpha A = X^{-1}(B^* + \alpha B)\ X$ …… (2). Since A is isometric then $A^*A = I$ substituting in the equality (1) we have
$I = A^*A = X^{-1}(B^*B)\ X$ which implies that $B^*B = I$. Thus, B is isometric.

Conversely: by the same way we can prove that A is isometric whenever B is isometric.

Proposition 1. 9: Let α∈ ℝ. A, B are two operators in B $(\mathcal{H})$ with $A \overset{\propto}{\approx} B$. Then:

(i) A^*A is onto if and only if B^*B is onto,

(ii) $A^* + \alpha A$ is onto if and only if $B^* + \alpha B$ is onto,

(iii) A^*A is one -to-one if and only if B^*B is one-to-one,

(iv) $A^* + \alpha A$ is one-to-one if and only if $B^* + \alpha B$ is one -to-one,

(v) A^*A is projection if and only if B^*B is projection.

Proof: Clearly.

Remark 1.10: Let α∈ ℝ. A, B are two operators in B $(\mathcal{H})$ with $A \overset{\propto}{\approx} B$. Then:

(vi) A^*A is one-to-one and onto if and only if B^*B is one-to-one and, onto.

(vii) $A^* + \alpha A$ is one-to-one and, onto if and only if $B^* + \alpha B$ is one-to-one and, onto.

Proof: immediately from proposition 1.9 above.

proposition 1.11: Let $A \in B\ (\mathcal{H})$ and $A \overset{\propto}{\approx} I$, then A is isometry.

Proof: Suppose that $A \overset{\propto}{\approx} I$. then there exists an invertible operator X such that $A^*A = X^{-1}(I^*I)\,X = X^{-1}(I) \quad X = X^{-1}X = I$…. (1). Then $A^*A = I$ ($i.e.\, A$ *is isometry*).

Proposition 1.12: Let $A, B \in B\,(\mathcal{H})$ and $A \overset{\propto}{\approx} B$ such that A is partially isometric then B is partially isometric.

Proof: $A \overset{\propto}{\approx} B$ means that there exists an invertible operator X such that

$A^*A = X^{-1}(B^*B)\; X$….. (1). Since A is parietally isometric then A^*A is projection (i.e. $(A^*A)^2 = A^*A$). By squaring both sides in (1) we have ($X^{-1}(B^*B)\,X$) $(X^{-1}(B^*B)\;X) = (A^*A)^2 = A^*A$. Then $X^{-1}(B^*B)$ $(B^*B)\,X = X^{-1}(B^*B)\,X$ … (2).

Pre-multiplying and post-multiplying (2) by X and X^{-1} respectively we have, $(B^*B)^2 = B^*B$(i.e. B^*B is projection). Which implies that B is partially isometric.

Proposition 1.13: Let α∈ ℝ. Then the transformation $\varphi : B\,(\mathcal{H}) \longrightarrow$ B $(\mathcal{H})$ that satisfies $\varphi(A^*A) = X^{-1}(B^*B)$ X, $\varphi(A^* + \alpha A) = X^{-1}(B^* + \alpha B)$ X is an automorphism. That is, it maps sums into sums, products into products and scalar multiples into scalar multiplies.

Proof: suppose that A, B, C and $D \in B(\mathcal{H})$ *such that* $\varphi(A^*A) = X^{-1}(B^*B)$ X and $\varphi(C^*C) = X^{-1}(D^*D)\,X$. Then

$\varphi(A^*A{+}\alpha C^*C) = X^{-1}(B^*B + \alpha D^*D) \quad X = X^{-1}(B^*B)\,X + \alpha \quad X^{-1}(D^*D) \quad X = \varphi(A^*A) + \alpha\, \varphi(C^*C)$ and, $\varphi((A^*A)\;(C^*C)) = X^{-1}((B^*B)(D^*D))\,X = X^{-1}(B^*B)XX^{-1}(D^*D)\,X$ $= (X^{-1}(B^*B)X)(X^{-1}(D^*D)\,X) = \varphi(A^*A)\,\varphi(C^*C)$.

Proposition 1.14: Let $A, B \in B\,(\mathcal{H})$ such that A, B are unitarily equivalent then $A \overset{\propto}{\approx}$ B for every α ∈ ℝ.

Proof: Since A and B are unitarily equivalent then there exists a unitary operator U such that $A = U^*BU$. Then $A^* = U^*B^*U$ which implies that $A^*A = (U^*B^*U)(U^*BU) = U^*B^*(UU^*)BU = U^*B^*(I)BU = U^*B^*BU$. And, $A^* + \alpha A = U^*B^*U + \alpha U^*BU = U^*B^*U + U^*\alpha BU = U^*(B^* + \alpha B)U$

Thus, $A \overset{\propto}{\approx} B$ for all α ∈ ℝ.

proposition 1.15: Let $A, B \in B\,(\mathcal{H})$ such that $A \overset{\propto}{\approx} B$ for every real α. Then $(A + \lambda I) \overset{\propto}{\approx} (B + \lambda I)$for every real λ.

Proof: $A \overset{\propto}{\approx}$ B means that there is an invertible operator X such that.

$A^*A = X^{-1}(B^*B)\,X$ ….. (1). And, $A^* + \alpha A = X^{-1}(B^* + \alpha B)\,X$ … (2).

From the equality (2) we have $A^* + \alpha A = X^{-1}B^*X + X^{-1}\alpha B\;X$, by post-adding to both sides $\lambda I + \alpha\lambda I$ which implies that $A^* + \alpha A + \lambda I + \alpha\lambda I = X^{-1}B^*X + X^{-1}\alpha B\; X{+}\lambda I + \alpha\lambda I$. Then we have $A^* + \lambda I + \alpha(A + \lambda I) = X^{-1}B^*X + X^{-1}\alpha B\; X{+}\lambda I + \alpha\lambda I$ which implies that

$(A + \lambda I)^* + \alpha(A + \lambda I) = X^{-1}(B + \lambda I)^*X{+}X^{-1}\,(\alpha B + \lambda I)X$…. (3). Since λ is real number. Now, we want to prove that $(A + \lambda I)^*(A + \lambda I) = X^{-1}(B + \lambda I)^*\,(A + \lambda I)X$. $(A + \lambda I)^*(A + \lambda I) = A^*A + \lambda A^* + \lambda A + \lambda^2 I = A^*A + \lambda(A^* + A) + \lambda^2 I$

$= X^{-1}(B^*B)\;X + \lambda X^{-1}(B^* + B)\;X + \lambda^2 X^{-1}X$ (since (1) and (2) are satisfies when α= 1) $= X^{-1}[(B^*B) + \lambda(B^* + B) + \lambda^2]X = X^{-1}[(B^* + \lambda I)(B + \lambda I)]\;X$ $= X^{-1}[(B + \lambda I)^*\;(B + \lambda I)]\;X$, since λ is real number.

Then $(A + \lambda I)^*(A + \lambda I) = X^{-1}[(B + \lambda I)^*\;\;(B + \lambda I)]\;X$…. (4).

From the equality (3) and the equality (4) we have $(A + \lambda I) \overset{\propto}{\approx} (B + \lambda I)$for every real λ.

proposition 1.16: Let $A, B \in B\,(\mathcal{H})$ be projections such that $A \overset{\propto}{\approx} B$ and $(A + \lambda I) \overset{\propto}{\approx} (B + \lambda I)$. Then: $\sigma(A) = \sigma(B), \sigma_p\,(A) = \sigma_p(B)\, and\; \sigma_{ap}\,(A) = \sigma_{ap}(B)$.

Proof: $A \overset{\propto}{\approx} B$ means that there is an invertible operator X such that.

$A^*A = X^{-1}(B^*B)\;X$ ……… (1). And, $A^* + \alpha A = X^{-1}(B^* + \alpha B)\,X$ ……. (2).

Since A and B are projection then A and B are self-adjoints. Then (2) becomes $(1 + \alpha)A = X^{-1}(1 + \alpha)B\;X$ which implies that $A = X^{-1}B\;X$. This means that $A \sim B$, then $\sigma(A) = \sigma(B), \sigma_p\,(A) = \sigma_p(B)\; and, \sigma_{ap}\,(A) = \sigma_{ap}(B)$ [6].

Theorem 1.17 [10]: the operator $A \in B\,(\mathcal{H})$ is compact if and only if A^*A is compact.

Proposition 1.18: Let α∈ ℝ. $A, B \in B\,(\mathcal{H})$ and $A \overset{\propto}{\approx} B$. If A is compact then B is compact.

Proof: since $A \overset{\propto}{\approx} B$ then there exsist an invertible operator X such that

$A^*A = X^{-1}B^*B\,X$ pre-multiplying and post-multiplying both sides by X and X^{-1} respectively, we have $X\,A^*A\,X^{-1} = B^*B$. Since A is compact then $X\,A^*A\,X^{-1}$is also compact. By theorem 1.17 above then B is compact.

Theorem 1.19: Let α∈ ℝ. $A, B \in B\,(\mathcal{H}), X$ be an invertible operator. If $XA = BX$ and, $XA^* = B^*X$. Then A and B are α-almost similar.

Proof: by hypothesis $XA = BX$ and, $XA^* = B^*X$ then we have $A = X^{-1}BX$ and, $A^* = X^{-1}B^*X$. Now, $A^*A = (X^{-1}B^*X)(X^{-1}BX) = X^{-1}B^*(X\,X^{-1})BX = X^{-1}B^*B\;X$ and,

$A^* + \alpha A = X^{-1}B^*X + X^{-1}(\alpha B)X = X^{-1}(B^* + \alpha B\;)\;X$. Then A and B are α-almost similar.

Proposition 1.20: If A ,$B \in$ B $(\mathcal{H})$ are similar normal operators, then $A \overset{\propto}{\approx} B$.

Proof: suppose that A and B are similar normal operators then there exists an invertible operator X such that $XA = BX$. Then $XA^* = B^*X$ by Fuglede-Putnam theorem [6].

Now, by using theorem 1.20 we have, A and B are α-almost similar.

Remark 1.21: The converse of the proposition 1.20 is not true in general.

Consider the following example: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $X = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, be the operators on two-dimensional Hilbert space $\mathbb{C}^2$, take $\alpha = 2$,then

$A \overset{2}{\approx} B$. Also A is similar to B (I.e. $XA = BX$) but $A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = AA^*$ and, $B^*B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = BB^*$. Then A and B are not normal operators.

2. The properties of self-adjoint operator on α-almost similarity.

Proposition 2.1: Suppose that A, B are self-adjoint operators in $B\,(\mathcal{H})$ with $A \sim B$ (i.e. A is similar to B), then $A \overset{\propto}{\approx} B$, for every α∈ ℝ.

Proof: Since A and B are similar operators, then there exists an invertible operator X such that $XA = B\,X$ (i.e. $A = X^{-1}B\,X$).

Also, A and B are self-adjoint operators in B $(\mathcal{H})$, then

$A^*A = X^{-1}B^*B\,X$ ……….(1). Also, $A^* + \alpha A = A + \alpha A = X^{-1}B\,X + \alpha\, X^{-1}B\,X = X^{-1}(B + \alpha B)\,X = X^{-1}(B^* + \alpha B)X$ ……… (2). From (1) and (2) we have $A \overset{\propto}{\approx} B$.

Remark 2.2: The converse of the Proposition 2.1. above is not true in general.

For example: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $X = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$ take $\alpha = 2$. We know that $A \overset{2}{\approx} B$ as in example 1.2. Moreover $A \sim B$. But $A \neq A^*$, also $B \neq B^*$.Thus, A and B are not self-adjoint operators.

Proposition 2.3: Let α= $-1 \in \mathbb{R}$. $A, B \in B\,(\mathcal{H})$ and $A \overset{\propto}{\approx} B$. If A is self-adjoint then B is self-adjoint.

Proof: Since $A \overset{-1}{\approx} B$, then there exist an invertible operator X such that $A^* - A = X^{-1}(B^* - B)\,X$. Which implies that $0 = X^{-1}(B^* - B)\,X$ …… (1). Pre-multiplying and post multiplying (1) by X and X^{-1} respectively we have $0 = B^* - B$. Then $B = B^*$.

Remark 2.4: The converse of proposition 2.3 above is not true in general for example $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = A^*, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B^* and, X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$, take $\alpha \in \mathbb{R}$

Then $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix} \neq X^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = X^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = X^{-1} I\, X = I$ .Thus, A is not α-almost similar to B. Then A is not (-1)-almost similar to B.

Theorem 2.5 [4]: (Cartesian form) let T be any operator, then there exist self-adjoint operators A and B such that $T = A + iB$. When $A = \frac{1}{2}(T + T^*)$ and, $B = \frac{1}{2i}(T - T^*)$.

Theorem 2.6: Let $T \in$B $(\mathcal{H})$ then $T = T^*$ if and only if T is normal and

$(T + T^*)^{\,2} = 4T^*$T.

Proof: If $T = T^*$ then clearly $(T + T^*)^{\,2} = 4T^*$T and T is normal.

Conversely: If $4T^*T = (T^* + \mathrm{T})^{\,2} = (T^* + T)(T^* + T) = T^{*2} + 2T^*T + T^2$. Hence, $T^{*2} - 2T^*T + T^2 = 0$ which implies that $(T^* - \mathrm{T})^{\,2} = 0$. $\Longrightarrow$ $-(T^* - \mathrm{T})^{\,2} = 0 \Longrightarrow (T^* - T)(T - T^*) = 0$. Let $S = T^* - T$ $\Longrightarrow SS^* = 0 \Longrightarrow 0 = \langle SS^*x|x\rangle = \langle S^*x|S^*x\rangle = \|S^*x\|^2$ for every x. Then $S^*x = 0$ for every $x \Longrightarrow S^* = 0 \Longrightarrow S = 0 \Longrightarrow T^* - T = 0 \Longrightarrow T^* = T$.

Remark 2.6: If $T = T^*$ then $(T^* + \alpha\mathrm{T})^{\,2} = (1 + \alpha\,)^2T^*$T for every α∈ ℝ.

Proposition 2.7: Suppose that $(T^* + \alpha\mathrm{T})^{\,2} = (1 + \alpha\,)^2T^*$T then:

(i) If α=1 and T is normal then $T = T^*$.

(ii) If α=-1 then $T = T^*$.

(iii) If α≠1, -1 then $T^{*2} = T^2$.

Proof: (i) directly as in theorem 2.6. And (ii) clearly.

Now to prove (iii) let α≠1, -1. $(T^* + \alpha\mathrm{T})^{\,2} = (1 + \alpha\,)^2T^*$T by taking adjoint to both sides we have $(T + \alpha T^*)^{\,2} = (1 + \alpha\,)^2T^*$T. Then $T^{*2} + \alpha T^*T + \alpha TT^* + \alpha^2T^2 = T^2 + \alpha TT^* + \alpha^2T^{*2} \Longrightarrow T^{*2} = T^2$.

Theorem 2.8 [4]: If T is normal operator, then there exists a unitary operator U such that $T^* = UT$.

3. The properties of $\boldsymbol{\beta - operator}$ on α-almost similarity.

Definition 3.1: let $A \in B(\mathcal{H})$, then A is called an $\beta - operator$ if A^*A commutes with A^*+αA. The class of all $\beta - operator$ in a Banach algebra on a Hilbert space $\mathcal{H}$ is denoted by β i.e. β= $\{A: A \in B\,(\mathcal{H})\ such\ that\ [A^*A, A^*{+}\alpha A] = 0\}$.

Example 3.2: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and take α=3 then $[A^*A, A^*{+}3A] = 0$

i.e. $(A^*A)(A^*{+}3A) = (A^*{+}3A)(A^*A)$ which implies that A is β-operator.

Proposition 3.3: If $A \in B\,(\mathcal{H})$ is β- operator then kA is β- operator for every real number k.

Proof: Clearly.

Proposition 3.4: If $A, B \in B\,(\mathcal{H})$ and $A \overset{\propto}{\approx} B$ such that B is $\beta - operator$ then A is $\beta - operator$.

Proof: $A \overset{\alpha}{\approx} B$ means that there exists an invertible operator X such that

A^*A = $X^{-1}(B^*B)\,X$. And, $A^* + \alpha$A = $X^{-1}(B^* + \alpha B)\,X$.

Then, $[\,X^{-1}(B^* + \alpha B)\,X]\,[X^{-1}(B^*B)\,X] = [\,A^* + \alpha\mathrm{A}]\,A^*A$ …... (1)

And $[X^{-1}(B^*B)\ X]\ [\ X^{-1}(B^* + \alpha B)\ X] = A^*\mathrm{A}\ [\,A^* + \alpha\mathrm{A}]$ …. (2). From the equality (1) we have: $[\,X^{-1}(B^* + \alpha B)\,(B^*B)\,X] = [\,A^* + \alpha\mathrm{A}]\,A^*A$ …… (3). Also, from the equality (2) we have: $[X^{-1}(B^*B)\,(B^* + \alpha B)\,X] = A^*A\,[\,A^* + \alpha\mathrm{A}]$……..… (4).

Since B is β- operator then the left-hand side of the equality (3) and the equality (4) are equal. which imply that the right-hand side of the equality (3) and the equality (4) are equal. Hence A is β- operator.

4. The relation among similarity, unitarily equivalent, quasi similarity and almost similarity with α-almost similarity.

Proposition 4.1: Let $A, B \in B\,(\mathcal{H})$ are orthogonal projection then A and B are α-almost similar if and only if A and B are similar.

Proof: Suppose that $A \overset{\propto}{\approx} B$ and A, B are projection then by proposition 1.16 we get $A{\sim}B$.

Conversely, suppose that A and B are similar operators then there exists invertible operator X such that $A = X^{-1}B\ X$, since A and B are orthogonal projection then $A = A^* = A^2$, $B = B^* = B^2$. Which implies that $A^2 = X^{-1}B^2\ X$ then we have $A^*A = X^{-1}B^*B\ X$.

On the other hand, the second inequality follows from the fact that

$A^* + \alpha A = (1+\alpha)A = (1+\alpha)X^{-1}B\ X = X^{-1}(B^* + \alpha B)X$. Thus, $A \overset{\propto}{\approx} B$.

Proposition 4.2: Let $\alpha \in \mathbb{R}$. $A, B \in B(\mathcal{H})$ and A, B are self-adjoint then A and B are unitarily equivalent if and only if $A \overset{\propto}{\approx} B$.

Proof: Suppose that A and B are unitarily equivalence then by proposition 1.14we have $A \overset{\propto}{\approx} B$.

Conversely: Suppose that $A, B \in B(\mathcal{H})$ are self-adjoint with $A \overset{\propto}{\approx} B$.

Now, $A \overset{\propto}{\approx} B$ means that there exists an invertible operator X such that

$A^*A = X^{-1}(B^*B)\ X$……. (1), and $A^* + \alpha A = X^{-1}(B^* + \alpha B)\ X$ ……… (2).

Since A, B are self-adjoint and $A \overset{\propto}{\approx} B$ then they are similar operates($i.e\ A = X^{-1}B\ X$). Then A and B are both similar and self-adjoint operators then A and B are normal. Thus A and B are unitarily equivalence.

Corollary 4.3: Let $\alpha \in \mathbb{R}$. $A, B \in B(\mathcal{H})$ are self-adjoint and $A \overset{\propto}{\approx} B$. Then A and B are unitarily equivalent.

Proof: directly from proposition 4.2 above.

Proposition 4.4: Let $A, B \in B(\mathcal{H})$ are self-adjoint operators then, A and B are α- almost similar if and only if A and B are almost similar.

Proof: Suppose that A, B are α-almost similar then there is an invertible operator X such that. $A^*A = X^{-1}(B^*B)\ X$ ……. (1), and $A^* + \alpha A = X^{-1}(B^* + \alpha B)\ X$ ……. (2).

Since A and B are self-adjoint Then $A = A^*$, $B = B^*$ then (2) becomes

$(1+\alpha)A = (1+\alpha)X^{-1}B\ X$. Now pre-multiplying both sides by $\frac{2}{(1+\alpha)}$, $\alpha \neq -1$.Which implies that $2A = 2X^{-1}BX \Longrightarrow A + A^* = X^{-1}(B + B^*)X$ …….. (3).

From (1) and (3) we have A and B are almost similar.

Conversely, suppose that A, B are almost similar then (1) and (3) satisfies. Since A and B are self-adjoint Then (3) becomes $2A = 2X^{-1}BX$. pre-multiplying both sides by $\frac{1+\alpha}{2}$which implies that $(1+\alpha)A = (1+\alpha)X^{-1}BX \Longrightarrow A + \alpha A = X^{-1}(B + \alpha B)\ X \Longrightarrow A^* + \alpha A = X^{-1}(B^* + \alpha B)\ X$. Thus, A and B are α-almost similar.

Remark 4.7: the converse of proposition 4.6 is not true in general consider the following example: Let $A = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$ be the operators on the two-dimensional Hilbert space $\mathbb{C}^2$, and define the invertible operator on $\mathbb{C}^2$ as follows: $X = X^{-1} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$,take $\alpha = 2$. then $A \overset{2}{\approx} B$. As in example 1.2. Also, $A \overset{a.s}{\approx} B$. But $A \neq A^*$ and, $B \neq B^*$.

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المؤثرات الخطية المتشابهة تقريبا من النمط- α

امجد حمد عبد المجيد ، ليث خليل شاكر

قسم الرياضيات ، كلية علوم الحاسوب والرياضيات ، جامعة تكريت ، تكريت ، العراق

الملخص

درسنا في هذه البحث المؤثرات الخطية المقيدة المتشابهة تقريبا من النمط-α وهو مفهوم جديد لنظرية المؤثرات الخطية, كذلك بعض المفاهيم الاساسية المتعلقة بمفهوم المؤثرات الخطية المقيدة المتشابهة تقريبا من النمط-α. كذلك عرفنا مفهوما جديدا والذي اطلقنا عليه اسم المؤثر من النمط-β والذي يعتبر توسيعا للمؤثر من النمط-θ وعلاقة هذا المؤثر بالمؤثرات الخطية المتشابهة تقريبيا من النمط-α. في نهاية هذا البحث درسنا بعض العلاقات المهمة بين لتشابه, والمؤثرات الاحادية المتكافئة, والتشابه التقريبي من جهة وبين المؤثرات الخطية المتشابهة تقريبا من النمط-α من الجهة الاخرى.