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Weakly Nearly Quasi Prime Submodules

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ABSTRACT

In this paper, all rings R are commutative with identity, and all R-modules are unitary Left R-modules. We introduce the concept WNQP submodule as new generalizations of weakly quasi prime submodule and give basic properties, examples and characterizations of this concept.

1- Introduction

Weakly quasi prime submodule was introduced and studied by [1] in 2013,as generalization of weakly prime submodule, where a proper submodule N of an R-module M is a weakly prime , if whenever $0 \neq am \in N$, for $a \in R$, $m \in M$, implies that either $m \in N$ or $am \subseteq N$ [2] ,and a proper submodule N of an R-module M is a weakly quasi prime, if whenever $0 \neq abm \in N$, for $a, b \in R$,$m \in M$,implies that $am \in N$ or $bm \in N$, Recently, this concept generalized to weakly approximatitly quasi prime submodules and weakly pseudo quasi 2-abserbing submodules see [3,4]. The Jacobson radicale of a module M denoted by $J(M)$ is the intersection of all maximal submodule 0f M. A submodule H of an R-module M is called coclosed if for any submodule L of M with $L \subseteq H$ we have $\frac{H}{L}$ is small in $\frac{M}{L}$,implies that $N = L$.[5] . Recall that a submodule H of an R-module M is small if $H + K = M$, implies that $K = M$ for any proper submodule K of M. [8] A non- zero R-module M is called hollow, if every proper submodule of M is small [8].

2- Basic properties and characterizations of weakly nearly quasi prime submodules.

In this section we introduce the definition of weakly quasi prime submodule and give examples, and basic properties and characterizations, of this concept.

Definition (2.1)

A proper submodule H of an R-module M is called weakly nearly quasi prime submodule of M ($for\ short\ WNQP\ submodule\ of\ M$), if whenever $0 \neq cdm \in H$ for $c, d \in R, m \in M$, implies that either $cm \in H + J(M)$ or $dm \in H + J(M)$. And an ideal A of a ring R is called a WNQP ideal of R if A is a WNQP submodule of an R-module R.

Examples and Remarks (2.2)

1. Every weakly quasi prime submodule of an R-module M is a WNQP submodule, but not conversely

Proof: It is obvious

For the converse consider the following Example:

Consider the Z-module Z_{48}, the submodule $\langle\bar{4}\rangle$ is a WNQP submodule of Z_{48} but not weakly quasi prime submodule. Since $J(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$. For $0 \neq 2.2.\bar{1} \in \langle\bar{4}\rangle$, for $2 \in Z$, $\bar{1} \in Z_{48}$, but $2.\bar{1} \notin \langle\bar{4}\rangle$. So $\langle\bar{4}\rangle + J(Z_{48}) = \langle\bar{4}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$.

Thus for $a,b \in Z$, $m \in Z_{48}$ if $0 \neq abm \in \langle\bar{4}\rangle$, it follows that either am $am \in \langle\bar{4}\rangle + J(Z_{48}) = \langle\bar{2}\rangle$ or $bm \in \langle\bar{4}\rangle + J(Z_{48}) = \langle\bar{2}\rangle$

2. Every maximal submodule of an R-module M is a WNQP submodule but not conversely.

Proof: Let H be amaximale submodule of M then by $[1, prop. (3.1.8)]$ H is a weakly quasi prime. Hence by (1) H is a WNQP submodule of M.For the converse, the submodule $\langle\bar{4}\rangle$ of Z-module Z_{48} is a WNQP submodule but not maximal.

3. Every weakly prime submodule of M is a WNQP submodule, but not conversely.

Proof:

Let H be a weakly prime submodule of M, then by [1] H is a weakly quasi prime Thus by part (1) H is a WNQP submodule of M. For the converse consider the following example:

Let $M = Z_{48}$, $R = Z$, $H = \langle\bar{8}\rangle$ is not weakly prime submodule of M, since $0 \neq 2.\bar{4} \in \langle\bar{8}\rangle$ for $2 \in Z$, $\bar{4} \in Z_{48}$,but $\bar{4} \notin \langle\bar{8}\rangle$ and $2 \notin [\langle\bar{8}\rangle :_Z Z_{48}] = 8Z$ But $H = \langle\bar{8}\rangle$ is WNQP submodule if whenever $0 \neq 2.4.\bar{1} \in \langle\bar{8}\rangle$, for, $2,4 \in Z$, $\bar{1} \in Z_{48}$, implies that $2.\bar{1} \in \langle\bar{8}\rangle + J(Z_{48}) = \langle\bar{2}\rangle$ or $4.\bar{1} \in \langle\bar{8}\rangle + J(Z_{48}) = \langle\bar{2}\rangle$.

4. The intersection of two WNQP submodules of M need not to be WNQP submodule of M The following example shows that:

The submodules $\langle\bar{3}\rangle, \langle\bar{4}\rangle$ of the Z-module Z_{48} are WNQP submodule of Z_{48}, but $\langle\bar{3}\rangle \cap \langle\bar{4}\rangle = \langle\overline{12}\rangle$ is not WNQP submodule of Z_{48}. Since $0 \neq 3.4.\bar{1} \in \langle\overline{12}\rangle$, $3,4 \in Z$, $\bar{1} \in Z_{48}$, but $3.\bar{1} \notin \langle\overline{12}\rangle + J(Z_{48}) = \langle\bar{6}\rangle$ and $4.\bar{1} \notin \langle\overline{12}\rangle + J(Z_{48}) = \langle\bar{6}\rangle$.

Now, we introduce many characterizations of WNQP submodules.

Proposition (2.3)

A proper submodule H of an R-module M is a WNQP submodule of M if and only if $[H:_M cd] \subseteq [0:_M cd] \cup [H + J(M):_M c] \cup [H + J(M):_M d]$

Proof:

($\Rightarrow$) Let n $n \in [H:_M cd]$,implies that $cdn \in H$. If $cdn = 0$, then $n \in [0:_M cd]$, so $n \in [0:_M cd] \cup [H + J(M):_M c] \cup [H + J(M):_M d]$

If $0 \neq cdn \in H$ and H is a WNQP submodule of M implies that either $cn \in H + J(M)$ or $dn \in H + J(M)$, it follows that $n \in [H + J(M):_M c]$ or $n \in [H + J(M):_M d]$, that is $n \in [H + J(M):_M c] \cup [H + J(M):_M d]$. Thus, $[H:_M cd] \subseteq [0:_M cd] \cup [H + J(M):_M c] \cup [H + J(M):_M d]$.

($\Leftarrow$) Assume that $0 \neq cdn \in H$, for $c,d \in R$, $n \in M$, implies that $n \in [H:_M cd]$, by hypothesis , $n \in [0:_M cd] \cup [H + J(M):_M c] \cup [H + J(M):_M d]$. But $0 \neq cdn$,it follows that $n \notin [0:_M cd]$.

Hence $n \in [H + J(M):_M c] \cup [H + J(M):_M d]$,that is either $n \in [H + J(M):_M c]$ or $n \in [H + J(M):_M d]$, thus either $cn \in H + J(M)$ or $dn \in H + J(M)$.

Hence H is a WNQP submodule of M.

Proposition (2.4)

A proper submodule H of an R-module M is a WNQP submodule of M if and only if for each $c \in R$ and $n \in M$ with $cn \notin H + J(M)$, $[H:_R cn] \subseteq [0:_R cn] \cup [H + J(M):_R n]$.

Proof:

($\Rightarrow$) Let $d \in [H:_R cn]$, implies that $cdn \in H$, If $cdn = 0$, then $d \in [0:_R cn]$, it follows that $d \in [0:_R cn] \cup [H + J(M):_R n]$. If $0 \neq cdn \in H$ for $c,d \in R$, $n \in M$ with $cn \notin H + J(M)$, it follows that $dn \in H + J(M)$, that is $d \in [H + J(M):_R n]$ so $d \in [0:_R cn] \cup [H + J(M):_R n]$, it follows $[H:_R cn] \subseteq [0:_R cn] \cup [H + J(M):_R n]$.

($\Leftarrow$) Let $0 \neq cdn \in H$, with $cn \notin H + J(M)$ for $c,d \in R$, $n \in M$, implies that $d \in [H:_R cn]$ [, it follows by hypothesis, we get $d \in [0:_R cn] \cup [H + J(M):_R n]$. Since $0 \neq cdn$, implies that $d \notin [0:_R cn]$. Thus $d \in [H + J(M):_R n]$, implies that $dn \in H + J(M)$. Hence H is a WNQP submodule of M.

The following corollaries are direct application of proposition (2.4)

Corollary (2.5)

A proper submodule H of an R-module M is a WNQP submodule of M, if and only if for every $c \in R$ and every submodule L of M with $cL \nsubseteq H + J(M)$, $[H:_R L] \subseteq [0:_R L] \cup [H + J(M):_R L]$.

Corollary (2.6)

A proper submodule H of an R-module M is a WNQP submodule of M if and only if for every ideal A of R and every submodule L of M with $AL \nsubseteq H + J(M)$.$[H:_R AL] \subseteq [0:_R AL] \cup [H + J(M):_R L]$.

Proposition (2.7)

A proper submodule H of an R-module M is a WNQP submodule of M if and only if whenever $\{0\} \neq cdL \subseteq H$ for $c,d \in R$, L is a submodule of M , implies that either $cL \subseteq H + J(M)$ or $dL \subseteq H + J(M)$.

Proof:

($\Rightarrow$) Let $\{0\} \neq cdL \subseteq H$ for $c,d \in R$, L is a submodule of M with $cL \nsubseteq H + J(M)$ and $dL \nsubseteq H + J(M)$, that is there exists a non-zero elements $e_1, e_2 \in L$ with $ce_1 \notin H + J(M)$ and $de_2 \notin H + J(M)$. Thus $0 \neq cde_1 \in H$ and H is a WNQP submodule of M and $ce_1 \notin H + J(M)$, implies that $de_1 \in H + J(M)$. Again $0 \neq cde_2 \in H$ and H is a WNQP submodule of M and $de_2 \notin H + J(M)$, implies that $ce_2 \in H + J(M)$. Also $0 \neq cd(e_1 + e_2) \in H$, implies that either $c(e_1 + e_2) \in H + J(M)$ or $d(e_1 + e_2) \in H + J(M)$. If $c(e_1 + e_2) = ce_1 + ce_2 \in H + J(M)$, but $ce_2 \notin H + J(M)$, implies that $ce_1 \in H + J(M)$ contradiction. If $d(e_1 + e_2) = de_1 + de_2 \in H + J(M)$ and $de_1 \in H + J(M)$, then $de_2 \in H + J(M)$ contradiction. Thus $cL \subseteq H + J(M)$ or $dL \subseteq H + J(M)$.

($\Leftarrow$) It is obvious.

Proposition (2.8)

A proper submodule H of an R-module M is a WNQP submodule of M if and only if whenever $\{0\} \neq CDL \subseteq H$, for C, D are ideals of R,and L is a submodule of M ,implies that either $CL \subseteq H + J(M)$ or $DL \subseteq H + J(M)$.

Proof:

$(\Longrightarrow)$ Let $\{0\} \neq CDL \subseteq H$, where C, D are ideals of R and L is a submodule of M with $CL \nsubseteq H + J(M)$ and $DL \nsubseteq H + J(M)$, it follows that there exists a non-zero elements $n_1, n_2 \in L$, and a non-zero elements $c \in C, d \in D$ such that $cn_1 \notin H + J(M)$ and $dn_2 \notin H + J(M)$. We have $0 \neq cdn_1 \in H$, and H is a WNQP with $cn_1 \notin H + J(M)$, implies that $dn_1 \in H + J(M)$. Also $0 \neq cdn_2 \in H$ and H is a WNQP, and $dn_2 \notin H + J(M)$ implies that $cn_2 \in H + J(M)$. Also , $0 \neq cd(n_1 + n_2) \in H$ and H is a WNQP then $c(n_1 + n_2) \in H + J(M)$ or $d(n_1 + n_2) \in H + J(M)$. If $c(n_1 + n_2) = cn_1 + cn_2 \in H + J(M)$ and $cn_2 \in H + J(M)$, implies that $cn_1 \in H + J(M)$ contradiction. If $d(n_1 + n_2) = dn_1 + dn_2 \in H + J(M)$ and $dn_1 \in H + J(M)$, implies that $dn_2 \in H + J(M)$ contradiction. Thus $CL \subseteq H + J(M)$ or $DL \subseteq H + J(M)$.

$(\Longleftarrow)$ Let $\{0\} \neq cdL \subseteq H$ for $c, d \in R$,L is a submodule of M , that is $\{0\} \neq \langle c\rangle\langle d\rangle L \subseteq H$, by hypothesis either $\langle c\rangle L \subseteq H + J(M)$ or $\langle d\rangle L \subseteq H + J(M)$. Thus either $cL \subseteq H + J(M)$ or $dL \subseteq H + J(M)$.

As direct consequence of proposision (2.8) we have the following corollaries.

Corollary (2.9)

A proper submodule H of an R-module M is a WNQP if and only if whenever $\{0\} \neq CDn \subseteq H$ for C, D are ideals of R, $n \in M$ implies that either $Cn \subseteq H + J(M)$ or $Dn \subseteq H + J(M)$.

Corollary (2.10)

A proper submodule H of an R-module M is a WNQP if and only if $aDn \subseteq H$ for $a \in R$, D is an ideal of R, $n \in M$, implies that either $cn \in H + J(M)$ or $Dn \subseteq H + J(M)$.

Corollary (2.11)

A proper submodule H of an R-module M is a WNQP if and only if whenever $\{0\} \neq aDL \subseteq H$,$a \in R$,D is an ideal of R and L is a submodule of M , implies that either $cL \subseteq H + J(M)$ or $DL \subseteq H + J(M)$

.The following results are some basic properties of WNQP submodules.

Proposition (2.12)

Let L , K are submodules of an R-module M with L is a proper submodules of K and $J(M) \subseteq J(K)$. If L is a WNQP submodule of M, then L is a WNQP submodule of K.

Proof:

Let $\{0\} \neq CDN \subseteq L$ for c, d are ideals of R ,N is a submodule of $L \subseteq M$. Since L is a WNQP submodule of M then by proposition (2.8) either $CN \subseteq L + J(M)$ or $DN \subseteq L + J(M)$, since $J(M) \subseteq J(K)$, we get either $CN \subseteq L + J(K)$ or $DN \subseteq L + J(K)$. Hence L is a WNQP submodule of K.

We need to recall the following Lemma

Lemma (2.13) $[5, \boldsymbol{prop.} (1.2.16)]$

"If H is a coclosed submodule of an R-module M, then $J(H) = H \cap J(M)$".

Proposition (2.14)

Let L and K are submodule of an R-module M with $K \nsubseteq L$ and K is a coclosed submodule of M such that $J(M) \subseteq K$. If L is a WNQP submodule of M, then $L \cap K$ is a WNQP submodule of K.

Proof:

Since $K \nsubseteq L$, then $L \cap K$ is a proper submodule of K , Let $\{0\} \neq CDH \subseteq L \cap K$ where C, D are ideals of R , H is a submodule of K , it follows that H is a submodule of M , so $\{0\} \neq CDH \subseteq L$ and $\{0\} \neq CDH \subseteq K$, since L is a WNQP submodule of M, then by proposition (2.8) either $CH \subseteq L + J(M)$ or $DH \subseteq L + J(M)$, it follows that either $CH \subseteq \big(L + J(M)\big) \cap K$ or $DH \subseteq \big(L + J(M)\big) \cap K$, but $J(M) \subseteq K$, then by modular Law , we get either $CH \subseteq (L \cap K) + \big(K \cap J(M)\big)$ or $DH \subseteq (L \cap K) + \big(K \cap J(M)\big)$, But L is a coclosed , then by Lemma (2.13) we get either $CH \subseteq (L \cap K) + J(K)$ or $DH \subseteq (L \cap K) + J(K)$. Thus $L \cap H$ is a WNQP submodule of K.

Proposition **(2.15)**

Let L , K be WNQP submodules of an R-module M with $K \nsubseteq L$ and either $J(M) \subseteq L$ or $J(M) \subseteq K$. Then $L \cap K$ is a WNQ P submodule of M.

Proof:

It is clear that $L \cap K$ is a proper submodule of M . Let $\{0\} \neq adH \subseteq L \cap K$ where $a, d \in R$, H is a submodule of M , implies that $\{0\} \neq adH \subseteq L$ and $\{0\} \neq adH \subseteq K$. But L ,K are WNQP, then by proposition (2.7) either $\big(aH \subseteq L + J(M)$ *or* $dH \subseteq L + J(M)\big)$ and either $\big(aH \subseteq K + J(M)$ *or* $dH \subseteq K + J(M)\big)$, it follows that either $aH \subseteq \big(L + J(M)\big) \cap \big(K + J(M)\big)$ or $dH \subseteq \big(L + J(M)\big) \cap \big(K + J(M)\big)$ .suppose that $J(M) \subseteq L$, it follows that $L + J(M) = L$. Thus either $aH \subseteq L \cap \big(K + J(M)\big)$ or $dH \subseteq L \cap \big(K + J(M)\big)$, Hence by modular law either $aH \subseteq (L \cap K) + J(M)$ or $dH \subseteq (L \cap H) + J(M)$. Therefor $L \cap H$ is a WNQP submodule of M.

Proposition (2.16)

Let M be an R-module with $J(M)$ is a weakly quasi prime submodule of M, and L is a proper submodule of M with $L \subseteq J(M)$. Then N is a WNQP submodule of M.

Proof:

Let $\{0\} \neq cdH \subseteq L$ for, $c, d \in R$, H is a submodule of , it follows that $\{0\} \neq cdH \subseteq J(M)$. But $J(M)$ is a weakly quasi prime submodule of M, then either $cH \subseteq J(M) \subseteq L + J(M)$ or $dH \subseteq$

$J(M) \subseteq L + J(M)$. Hence L is a WNQP submodule of M.

Corollary (2.17)

Let L be a small submodule of an R-module M with $J(M)$ is a weakly quasi prime submodule of M .Then L is a WNQP submodule of M.

Proof:

Since L is a small submodule of M, then by [6] $L \subseteq J(M)$, hence the proof follows by proposition (2.16).

Proposition (2.18)

Let M be an R-module, and L be a submodule of M with $J(M) \subseteq L$, Then L is a WNQP submodule of M if and only if $[L:_M J]$ is a WNQP submodule of M for any ideal J of R.

Proof:

$(\Rightarrow)$ Let $0 \neq cdm \in [L:_M J]$ for $c, d \in R$, $m \in M$, then $\{0\} \neq cd(Jm) \subseteq L$. Since L is a WNQP submodule of , then by proposition (2.8) either $c(Jm) \subseteq L + J(M)$ or $d(Jm) \subseteq L + J(M)$. But $J(M) \subseteq L$, implies that $L + J(M) = L$, it follows that either $cJm \subseteq L$ or $dJm \subseteq L$, implies that either $cm \in [L:_M J] \subseteq [L:_M J] + J(M)$ or $dm \in [L:_M J] \subseteq [L:_M J] + J(M)$. Thus $[L:_M J]$ is a WNQP submodule of M.

$(\Leftarrow)$ Direct by taken $J = R$.

Corollary (2.19)

Let M be an R-module, and L is a submodule of M with $J\left(\frac{M}{L}\right) = \{0\}$. Then L is a WNQP submodule of M if and only if $[L:_M J]$ is a WNQP submodule of M for any ideal J of R.

Proof:

Since $J\left(\frac{M}{L}\right) = \{0\}$, then by [7, prop. (9.1.4)] $J(M) \subseteq L$. Hence proof follows by proposition (2.18).

Proposition (2.20)

Let $f \in Hom(M, M')$ be an R-epimorphism and $Kerf$ is a small submodule of M and H is a WNQP submodule of M'. Then $f^{-1}(H)$ is a WNQP submodule of M.

Proof:

Let $0 \neq cdm \in f^{-1}(H)$, for $c, d \in R, m \in M$ implies that $0 \neq cdf(m) \in H$. Since H is a WNQP submodule of M', it follows that either $cf(m) \in H + J(M')$ or $df(m) \in H + J(M')$it follows by [7, coro. (9.1.5)(a)] either $cm \in f^{-1}(H) + f^{-1}(J(M')) = f^{-1}(H) + J(M)$ or $dm \in f^{-1}(H) + J(M)$. Hence $f^{-1}(H)$ is a WNQP submodule of M.

Proposition (2.21)

Let $f \in Hom(M, M')$ be an R-epimorphism and $Kerf$ is a small submodule of M, If H is aWNQP submodule of M with $Kerf \subseteq H$, Then $f(H)$ is a WNQP submodule of M'.

Proof:

Since $Kerf \subseteq H$, then it is clear that $f(H)$ is a proper submodule of M'. Let $0 \neq cdm' \in f(H)$, for $c, d \in R$, $m' \in M'$. But f is an epimorphism , then $0 \neq cdf(m) \in f(H)$ for some non-zero $m \in M$. Thus $0 \neq cdf(m) = f(n)$ for some non-zero $n \in H$, implies that $0 \neq cdm \in H$ $(since\ Kerf \subseteq H)$. But H is a WNQP submodule of M,then either $cm \in H + J(M)$ or $dm \in H + J(M)$, thus either $cf(m) \in f(H) + f(J(M))$ or $df(m) \in f(H) + f(J(M))$. Since $Kerf$ is small submodule of , then by [7, coro. (9.1.5)(b)] $f(J(M)) = J(M')$, hence either $cm' \in f(H) + J(M')$ or $dm' \in f(H) + J(M')$. Therefor $f(H)$ is a WNQP submodule of M'.

Proposition (2.22)

Let $M = M_1 \oplus M_2$ where M_1, M_2 are hollow R-modules , and H_1 , H_2 are WNQP submodule of M_1 , M_2 respectively , Then $H = H_1 \oplus H_2$ is a WNQP submodule of $M = M_1 \oplus M_2$.

Proof:

Since M_1, M_2 are hollow R-modules then H_1 , H_2 are small submodules of M_1 , M_2 respectively then by [9, prop. (5.20)(1)] $H_1 \oplus H_2$ is small submodule of M. Let $(0,0) \neq cd(m_1, m_2) \in H_1 \oplus H_2$ where $(m_1, m_2) \in M_1 \oplus M_2$ such that m_1 , m_2 are non-zero elements of M_1 , M_2 respectively, $c, d \in R$. Since $H_1 \oplus H_2$ small submodule of M then $H_1 \oplus H_2 \subseteq J(M) = J(M_1 \oplus M_2)$, implies that $H_1 \oplus H_2 + J(M_1 \oplus M_2) = J(M_1 \oplus M_2)$. Now, $(0,0) \neq cd(m_1, m_2) \in H_1 \oplus H_2$, implies that $0 \neq cdm_1 \in H_1$ and $0 \neq cdm_2 \in H_2$. But H_1 , H_2 are WNQP submodule of M_1 , M_2 respectively, then either $cm_1 \in H_1 + J(M_1)$ or $dm_1 \in H_1 + J(M_1)$ and either $cm_2 \in H_2 + J(M_2)$ or $dm_2 \in H_2 + J(M_2)$, But both H_1, H_2 are small submodules of M_1, M_2 respectively it follows that $H_1 \subseteq J(M_1)$, $H_2 \subseteq J(M_2)$ that is $H_1 + J(M_1) = J(M_1)$ and $H_2 + J(M_2) = J(M_2)$. Hence either $cm_1 \in J(M_1)$ or $dm_1 \in J(M_1)$ and either $cm_2 \in J(M_2)$ or $dm_2 \in J(M_2)$, implies that either $c(m_1, m_2) \in J(M_1) \oplus J(M_2)$ or $d(m_1, m_2) \in J(M_1) \oplus J(M_2)$. But $J(M_1 \oplus M_2) = J(M_1) \oplus J(M_2)$. Thus either $c(m_1, m_2) \in J(M_1 \oplus M_2) \subseteq (H_1 \oplus H_2) + J(M_1 \oplus M_2)$ or $d(m_1, m_2) \in J(M_1 \oplus M_2) \subseteq (H_1 \oplus H_2) + J(M_1 \oplus M_2)$. That is $H_1 \oplus H_2$ is a WNQP submodule of M.

Proposition (2.23)

Let $M = M_1 \oplus M_2$ be R-module, and H_1 is a small submodule of M_1, and M_2 has no maximal submodule. Then H_1 is a WNQP submodule of M_1 if and only if $H_1 \oplus M_2$ is a WNQP submodule of M.

Proof:

$(\Rightarrow)$ Let $(0,0) \neq cd(m_1, m_2) \in H_1 \oplus M_2$, for $c, d \in R, (m_1, m_2) \in M_1 \oplus M_2$, m_1, m_2 are non-zero elements of M_1, M_2 respectively , then it follows that $0 \neq cdm_1 \in H_1$. Since H_1 is a WNQP submodule of M_1, then either $cm_1 \in H_1 + J(M_1)$ or $dm_1 \in H_1 + J(M_1)$, but H_1 is small, so $H_1 \subseteq J(M_1)$, that is $H_1 + J(M_1) = J(M_1)$, and since M_2 has no maximal submodule, then $M_2 = J(M_2)$. Thus either $cm_1 \in J(M_1)$ or $dm_1 \in J(M_1)$, and since $m_2 \in M_2$ .It follows that either $c(m_1, m_2) \in J(M_1) \oplus J(M_2) = J(M_1 \oplus M_2) \subseteq (H_1 \oplus M_2) + J(M_1 \oplus M_2)$ or $d(m_1, m_2) \in J(M_1) \oplus J(M_2) = J(M_1 \oplus M_2) \subseteq (H_1 \oplus M_2) + J(M_1 \oplus M_2)$. That is $H_1 \oplus M_2$ is a WNQP submodule of M.

($\Leftarrow$) Let $0 \neq cdm_1 \in H_1$, for $c, d \in R$, m_1 is a non-zero element of M_1, it follows for each $m_2 \in M_2$, $(0,0) \neq cd(m_1, m_2) \in H_1 \oplus M_2$ but $H_1 \oplus M_2$ is a WNQP submodule of M, then either $c(m_1, m_2) \in (H_1 \oplus M_2) + J(M_1 \oplus M_2) = (H_1 \oplus M_2) + \left(J(M_1) \oplus J(M_2)\right)$ or $d(m_1, m_2) \in (H_1 \oplus M_2) + \left(J(M_1) \oplus J(M_2)\right)$. But H_1 is small submodule of M_1, then $H_1 \subseteq J(M_1)$, that is $H_1 + J(M_1) = J(M_1)$ and since M_2 has no maximal submodule, then $M_2 = J(M_2)$. Thus either $c(m_1, m_2) \in (H_1 \oplus M_2) + \left((H_1 + J(M_1)) \oplus M_2\right)$ or $d(m_1, m_2) \in (H_1 \oplus M_2) + \left((H_1 + J(M_1)) \oplus M_2\right)$. But $H_1 \oplus M_2 \subseteq (H_1 + J(M_1)) \oplus M_2$, implies that $(H_1 \oplus M_2) + (H_1 + J(M_1)) \oplus M_2 = (H_1 + J(M_1)) \oplus M_2$ so, either $c(m_1, m_2) \in (H_1 + J(M_1)) \oplus M_2$ or $d(m_1, m_2) \in (H_1 + J(M_1)) \oplus M_2$, that is either $cm_1 \in H_1 + J(M_1)$ or $dm_1 \in H_1 + J(M_1)$. Thus H_1 is a WNQP submodule of M_1.

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المقاسات الجزئية الظاهرية الاولية الضعيفة المتقاربة

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الملخص

في هذا البحث جميع الحلقات هي ابدالية بمحايد وجميع المقاسات المعرفة عليها هي مقاسات احادية يسارية. قدمنا مفهوم المقاسات الجزئية من نمط WNQP كأعمام جديد للمقاسات الظاهرية الاولية الضعيفة وأعطينا خواص الاساسية وأمثلة و تشخيصات حول هذا المفهوم.