Weakly Nearly Quasi Prime Submodules

Hero Jumaa Hassan, Haibat K. Mohammad Ali

Department of Mathematic, College of Computer Science and Math., University of Tikrit, Iraq

Article history:
- Received: 3/3/2022
- Accepted: 11/5/2022
- Available online: / / 2022

Keywords: Weakly Quasi Prime Submodules, Weakly nearly quasi prime submodules, Jacobson radical of Modules

Corresponding Author:
Name: Hero Jumaa Hassan
E-mail: j.hassan35436@st.tu.edu.iq
ali2013@gmail.com

1- Introduction

Weakly quasi prime submodule was introduced and studied by [1] in 2013, as generalization of weakly prime submodule, where a proper submodule \( N \) of an \( R \)-module \( M \) is a weakly prime, if whenever \( 0 \neq am \in N \), for \( a \in R \), \( m \in M \), implies that either \( m \in N \) or \( am \subseteq N \) [2]. And a proper submodule \( N \) of an \( R \)-module \( M \) is a weakly quasi prime, if whenever \( 0 \neq abm \in N \), for \( a, b \in R \), \( m \in M \), implies that \( am \in N \) or \( bm \in N \). Recently, this concept generalized to weakly approximately quasi prime submodules and weakly pseudo quasi 2-absorbing submodules see [3,4]. The Jacobson radical of a module \( M \) denoted by \( J(M) \) is the intersection of all maximal submodule \( 0 \) of \( M \). A submodule \( H \) of an \( R \)-module \( M \) is called closed if for any submodule \( L \) of \( M \) with \( L \subseteq H \) we have \( \frac{H}{L} \) is small in \( \frac{M}{L} \), implies that \( N = L \) [5]. Recall that a submodule \( H \) of an \( R \)-module \( M \) is small if \( H + K = M \), implies that \( K = M \) for any proper submodule \( K \) of \( M \) [8]. A non-zero \( R \)-module \( M \) is called hollow, if every proper submodule of \( M \) is small [8].

2- Basic properties and characterizations of weakly nearly quasi prime submodules.

In this section we introduce the definition of weakly quasi prime submodule and give examples, and basic properties and characterizations, of this concept.

Definition (2.1)

A proper submodule \( H \) of an \( R \)-module \( M \) is called weakly nearly quasi prime submodule of \( M \) (for short WNQP submodule of \( M \)), if whenever \( 0 \neq cdm \in H \) for \( c, d \in R, m \in M \), implies that either \( cm \in H + J(M) \) or \( dm \in H + J(M) \). And an ideal \( A \) of a ring \( R \) is called a WNQP ideal of \( R \) if \( A \) is a WNQP submodule of an \( R \)-module \( R \).

Examples and Remarks (2.2)

1. Every weakly quasi prime submodule of an \( R \)-module \( M \) is a WNQP submodule, but not conversely

Proof: It is obvious

For the converse consider the following Example:

Consider the \( Z \)-module \( Z_{48} \), the submodule \( (\bar{4}) \) is a WNQP submodule of \( Z_{48} \) but not weakly quasi prime submodule. Since \( J(Z_{48}) = (\bar{2}) \cap (\bar{3}) = (\bar{6}) \). For \( 0 \neq 2.2.\bar{1} \in (\bar{4}) \), for \( 2 \in Z \), \( \bar{1} \in Z_{48} \), but \( 2.\bar{1} \notin (\bar{4}) \). So \( (\bar{4}) + J(Z_{48}) = (\bar{4}) + (\bar{6}) = (\bar{2}) \).
Thus for \( a, b \in Z, m \in Z_{48}\) if \( \neq abm \in (4) \), it follows that either \( amn \in (4) + J(Z_{48}) = (2) \) or \( bn \in (4) + J(Z_{48}) = (2) \).

2. Every maximal submodule of an R-module \( M \) is a WNQP submodule but not conversely.

**Proof:** Let \( H \) be a maximal submodule of \( M \) then by \([1, \text{prop. (3.1.8)}]\) \( H \) is a weakly quasi prime. Hence by \((1) \) \( H \) is a WNQP submodule of \( M \). For the converse, the submodule \((4)\) of \( Z \)-module \( Z_{48}\) is a WNQP submodule but not maximal.

3. Every weakly prime submodule of \( M \) is a WNQP submodule but not conversely.

**Proof:**

Let \( H \) be a weakly prime submodule of \( M \), then by \([1]\) \( H \) is a weakly quasi prime. Thus by part \((1) \) \( H \) is a WNQP submodule of \( M \). For the converse consider the following example:

Let \( M = Z_{48}, R = Z, H = (8) \) is not weakly prime submodule of \( M \), since \( \neq 2, 4 \in (8) \) for \( 2 \in Z, 4 \in Z_{48}\) but \( 2 \notin (8) \) and \( 4 \notin (8)\). But \( H = (8) \) is WNQP submodule if whenever \( \neq 2, 4. \bar{I} \in (8) \), for \( 2, 4 \in Z ; \bar{I} \in Z_{48}\), implies that \( 2. \bar{I} \in (8) + (Z_{48}) = (2) \) or \( 4. \bar{I} \in (8) + (Z_{48}) = (2) \).

4. The intersection of two WNQP submodules of \( M \) need not to be WNQP submodule of \( M \). The following example shows that:

The submodules \((3), (4)\) of the \( Z \)-module \( Z_{48}\) are WNQP submodules of \( Z_{48}\), but \((3) \cap (4) = (12)\) is not WNQP submodule of \( Z_{48}\). Since \( \neq 3.4 \bar{I} \in (12), 3, 4 \in Z, \bar{I} \in Z_{48}\), but \( 3. \bar{I} \notin (12) + (Z_{48}) = (6) \) and \( 4. \bar{I} \notin (12) + (Z_{48}) = (6)\).

Now, we introduce many characterizations of WNQP submodules.

**Proposition (2.3)**

A proper submodule \( H \) of an \( R \)-module \( M \) is a WNQP submodule of \( M \) if and only if \([H: mcd] \subseteq [0: mcd] \cup [H + J(M); m]\) \( \cup [H + J(M); m] \).

**Proof:**

\((\Rightarrow)\) Let \( n \in [H: mcd] \), implies that \( cdn \in H \). If \( cdn = 0 \), then \( n \in [0: mcd] \), so \( n \in [0: mcd] \cup [H + J(M); m] \cup [H + J(M); m] \).

If \( \neq cdn \in H \) and \( H \) is a WNQP submodule of \( M \) implies that either \( cn \in H + J(M) \) or \( dni \in H + J(M), it follows that \( n \in [H + J(M); m] \cup [H + J(M); m] \), that is \( n \in [H + J(M); m] \cup [H + J(M); m] \). Thus, \([H: mcd] \subseteq [0: mcd] \cup [H + J(M); m] \cup [H + J(M); m] \).

\((\Leftarrow)\) Assume that \( \neq cdn \in H \), for \( c, d \in R, n \in M \), implies that \([H: mcd] \), by hypothesis. \( n \in [0: mcd] \cup [H + J(M); m] \cup [H + J(M); m] \).

But \( \neq cdn \), it follows that \( n \in [0: mcd] \).

Hence \( n \in [H + J(M); m] \cup [H + J(M); m] \). That is either \( n \in [H + J(M); m] \cup [H + J(M); m] \) or \( dni \in H + J(M) \).

Hence \( H \) is a WNQP submodule of \( M \).

**Proposition (2.4)**

A proper submodule \( H \) of an \( R \)-module \( M \) is a WNQP submodule of \( M \) if and only if for each \( c \in R \) and \( n \in M \) with \( cn \in H + J(M) \), \([H: cn] \subseteq [0: cn] \cup [H + J(M); n] \).

**Proof:**

\((\Rightarrow)\) Let \( d \in [H: cn] \), implies that \( cdn \in H \). If \( cdn = 0 \), then \( d \in [0: cn] \cup [H + J(M); n] \). If \( \neq cdn \in H \) for \( c, d \in R, n \in M \) with \( cn \notin H + J(M) \), it follows that \( dni \in H + J(M) \), that is \( d \in [H + J(M); n] \) so \( d \in [0: cn] \cup [H + J(M); n] \). Hence \([H: cn] \subseteq [0: cn] \cup [H + J(M); n] \).

\((\Leftarrow)\) Let \( \neq cdn \in H \), with \( cn \notin H + J(M) \) for \( c, d \in R, n \in M \), implies that \( d \in [H: cn] \), it follows by hypothesis, we get \( d \in [0: cn] \cup [H + J(M); n] \). Since \( \neq cdn \), implies that \( d \notin [0: cn] \). Thus \( d \in [H + J(M); n] \), implies that \( dni \in H + J(M) \). Hence \( H \) is a WNQP submodule of \( M \).

The following corollaries are direct application of proposition (2.4).

**Corollary (2.5)**

A proper submodule \( H \) of an \( R \)-module \( M \) is a WNQP submodule of \( M \), if and only if for every \( c \in R \) and every submodule \( L \) of \( M \) with \( cl \notin H + J(M) \), \([H: cl] \subseteq [0: L] \cup [H + J(M): L] \).

**Corollary (2.6)**

A proper submodule \( H \) of an \( R \)-module \( M \) is a WNQP submodule of \( M \) if and only if for every ideal \( A \) of \( R \) and every submodule \( L \) of \( M \) with \( AL \subseteq H + J(M) \), \([H: AL] \subseteq [0: L] \cup [H + J(M); L] \).

**Proposition (2.7)**

A proper submodule \( H \) of an \( R \)-module \( M \) is a WNQP submodule of \( M \) if and only if whenever \( \neq cdL \in H \), for \( c, d \in R, L \) is a submodule of \( M \), implies that either \( cl \subseteq H + J(M) \) or \( dL \subseteq H + J(M) \).

**Proof:**

\((\Rightarrow)\) Let \( \neq cdL \subseteq H \) for \( c, d \in R, L \) a submodule of \( M \) with \( cl \subseteq H + J(M) \) and \( dL \subseteq H + J(M) \). Thus \( \neq cdeL \in H \) and \( H \) is a WNQP submodule of \( M \) and \( cL \subseteq H + J(M) \), implies that \( dceL \in H + J(M) \). Again \( \neq cdeL \in H \), and \( H \) is a WNQP submodule of \( M \) and \( cL \subseteq H + J(M) \), implies that \( dceL \in H + J(M) \). Also \( \neq cdeL \in H + J(M) \), implies that either \( c(eL + eL) \in H + J(M) \) or \( d(eL + eL) \in H + J(M) \). If \( c(eL + eL) = cceL + cceL \in H + J(M) \), but \( cceL \notin H + J(M) \), implies that \( cceL \in H + J(M) \) contradiction. If \( d(eL + eL) = deL + deL \in H + J(M) \) and \( deL \in H + J(M) \), then \( deL \in H + J(M) \) contradiction. Thus \( cl \subseteq H + J(M) \) or \( dL \subseteq H + J(M) \).

\((\Leftarrow)\) It is obvious.
Proposition (2.8)
A proper submodule $H$ of an R-module $M$ is a WNQP submodule of $M$ if and only if whenever $\{0\} \neq CDL \subseteq H$, for $C, D$ are ideals of $R$, and $L$ is a submodule of $M$, implies that either $CL \subseteq H + J(M)$ or $DL \subseteq H + J(M)$.

Proof:
($\Rightarrow$) Let $\{0\} \neq CDL \subseteq H$, where $C, D$ are ideals of $R$ and $L$ is a submodule of $M$ with $CL \subseteq H + J(M)$ and $DL \subseteq H + J(M)$, it follows that there exists a non-zero elements $n_1, n_2 \in L$, and a non-zero elements $c \in C, d \in D$ such that $cn_1 \in H + J(M)$ and $dn_2 \in H + J(M)$. Thus we have $\{0\} \neq cdn_1 \subseteq H$, and $H$ is a WNQP submodule of $H + J(M)$, implies that $dn_1 \in H + J(M)$. Also, $\{0\} \neq cdn_2 \subseteq H$ and $H$ is a WNQP submodule of $H + J(M)$, and $dn_2 \in H + J(M)$. $\{0\} \neq cdn_1 \subseteq H$ and $H$ is a WNQP submodule of $H + J(M)$, implies that $dn_1 \in H + J(M)$. Also, $\{0\} \neq cdn_2 \subseteq H$, and $H$ is a WNQP submodule of $H + J(M)$, and $dn_2 \in H + J(M)$. Thus direct consequence of proposition (2.8) we have the following corollaries.

Corollary (2.9)
A proper submodule $H$ of an R-module $M$ is a WNQP if and only if whenever $\{0\} \neq CDL \subseteq H$ for $C, D$ are ideals of $R$, and $\{0\} \neq CDL \subseteq H$ for $C, D$ are ideals of $R$, and $n \in M$, implies that either $nL \subseteq H + J(M)$ or $DN \subseteq H + J(M)$.

Corollary (2.10)
A proper submodule $H$ of an R-module $M$ is a WNQP if and only if whenever $aDL \subseteq H$, $a \in R, D$ is an ideal of $R$, and $L$ is a submodule of $M$, implies that either $CL \subseteq H + J(M)$ or $DL \subseteq H + J(M)$.

The following results are some basic properties of WNQP submodules.

Proposition (2.12)
Let $L, K$ be submodules of an R-module $M$ with $L$ is a proper submodule of $K$ and $J(M) \subseteq J(K)$. If $L$ is a WNQP submodule of $M$, then $L$ is a WNQP submodule of $K$.

Proof:
Let $\{0\} \neq CDN \subseteq L$ for $c, d$ are ideals of $R$, $N$ is a submodule of $L \subseteq M$. Since $L$ is a WNQP submodule of $M$, then $L$ is a WNQP submodule of $K$.

Proposition (2.13)
Let $L \subseteq J(M)$ or $DN \subseteq L + J(M)$, since $J(M) \subseteq J(K)$, we get either $CN \subseteq L + J(K)$ or $DN \subseteq L + J(K)$. Hence $L$ is a WNQP submodule of $K$.

We need to recall the following Lemma

Lemma (2.13) [5, prop. (1.2.16)]
If $H$ is a coclosed submodule of an R-module $M$, then $H = H \cap J(M)$.

Proposition (2.14)
Let $L$ and $K$ are submodules of an R-module $M$ with $K \subseteq L$ and $K$ is a coclosed submodule of $M$ such that $J(M) \subseteq K$. If $L$ is a WNQP submodule of $M$, then $L \cap K$ is a WNQP submodule of $K$.

Proof:
Since $K \subseteq L$, then $L \cap K$ is a proper submodule of $K$. Let $\{0\} \neq CDH \subseteq L \cap K$ where $C, D$ are ideals of $R$, $H$ is a submodule of $K$, it follows that $H$ is a submodule of $M$, so $\{0\} \neq CDH \subseteq L$ and $\{0\} \neq CDH \subseteq K$. Since $L$ is a WNQP submodule of $M$, by then proposition (2.8) either $CH \subseteq L + J(M)$ or $DH \subseteq L + J(M)$, it follows that either $CH \subseteq L + J(M) \cap K$ or $DH \subseteq L + J(M) \cap K$. But $J(M) \subseteq K$, then by modular law, we get either $CH \subseteq (L \cap K) + (K \cap J(M))$ or $DH \subseteq (L \cap K) + (K \cap J(M))$. Thus $L \cap K$ is a WNQP submodule of $K$.

Proposition (2.15)
Let $L, K$ be WNQP submodules of an R-module $M$ with $K \subseteq L$ and either $J(M) \subseteq L$ or $J(M) \subseteq K$. Then $L \cap K$ is a WNQP submodule of $M$.

Proof:
It is clear that $L \cap K$ is a proper submodule of $M$. Let $\{0\} \neq dHL \subseteq L \cap K$ where $a, d \in R$, $H$ is a submodule of $M$, implies that $\{0\} \neq dHL \subseteq L$, and $\{0\} \neq dHL \subseteq K$. But $L, K$ are WNQP, then by proposition (2.7) either $aHL \subseteq L + J(M)$ or $dHL \subseteq L + J(M)$ and either $aKL \subseteq K + J(M)$ or $dHL \subseteq K + J(M)$. It follows that either $aHL \subseteq L + J(M) \cap (K + J(M))$ or $dHL \subseteq L + J(M) \cap (K + J(M))$, suppose that $J(M) \subseteq L$, it follows that $L + J(M) = L$. Thus either $aHL \subseteq L + (K + J(M))$ or $dHL \subseteq L + (K + J(M))$. Hence by modular law either $aHL \subseteq (L \cap K) + J(M)$ or $dHL \subseteq (L \cap K) + J(M)$. Therefore $L \cap K$ is a WNQP submodule of $M$.

Proposition (2.16)
Let $M$ be an R-module with $J(M)$ is a weakly quasi prime submodule of $M$, and $L$ is a proper submodule of $M$ with $L \subseteq J(M)$. Then $N$ is a WNQP submodule of $M$.

Proof:
Let $\{0\} \neq cdH \subseteq L$, for $c, d \in R$, $H$ is a submodule of $M$, either $\{0\} \neq cdH \subseteq J(M)$. But $J(M)$ is a weakly quasi prime submodule of $M$, then either $cH \subseteq J(M) \subseteq L + J(M)$ or $dH \subseteq L + J(M)$.
$J(M) \subseteq L + J(M)$. Hence $L$ is a WNQP submodule of $M$.

**Corollary (2.17)**
Let $L$ be a small submodule of an $R$-module $M$ with $J(M)$ is a weakly quasi prime submodule of $M$. Then $L$ is a WNQP submodule of $M$.

**Proof:**
Since $L$ is a small submodule of $M$, then by [6] $L \subseteq J(M)$, hence the proof follows by proposition (2.16).

**Proposition (2.18)**
Let $M$ be an $R$-module, and $L$ be a submodule of $M$ with $J(M) \subseteq L$. Then $L$ is a WNQP submodule of $M$ if and only if $[L:M]$ is a WNQP submodule of $M$ for any ideal $J$ of $R$.

**Proof:**
($\Rightarrow$) Let $0 \neq cd m \in [L:M]$ for $c, d \in R, m \in M$, then $\{0\} \neq cd f(m) \subseteq L$. Since $L$ is a WNQP submodule of $M$, then by proposition (2.8) either $cJ(m) \subseteq L + J(M)$ or $dJ(m) \subseteq L + J(M)$. But $J(M) \subseteq L$, implies that $L + J(M) = L$, it follows that either $cdm \in [L:M]$, or $dJ(m) \subseteq L$, implies that either $cJ(m) \subseteq L + J(M)$ or $dJ(m) \subseteq L + J(M)$ or $dm \in [L:M]$. Thus $[L:M]$ is a WNQP submodule of $M$.

($\Leftarrow$) Direct by taken $J = R$.

**Corollary (2.19)**
Let $M$ be an $R$-module, and $L$ is a submodule of $M$ with $J(M) = \{0\}$. Then $L$ is a WNQP submodule of $M$ if and only if $[L:M]$ is a WNQP submodule of $M$ for any ideal $J$ of $R$.

**Proof:**
Since $J(M) = \{0\}$, then by [7,prop.(9.14)] $J(M) \subseteq L$. Hence proof follows by proposition (2.18).

**Proposition (2.20)**
Let $f \in \text{Hom}(M,M')$ be an $R$-epimorphism and $\text{Ker} f$ is a small submodule of $M$ and $H$ is a WNQP submodule of $M'$. Then $f^{-1}(H)$ is a WNQP submodule of $M$.

**Proof:**
Let $0 \neq cd f(m) \in f^{-1}(H)$, for $c, d \in R, m \in M$ implies that $0 \neq cd f(m) \in H$. Since $H$ is a WNQP submodule of $M$, it follows that either $cf(m) \in H + J(M')$ or $df(m) \in H + J(M')$ it follows by [7, coro. (9.15)(a)] either $cm \in f^{-1}(H) + f^{-1}(J(M')) = f^{-1}(H) + J(M')$ or $dm \in f^{-1}(H) + J(M')$. Hence $f^{-1}(H)$ is a WNQP submodule of $M$.

**Proposition (2.21)**
Let $f \in \text{Hom}(M,M')$ be an $R$-epimorphism and $\text{Ker} f$ is a small submodule of $M$. If $H$ is a WNQP submodule of $M'$, then it is clear that $f(H)$ is a proper submodule of $M'$. Let $0 \neq cd m' \in f(H)$, for $c, d \in R, m' \in M'$. But $f$ is an epimorphism, then $0 \neq cd f(m) \in f(H)$ for some non-zero $m \in M$. Thus $0 \neq cd f(m) = f(n)$ for some non-zero $n \in H$, implies that $0 \neq cd m \in H$ (since $\text{Ker} f \subseteq H$). But $H$ is a WNQP submodule of $M$, then either $cm \in H + J(M)$ or $dm \in H + J(M)$, thus either $cf(m) \in f(H) + f(J(M))$ or $df(m) \in f(H) + f(J(M))$. Since $\text{Ker} f$ is a small submodule of $M$, then by [7,coro.(9.15)(b)] $f(J(M)) = J(M')$, hence either $cm' \in f(H) + J(M')$ or $dm' \in f(H) + J(M')$. Therefore $f(H)$ is a WNQP submodule of $M'$.

**Proposition (2.22)**
Let $M = M_1 \oplus M_2$ where $M_1, M_2$ are hollow $R$-modules, and $H_1, H_2$ are WNQP submodule of $M_1, M_2$ respectively. Then $H = H_1 \oplus H_2$ is a WNQP submodule of $M = M_1 \oplus M_2$.

**Proof:**
Since $M_1, M_2$ are hollow $R$-modules then $H_1, H_2$ are small submodules of $M_1, M_2$ respectively then by [9,prop.(5.20)(1)] $H_1 \oplus H_2$ is a submodule of $M$. Let $(0,0) \neq cd (m_1, m_2) \in H_1 \oplus H_2$ where $(m_1, m_2) \in M_1 \oplus M_2$ such that $m_1, m_2$ are non-zero elements of $M_1, M_2$ respectively, $c, d \in R$. Since $H_1 \oplus H_2$ small submodule of $M$ then $H_1 \oplus H_2 \subseteq J(M) = (J(M_1 \oplus M_2))$, implies that $H_1 \oplus H_2 + J(M_1 \oplus M_2) = J(M_1 \oplus M_2)$. Now, $(0,0) \neq cd (m_1, m_2) \in H_1 \oplus H_2$, implies that $0 \neq cd m_1 \in H_1$ and $0 \neq cd m_2 \in H_2$. But $H_1, H_2$ are WNQP submodule of $M_1, M_2$ respectively, then either $cm_1 \in H_1 + J(M_1)$ or $dm_1 \in H_1 + J(M_1)$ and either $cm_2 \in H_2 + J(M_2)$ or $dm_2 \in H_2 + J(M_2)$. But both $H_1, H_2$ is small submodules of $M_1, M_2$ respectively it follows that $H_1 \subseteq J(M_1)$, $H_2 \subseteq J(M_2)$ that is $H_1 + J(M_1) = J(M_1)$ and $H_2 + J(M_2) = J(M_2)$. Hence either $cm_1 \in J(M_1)$ or $dm_1 \in J(M_1)$ and either $cm_2 \in J(M_2)$ or $dm_2 \in J(M_2)$, implies that either $c(m_1, m_2) \in J(M_1) \oplus J(M_2)$ or $d(m_1, m_2) \in J(M_1) \oplus J(M_2)$. Thus either $c(m_1, m_2) \in J(M_1) \oplus J(M_2)$ or $d(m_1, m_2) \in J(M_1) \oplus J(M_2)$.

**Proposition (2.23)**
Let $M = M_1 \oplus M_2$ be $R$-module, and $H_1$ is a small submodule of $M_1$, and $M_2$ has no maximal submodule. Then $H_1$ is a WNQP submodule of $M_1$ if and only if $H_1 \oplus M_2$ is a WNQP submodule of $M$.

**Proof:**
($\Rightarrow$) Let $(0,0) \neq cd (m_1, m_2) \in H_1 \oplus M_2$, for $c, d \in R, (m_1, m_2) \in M_1 \oplus M_2$, $m_1, m_2$ are non-zero elements of $M_1, M_2$ respectively, then it follows that $0 \neq cd m_1 \in H_1$. Since $H_1$ is a WNQP submodule of $M_1$, then either $cm_1 \in H_1 + J(M_1)$ or $dm_1 \in H_1 + J(M_1)$, but $H_1$ is small, so $H_1 \subseteq J(M_1)$, that is $H_1 + J(M_1) = J(M_1)$, and since $M_2$ has no maximal submodule, then $M_2 = J(M_2)$. Thus either $cm_1 \in J(M_2)$ or $dm_1 \in J(M_2)$, and $d(m_1, m_2) \in J(M_1) \oplus J(M_2)$. That is $H_1 \oplus M_2$ is a WNQP submodule of $M$. 

98
since \( m_2 \in M_2 \). It follows that either \( c(m_1, m_2) \in J(M_1) \oplus J(M_2) = (H_1 \oplus M_2) + J(M_1) \oplus J(M_2) + J(M_1) \oplus J(M_2) = J(M_1) \oplus J(M_2) \subseteq (H_1 \oplus M_2) + J(M_1) \oplus J(M_2) \). That is \( H_1 \oplus M_2 \) is a WNQP submodule of \( M \).

\((\Leftarrow)\) Let \( 0 \neq cdm_1 \in H_1 \), for \( c,d \in R \), \( m_1 \) is a non-zero element of \( M_1 \); it follows for each \( m_2 \in M_2 \), \( (0,0) \neq cd(m_1, m_2) \in H_1 \oplus M_2 \) but \( H_1 \oplus M_2 \) is a WNQP submodule of \( M \), then either \( c(m_1, m_2) \in (H_1 \oplus M_2) + J(M_1) \oplus J(M_2) \) or \( d(m_1, m_2) \in (H_1 \oplus M_2) + J(M_1) \oplus J(M_2) \). But \( H_1 \) is a small submodule of \( M_1 \), then \( H_1 \subseteq J(M_1) \), that is \( H_1 + J(M_1) = J(M_1) \) and since \( M_2 \) has no maximal submodule, then \( M_2 = J(M_2) \). Thus either \( c(m_1, m_2) \in (H_1 \oplus M_2) + (H_1 + J(M_1)) \oplus M_2 \) or \( d(m_1, m_2) \in (H_1 \oplus M_2) + (H_1 + J(M_1)) \oplus M_2 \).

References